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CIRCUITS IN ELECTRICAL ENGINEERING

BY

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To Helen

and

The Children

PREFACE

Two seemingly conflicting demands are made of electric circuits courses by the typical modern electrical engineering curriculum:

1. Coverage of a wide variety of topics considered basic to the diversity of specialized applications that follow in later courses and in practice.
2. A rigorous and fairly detailed analytical treatment of all basic principles.

The problem facing all instructors of electric circuits is to satisfy both demands, so far as is possible within the brief time allotted by the curriculum to the study of circuits. This book has been written as one answer to the problem.

Tradition has long decreed that the formal study of electric circuit theory be divided into the separate studies of direct-current and alternating-current circuits. The result has been an undue allotment of time to the study of "d-c," and an insufficiency of time for the study of "a-c." It has also led to an unwarranted compartmentalizing of thinking on the part of many students. This book seeks to avoid these difficulties by taking the logical step of employing a *unified* approach to the study of electric circuits. *This approach provides one set of steady-state circuit principles sufficiently broad to apply to all frequencies to which the circuit viewpoint is applicable—including the zero-frequency case commonly accorded separate treatment under the name "direct current."* As a result more time can be devoted to thorough study of a greater variety of principles, and the attainment of a broader student viewpoint.

A major portion of this book is devoted to a study of steady-state circuit principles through the use of complex numbers. However, a study in terms of instantaneous values of both sinusoidal and nonsinusoidal wave relations and a study of transients are also important parts of the book. The unusual features of this book are:

1. A unified presentation of steady-state circuit principles that treats "d-c" as zero-frequency "a-c."
2. Enough fundamental physical theory so that an introductory course in electrical engineering is not absolutely essential to successful use of the book. A knowledge of mathematics through basic calculus is assumed.
3. Equal emphasis on the loop and node methods of solution.
4. Application of a variety of modern network theorems to both power and communications circuits.
5. An introduction to the concept of duals.
6. Direct use of AIEE *American Standard Definitions of Electrical Terms* throughout, with clarifying discussions where necessary.
7. Rationalized mks system of units discussed and employed.

The author acknowledges his indebtedness to many people without whose cooperation or assistance this book could never have been completed. Inevitably, he is indebted to the many writers who have preceded him in the field. His publishers, and his editor, Dr. W. L. Everitt, have been most helpful and cooperative. Many valuable suggestions have come from his students who have used this book in an earlier mimeographed version. He wishes to acknowledge particularly the cooperation of Dean W. H. Hall and Professor W. J. Seeley, as well as the encouragement and suggestions of Professor E. K. Kraybill. To Mr. Jack B. Harper, who computed the data for many of the curves, and to the typists—too numerous to mention by name—who worked so faithfully on the several drafts of the manuscript, many thanks!

CHARLES R. VAIL

Durham, North Carolina

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PART I

INTRODUCTION

CHAPTER 1

GENERAL BACKGROUND

1.01. The Engineering Profession

It is by no accident that the term "engineer" stems from the same Latin root as does the word "ingenious." The ingenuity of the engineer is traditional. Incompletely but humorously an engineer has been defined as "one who can tell whether a new machine will run before it is built—any darned fool can tell afterward!" A more formal definition states that engineering is the art and science which utilizes the properties of matter and the sources of energy in nature to provide structures, machines, and manufactured products for the benefit of mankind. Both of these definitions leave much to be desired; for no mention is made of the fact that engineering provides both products and services; that, under any economic system, it must function on a sound economic basis; and that it involves the organization and the direction of the human effort needed to accomplish the technical objectives.

So rapidly has engineering developed during the past few decades, and so significant has the position of the engineer become in our social structure, that the original overalls-and-wrench concept of the engineer is rapidly yielding in the public mind to a concept of the engineer as a responsible member of an important and respected profession. W. E. Wickenden, late president of the Case School of Applied Science, has said: "The essence of a profession is social trusteeship. A physician is expected not only to be skilled in the arts of diagnosis and of therapy, but also to guarantee in company with his fellow physicians the standards and facilities of the health service offered to his community. . . . Likewise, an engineer is asked not only to be expert in the technique and economy of materials and energy, of structures and products and utility services, but also to guarantee that society will reap the fruits of scientific advance and technical progress in general well-being."[1] It is to be hoped that the student will feel a sense of self-dedication to a high calling as he undertakes to master the basic technical principles outlined in this book.

1.02. Electrical Engineering

The branch of the engineering profession known as electrical engineering enters into every form of industry and public service in which power is utilized, intelligence is transmitted, and precise control is exercised over physical, chemical, or mechanical operations. Electrical engineering is concerned with the generation, transmission, distribution, and utilization of electric power; with communications, embracing telegraphy, telephony, radio, television, and radar; with illumination; with electric transportation; and with industrial processes and their control. In any one of these varied fields the electrical engineer may engage in work which ranges over a variety of activities, such as:

Research
Development
Design
Manufacture
Application
Sales
Operation
Maintenance
Management
Teaching

Although some of these activities require a more detailed technical knowledge than others, all require a grasp in common of certain basic scientific principles. The object of this book is to present those electrical principles which may be considered basic to nearly all phases of electrical engineering activity.

1.03. Physical Concepts Basic to All Engineering

Matter, *energy*, *distance*, and *time* are the fundamental physical quantities with which the engineer must deal in his technical activities. The engineer is provided with a knowledge of these quantities by the chemist and by the physicist, who put their information into tables of material properties and statements of physical laws governing the interrelationships of these fundamental quantities.

The average person is fairly conversant in a qualitative sort of way with many of the elementary laws of physics as a logical consequence of his everyday experiences. Simple laws of mechanics are perceived and become almost second nature to him through such common acts as opening doors, riding elevators, driving automobiles, and hammering nails. To that same person, however, electricity may be surrounded with mystery. The idea that "not even the scientists know what electricity is" has a grip on the common mind, which then proceeds to ignore the fact that there is nothing fundamental in the universe which scientists can "explain."

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[1] Wickenden, W. E., "Goals in Engineering Education," *Electrical Engineering*, February 1945, Vol. 64, No. 2, page 67.

All physical phenomena must be accepted on faith: we observe them and must therefore accept their existence. At best, science can merely seek to discover how they operate and must treat the question "Why?" as being scientifically unanswerable. Electricity is no more of a scientific mystery than any of the other phenomena of nature; they are all, in a sense, mysteries. To put it another way, we know as much about electricity as we do about any other basic physical phenomenon. In fact, electric, hydraulic, dynamic, and chemical forces have been found to be governed by the same general laws. The sooner the student appreciates this fact and the sooner he begins relating in his mind corresponding phenomena from all branches of physical science, the sooner he will arrive at a good understanding of the principles of electricity. Since understanding gained in one branch of science or of engineering can easily be translated into understanding of other branches, the principles governing electricity can almost be inferred by an alert mind from everyday experiences with other physical laws in our daily living. When the student, upon experiencing variations in flow of water while taking his morning shower, finds himself comparing this occurrence with varying potential drop in an electric supply line, he can congratulate himself. To him, electricity, too, is well on the way to becoming "almost second nature."

1.04. General Scientific Procedure

As applied scientists, engineers can profitably employ in their attack on technical problems the same general procedure followed by workers in the field of pure science. This general scientific procedure can be divided into the following six steps: *definition, observation, mathematical formulation, mathematical manipulation, interpretation of results, and experimental verification of results.*

1. *Definition.* In engineering, as in other fields of human endeavor, the transmission of information from the mind of one individual to that of another cannot be accomplished effectively without the use of the spoken or written word. This fact in itself constitutes a powerful argument for the development of proficiency in the use of the English language on the part of every engineering student. Unless two individuals agree on a common definition of all words employed in a mutual exchange of ideas and information, an accurate exchange is impossible. Definition must therefore be the first step in the scientific procedure. It is important to note that after repeated use, definitions in the literal form give way to concepts in the mind of the user; they acquire far more significance than can be found in the bare words of the original definitions. Although these concepts ultimately form the pattern of our thinking, their origin in definitions cannot be overlooked.

In this text, in order to be consistent and to employ the latest and most generally accepted terminology and notation, the American Institute of Electrical Engineers (AIEE) *American Standard Definitions of Electrical Terms* and the *American Standard Graphical Symbols* sponsored by the American Institute of Electrical Engineers and the American Society of Mechanical Engineers and approved by the American Standards Association are used throughout.

2. *Observation,* the second step of the general scientific procedure, involves the gathering of statistical information based on repeated observations of the occurrence of physical phenomena. From these observations, the chemist, the physicist, the biologist, and the other basic scientists can eventually discern a common behavior pattern which we term a "natural law."

3. *Mathematical formulation,* or reduction of the statements of the observed laws to a "shorthand" form involving symbolic notation, is the third step of the general scientific procedure.

4. *Mathematical manipulation* of the formulas obtained in the preceding step is necessary when several phenomena are interrelated, that is, when there are a number of variables. The unaided human mind is limited to handling at any one time relations involving only a few interrelated variables. Solution of the more complex problems requires the aid of a powerful analytical tool, mathematics. Were it not for the fact that mathematics enables man to reason far beyond the inherent capacity of his unaided mind,

scientific progress would be virtually impossible. Mathematics can serve as a means of amplifying almost limitlessly man's basically feeble reasoning power. The result of such mathematical manipulation is a derived relation which may have startling significance, following completion of the fifth step.

5. *Interpretation* of the derived relations is the fifth step of the scientific procedure. By such interpretation, new insights are obtained into the phenomena observed, and the way is often pointed to the discovery of still further phenomena. This step can reduce the search for new principles from a random fine-tooth combing of all observable phenomena to an intelligently guided study of a relatively few closely related phenomena.

6. *Experimental verification* of the newly derived relations and their interpretation closes the cycle of procedure leading to the making of a new scientific discovery.

The preceding tabulation of steps in the general scientific procedure suggests a deliberate, plodding sort of progress. Much scientific progress is of that sort. Occasionally a flash of inspiration can cause a brilliant mind apparently to jump one or more of these steps. Even to genius of this sort, however, a scientific discovery is, in the words attributed to Edison, "one per cent inspiration and ninety-nine per cent perspiration."

1.05. Field and Circuit Viewpoints

The electrical engineer, in his technical activities, is called upon to deal with a broad array of related physical phenomena which either are electrical in their nature or are brought about by or give rise to electrical phenomena. The study of electrical phenomena and of the inseparably related magnetic phenomena is usually approached from one of two viewpoints, depending on the type of immediate application: the field viewpoint or the circuit viewpoint.

The field viewpoint of electrical and magnetic phenomena is the more general. A field may be defined, in broadest terms, as a physical entity in which varying magnitudes must be specified in terms of time and three-dimensional space. A better picture of what is meant by the term "field" as used in electrical engineering can be obtained by defining more specifically an electric field and a magnetic field. *An electric field can be said to exist in any portion of space or matter if an electric charge when suspended therein experiences* *a force.* Any electrically charged conductor is seen to possess a surrounding electric field, because in its presence another charged conductor is either repelled or attracted. Similarly, *a magnetic field is said to exist in any portion of space or matter if a (hypothetical) isolated north pole when suspended therein experiences a force, or if a magnetized particle when suspended therein experiences a torque.* A magnetic field is seen to exist between the poles of the familiar horseshoe magnet, since a compass needle placed between the poles will experience a definite torque. Both of these fields possess properties whose magnitudes may be functions of both time and three-dimensional space.

The circuit viewpoint of electrical and magnetic phenomena is, in reality, merely a specialized modification of the field viewpoint. *A circuit may be defined as a physical entity in which varying magnitudes can be sufficiently specified in terms of time and, at most, one space dimension.* Movement or variation along a path is usually implied.

As an example, the analysis of water flow through a uniform pipe can be treated, for most engineering purposes, as a circuit problem, whereas the analysis of water currents in a large swimming pool fed and drained by two or more pipes is a field problem. The flow of water through the pipe can be assumed, with sufficient engineering accuracy, to involve the same pressure and velocity at a given instant for all points on a given cross section of the pipe. The solution of a problem based on this system requires only the specification of time variation of either pressure or velocity axially along the pipe, involving only a single space dimension. In the swimming-pool problem, however, a three-dimensional description involving every point of the pool is necessary. Thus the pipe exemplifies a circuit problem, and the swimming pool exemplifies a field problem.

From the field viewpoint, the energy supplied to an electric lamp by a generator is borne through the intervening space by three-dimensional electromagnetic waves guided, perhaps, by a system of electric conductors. The circuit viewpoint recognizes that the guiding of electromagnetic waves by a system of conductors is always accompanied by the movement of electric charges through the conductors at a rate which is rather simply related to the rate of guided energy flow through the surrounding space. This phenomenon suggests the possibility of treating such guided energy flow in terms of one-dimensional mass movement of charges along electric conductors, rather than by the more complex analysis of field phenomena occurring in the surrounding three-dimensional space.

The decision as to which viewpoint is preferable in a given situation depends, in part, on the nature of the energy loss associated with the system under consideration. If the predominant energy loss is within the system itself, in the form of heat, then the circuit viewpoint is applicable. If, however, energy loss in the form of electromagnetic radiation is a predominant feature of the problem, then certainly the field viewpoint must be employed. The relative simplicity of the circuit viewpoint justifies its use wherever it is applicable. Fortunately, a major portion of electrical engineering practice can be and is based on the principles of circuit analysis. Electric machinery and other electrical devices, including those employed in communications, are essentially assemblages of interrelated circuits.

Because most technical problems of electrical engineering are amenable to circuit analysis, it is in electrical engineering that the techniques of circuit analysis have been developed to a fine point. In all branches of engineering, however, an amazingly large amount of analysis is concerned with some form of circuit. As a consequence, in recent years electrical techniques and terminology have crept into other branches of engineering. The individual who can analyze circuits with facility under conditions of all types has mastered a very important part of engineering technique. This book is designed to present the techniques which are basic to the analysis of circuits encountered in electrical engineering but which also can be applied successfully to the circuits encountered in such diverse branches of engineering as hydraulics, mechanics, and acoustics, and even to chemistry.

1.06. Open and Closed Circuits

A *closed circuit* is one which provides a cyclic space configuration such that the path is continuous. By proceeding along the path, one can return to the starting point. The path followed by the circulating refrigerant of an electric refrigerator exemplifies one type of closed circuit. Magnetic circuits are always closed, because magnetic flux lines are always continuous. An electric circuit may be either open or closed. An example of an *open circuit* is a long steel poker one end of which is thrust into the glowing firebed of a furnace, the other end held in the hand of a person. This is a thermal circuit in which temperature and rate of heat flow are, effectively, functions of the single dimension of distance along the rod. There is no closed path, and so, obviously, the circuit is open. Nevertheless, flow of heat does occur along the poker, as the person holding that poker will testify if he holds it in the fire very long.

1.07. Electric and Magnetic Circuits

The American Institute of Electrical Engineers makes the following definition of an electric circuit: "*A circuit is a conducting part or a system of conducting parts through which an electric current is intended to flow.*"[2]

For purposes of this chapter, the magnetic circuit can be defined as a system of parts through which magnetic flux lines can be established.

Theoretical and experimental investigations show that a remarkable interdependence exists between electric and magnetic circuits. Every electric circuit through which electric charges are moving establishes interlinking magnetic flux lines; and, conversely, every magnetic circuit in which the magnetic flux is changing tends to establish movement of electric charges in all electric circuits with which it interlinks. These interdependent circuit phenomena (or their paralleling field phenomena) underlie virtually every practical application of electricity. So important to electrical engineering is the concept of interlinking electric and magnetic circuits that it is symbolized in the official emblem of the American Institute of Electrical Engineers by two interlinking circles enclosing the initials "E. E." The emblem is reproduced below by courtesy of the Institute.

1.08. Conservation Principles

If we exclude those processes in which the nucleus of an atom is altered, we can employ advantageously two well-known principles which appear to operate in all physical processes. These are the

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[2] *American Standard Definitions of Electrical Terms*, No. 35.40.010. New York: American Institute of Electrical Engineers, 1942.

principle of conservation of matter, that matter can be changed in form but not in amount; and the *principle of the conservation of energy,* that energy can be changed in form but not in amount. They can be applied in all engineering work except that having to do with the applications of nuclear physics. Directly from these two principles can be derived the fundamental laws on which all electric circuit analysis may be based.

As an outgrowth of the two conservation principles, we can state, for all circuits which guide the movement of matter (or energy):

$$\left\{\begin{matrix}\text{Average flow of matter}\\ \text{(or energy) into circuit}\end{matrix}\right\} = \left\{\begin{matrix}\text{average flow of matter}\\ \text{(or energy) from circuit}\end{matrix}\right\}. \qquad (1\text{-}1)$$

This statement assumes, of course, no net storage of matter (or energy) in the circuit itself.

The statement of (1-1) can be illustrated by the flow of water through an inelastic pipe. Since water is virtually incompressible, regardless of fluctuations in rate of flow, then as long as the pipe is completely filled with water a new quantity of water cannot be forced into the pipe by the application of pressure at one end until a similar or greater quantity has left the pipe at the other end. As a matter of fact, even if the water were compressible or the side walls of the pipe elastic, the statement would still hold over any complete cycle of pressure variation.

Electric charges can enter a conductor no faster than they can leave that conductor, if no net charge is to be placed on the conductor.

1.09. Energy Sources and Sinks

Equation (1-1), in the energy sense, brings us directly to a concept fundamental to the analysis of energy-guiding circuits. By definition, we say that in all physical processes energy flows from a region of higher energy level to a region of lower energy level. Through the operation of this principle, heat flows from a region of higher temperature to a region of lower temperature, and water flows downhill, not uphill. Electric charges move from a region of higher absolute electric potential to a region of lower absolute electric potential. In general terms, in any physical situation, the region of higher energy level is called the *energy source,* and that of lower energy level is called the *energy sink*, or *receiver*. Unless a difference of energy level exists, there exists, by our definition, neither source nor sink, and energy flow cannot occur.

In terms of an electric circuit, all parts at which energy enters the circuit are termed sources, and all parts at which energy leaves the circuit are termed sinks, or receivers. The circuit serves as a guide to the flow of energy between the sources and the sinks. The rate of flow of energy from source to sink is termed the *power* of the electric circuit.

In general, transformations of power from one form to another occur at electric-circuit sources and sinks. At a source, power is converted from some other form to the form of electric power. Examples of devices which bring about this change are generators which convert mechanical power and chemical cells which convert chemically stored energy. At a sink, power is converted from the electric form to some other form. Thus, motors convert to mechanical power, and resistors convert to heat.

1.10. Reversible and Irreversible Transformation

The transformations of energy at energy sinks can be divided into two classes: those which are reversible and those which are irreversible.

An *irreversible* energy transformation is one which operates only "one way." The functioning of brakes on an automobile furnishes an example: by draining kinetic energy from the moving vehicle and dissipating it as heat, they can bring the car to rest, but no amount of heating of the brake drums by external means can set the car in motion again. The dissipation of electric energy as heat in an electric toaster exemplifies an irreversible electric-circuit energy transformation.

A *reversible* energy transformation is one in which energy converted from one form to another can be converted back again by a reversal of the transformation process. Two general types of reversible energy transformation must be noted, one involving potential energy and one involving kinetic energy.

In a mechanical system, the type of reversible energy storage which is due to the *position* of a body relative to the rest of the system is termed *potential mechanical energy*. That which is due to the *motion* of the body relative to the rest of the system is termed *kinetic mechanical energy*. The energy associated with a frictionless swinging pendulum oscillates between these two forms as the pendulum endlessly swings back and forth. At each peak of its swing, the pendulum possesses potential mechanical energy which has become, by the bottom of the swing, kinetic mechanical energy. The storage of mechanical energy in the system containing the frictionless pendulum is, in either form, a reversible process, as is evidenced by the fact that the pendulum continues to swing without the addition of external energy. In a practical pendulum, however, an irreversible transformation will also be present. Bearing and air friction will exact their toll from the moving pendulum, continually draining the initial store of energy, until, with swings of ever-decreasing amplitude, the pendulum ultimately comes to rest, its energy completely—and irreversibly—dissipated as heat.

The oscillatory discharging of a capacitor through an inductance coil is the electrical analogue of the swinging pendulum. Associated with the charged capacitor is electric energy stored in the potential form. This potential energy is stored by virtue of the *position* of the electric charges on the capacitor plates. Also, associated with the *moving* charges of the electric current is electric energy in the kinetic form (magnetic energy). The transformation of electric energy into either of these forms is a reversible process: the capacitor can be discharged, or the magnetic field can be caused to collapse, yielding a flow of charge. The heat-dissipating action of electric-circuit resistance, on the other hand, is an irreversible transformation analogous to the effect of mechanical friction. All three of these forms of electric-energy transformation at sinks are of fundamental importance in electric-circuit theory and will shortly be treated in more detail.

One of the inherent characteristics of sources in electric circuits is that the fundamental energy transformatians which they perform are reversible. Thus, generator action can be reversed to produce motor action, a discharging storage battery can be recharged, and so on. In practical devices, secondary effects may disguise partially or completely this inherent reversibility which, nevertheless, is a fundamental characteristic of a source.

1.11. Electricity in Atoms

As a final step in preparing a background for the study of electric circuit principles, it seems appropriate to discuss briefly the modern concept of the atom.

Theory and experiment indicate that all matter contained in the universe is electric in nature. In 1913, Niels Bohr and Sir Ernest Rutherford gave us a planetary concept of the atom, in which each atom consists of an electrically charged central nucleus about which revolve electrically charged satellites. Although modern researches have suggested many modifications of this concept, it still remains a very satisfactory means of picturing the atomic basis of most phenomena encountered in electrical engineering. Over a period of years, a number of elementary particles entering into atomic structure have been identified. The electric charge and the mass of each of these have been determined as closely as possible by existing means. Three of these particles which appear most prominently in the theory underlying the study of electrical engineering are the *electron*, the *proton*, and the *neutron*. For our purposes, each atomic nucleus can be considered to consist of protons and neutrons grouped in various proportions. The satellites revolving around each nucleus are electrons.

The *electron* is the smallest indivisible increment of electricity. It has been observed to possess both the properties of a particle, or corpuscle, and the properties of waves, but no satisfactory explanation of this duality exists. We shall treat the electron as a small particle possessing a mass which, when at rest, has a value of 9.10×10^{-31} kilogram and an electric charge which has a magnitude of $e = 1.601 \times 10^{-19}$ coulomb. By definition, the charge of the electron is said to be *negative*. It is observed that forces of mutual repulsion exist between each electron and all other electrons.

Experiment has confirmed a requirement of the principle of relativity that the mass of all bodies must vary with velocity. The actual mass is given by the relation

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{1-2}$$

where:

m = mass of the body at velocity v,
m_0 = mass of the body at rest,
v = velocity of the body,
c = velocity of light $\simeq 3 \times 10^8$ meters per second.

Unless the body is moving with a velocity which is comparable to that of light, the relativistic increase of mass is negligible. In many engineering applications, however, the velocity of a moving electron may actually approach that of light, with a consequent appreciable increase in mass. In such cases, equation (1-2) must be used to modify the value of electronic mass given in the preceding paragraph.

The *proton* is a particle possessing a mass 1849 times as great as that of the electron, although carrying a charge of equal magnitude but opposite algebraic sign. The proton is said to carry a *positive* charge. This reversal of sign arises from the fact that a proton is attracted by an electron. Forces of mutual repulsion exist between each proton and all other protons.

The *neutron* has a mass about 0.08 per cent larger than that of the proton, and carries no electric charge. Electrically, it is neither attracted nor repelled by charged particles.

Still other particles of matter have been identified through atom-splitting experiments. The *positron* is the "positive electron," or the positive charge of the proton. When a positron meets an electron, they disappear, their masses being converted into gamma rays. The positron appears, usually as the result of atomic disintegration of artificially radioactive materials, in the following process:

$$\text{Proton} \rightarrow \text{neutron} + \text{positron} + \text{neutrino}. \tag{1-3}$$

The *neutrino* has been postulated to ensure the conservation of energy and momentum during the process given above. It has no charge and virtually no mass. These statements are summarized in Table I.

TABLE I

PROPERTIES OF FUNDAMENTAL PARTICLES

Name	Electric Charge	Mass
Electron	$-e$	m
Proton	$+e$	$1849m$
Neutron	0	$1850m$
Positron	$+e$	m
Neutrino	0	Almost zero

$e = 1.601 \times 10^{-19}$ coulomb,
$m = 9.10 \times 10^{-31}$ kilogram.

The *photon*, or corpuscle of light, is of increasing importance in electrical engineering. It can be pictured as the smallest indivisible increment of radiant energy, having an energy value dependent in a given case on the frequency, or color, of radiation, and having a mass of zero. Visible light, heat, radio waves, and X rays convey energy made up of groups of photons.

On the basis of the atomic theory, we are now ready to define the term "electricity": "*Electricity is a physical agent pervading the atomic structure of matter and characterized by being separable, by the expenditure of energy, into two components designated as positive and negative electricity, in which state the electricity possesses recoverable energy.*"[3]

1.12 Units

The quantitative measure of any physical quantity requires two factors: first, the unit, composed of a certain definite amount of the quantity measured; and second, a specification of the number of times this unit is taken. All physical quantities could be expressed in terms of units arbitrarily chosen without reference to the possible physical interrelationship of these quantities. In such a case, all equations expressing physical relations among the quantities would require the inclusion of numerical constants. A more systematic scheme would define arbitrarily as few units as possible and would allow equations relating the physical quantities to define all other units. Under such a scheme, all numerical constants could be avoided except those due to geometric relations.

The number of quantities used in describing a given physical phenomenon is more or less arbitrary and can be decided by convenience. For example, velocity can be described either as a single separate quantity or in terms of time and distance. The latter convention would be undesirably cumbersome for most work. A study of the thirty or so quantities commonly used in the equations of electricity and magnetism reveals that only four—any four—could serve as fundamental quantities, or dimensions, in terms of which all the others could then be expressed.

In the study of electricity and magnetism, the four quantities usually selected as fundamental are *distance, mass, time,* and either *electric permittivity,* ϵ, or *magnetic permeability,* μ.

In the past, a number of systems of units have grown up. Much scientific work has been based on a pair of "scientific" or "cgs" systems, each of which employs as three of its basic units the *centimeter*, the *gram*, and the *second*. For the fourth fundamental unit, the *cgs electrostatic system*, whose unique units are recognizable by the prefix "stat-," as in statvolt, uses the electric permittivity of free space. The *cgs electromagnetic system*, whose unique units are recognizable by the prefix "ab-," as in abvolt, employs as the fourth fundamental unit the magnetic permeability of free space. In each of these systems, the only unavoidable dimensionless factor which appears because of the geometry is 4π. Because of the presence of this factor in certain equations, these systems are said to be *unrationalized*. These systems can be *rationalized* by arbitrarily placing a 4π in the denominator of Coulomb's-law expressions for electric charges and magnetic poles, thus removing the 4π from the equations in which they had originally appeared. One inconvenience found in using these scientific systems is that corresponding units are related by powers of 10 and by powers of the velocity of light, $c = 3 \times 10^{10}$ centimeters per second, with a consequent tedious handling of conversion factors.

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[3] *Ibid.*, No. 05.10.015.

Because both of the scientific systems yield units which are, in many cases, impractical in size, a partial system of "practical" electric units, including the familiar volt, ampere, and ohm, have long been in use. The practical system is related to the cgs electromagnetic system by powers of 10 and to the cgs electrostatic system by both powers of 10 and powers of c. In past years, then, we have been in the absurd position of employing three different systems of units in our work with electricity and magnetism. As a result, many of the equations used in electrical engineering have been scrambled-up combinations of units from all of these systems.

This situation was partially remedied in June 1935, when the International Electrotechnical Commission unanimously adopted a new system proposed by Giorgi, of Italy.[4] This *mks system* employs the *meter*, the *kilogram*, and the *second* as three of its fundamental units. The fourth basic unit has not as yet been determined, although the unit of resistance, the *ohm*, appears a possible choice. Regardless of which of the remaining units is selected as fundamental, they now are all defined in size and name. The remarkable result is a self-consistent system of units containing

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[4] Kennelly, A. E., "I.E.C. Adopts Mks System of Units," *Electrical Engineering* (*Transactions of the AIEE*), December 1935, Vol. 54, page 1337.

TABLE II

CORRESPONDENCE OF MAJOR SYSTEMS OF UNITS

(See also Table A10 in the Appendix)

Quantity	Rationalized MKS Practical Units	Unrationalized CGS Units	
		Electrostatic	Electromagnetic
Length	Meter	Centimeter	
Mass	Kilogram	Gram	
Time	Second	Second	
Force	Newton	Dyne	
Energy	Joule	Erg	
Power	Watt	Erg per second	
Charge	Coulomb	Statcoulomb	Abcoulomb
Current	Ampere	Statampere	Abampere
Charge density	Coulomb per cubic meter	Statcoulomb per cubic centimeter	Abcoulomb per cubic centimeter
Current density	Ampere per square meter	Statampere per square centimeter	Abampere per square centimeter
Resistance	Ohm	Statohm	Abohm
Conductivity	Mho per meter	Statmho per centimeter	Abmho per centimeter
Electric potential	Volt	Statvolt	Abvolt
Electric field intensity	Volt per meter	Statvolt per centimeter	Abvolt per centimeter
Capacitance	Farad	Statfarad	Abfarad
Dielectric displacement	Coulomb per square meter	Statcoulomb per square centimeter	Abcoulomb per square centimeter
Permittivity	Farad per meter	....	....
Electric dipole moment	Coulomb-meter	Statcoulomb-centimeter	Abcoulomb-centimeter
Magnetic flux	Weber	....	Maxwell
Flux density	Weber per square meter	....	Gauss
Magnetomotive force	Ampere-turn	....	Gilbert
Magnetic field intensity	Ampere-turn per meter	....	Oersted
Inductance	Henry	Stathenry	Abhenry
Permeability	Henry per meter	....	....
Magnetic dipole moment	Ampere-square meter	Statampere-square centimeter	Abampere-square centimeter

the "practical" units of electrical engineering: the volt, ampere, ohm, and so on. A further unsettled question is whether the new system shall be rationalized or unrationalized. Among American authors of electrical engineering textbooks, a strong tendency to adopt the *rationalized mks system* is becoming apparent. The present author prefers to follow this trend.

Table II shows, for a number of physical quantities, the correspondence between the rationalized mks system of units and the unrationalized cgs electrostatic and electromagnetic systems. Each is completely self-consistent. Except for certain relations in which the 4π has been suppressed by rationalization, any of the equations of this book may be used equally well with any one of these systems of units, providing that system is employed exclusively. For convenience, throughout the book, many of the most-used equations are followed by a notation of the rationalized mks units which apply. Table A11 of the Appendix carries a table of conversion factors so arranged as to indicate clearly the quantities which are affected by rationalization.

It is often convenient to employ the following prefixes and symbols in stating the values of physical quantities:

micro- = "millionth" $= 10^{-6}$; symbol, μ,
milli- = "thousandth" $= 10^{-3}$; symbol, m,
kilo- = "thousand" $= 10^{3}$; symbol, k,
mega- = "million" $= 10^{6}$; symbol, M.

The following examples illustrate their use:

$$\begin{aligned}
0.000{,}005 \text{ volt} &= 5 \times 10^{-6} \text{ volt} = 5 \text{ microvolts} = 5\mu\text{v},\\
0.005 \text{ volt} &= 5 \times 10^{-3} \text{ volt} = 5 \text{ millivolts} = 5 \text{ mv},\\
5000 \text{ volts} &= 5 \times 10^{3} \text{ volts} = 5 \text{ kilovolts} = 5 \text{ kv},\\
5{,}000{,}000 \text{ volts} &= 5 \times 10^{6} \text{ volts} = 5 \text{ megavolts} = 5 \text{ Mv}.
\end{aligned}$$

1.13. A Note Concerning Symbols

The choice of letters to symbolize the various physical quantities appearing in the equations of this textbook agrees, in general, with the notation followed in publications of the American Institute of Electrical Engineers, including the *American Standard Definitions of Electrical Terms*, published in 1942. The few exceptions are in the interest of convenience and clarity. A list of symbols is included in Table A11, Appendix.

For potential, current, charge, magnetic flux, and time, capital letters generally represent fixed quantities, and lower-case letters generally represent variable quantities. For example: I, i.

Very often, subscripts or superscripts are added to give special meanings to the symbols used for quantities. In such cases, the text discussion defines the meaning intended when the designation is used for the first time.

CHAPTER 2

ELECTRIC CURRENT

2.01. Free Electrons and Ions

If we are to treat the flow of guided electromagnetic energy in terms of the movement of electric charges through the guiding conductors, we must investigate the nature of these charges. We shall begin with the satellite electrons of the planetary atomic model.

The various materials employed in electrical engineering differ in their ability to support the movement of electric charges, because of the differing normal distributions of the satellite electrons about the nuclei of the atoms comprising these different materials. In the normal state, the electrons appear to be grouped about the nucleus of an atom in various *shells*, each of which represents a specific energy *level*, or energy content per electron. The energy content of a given electron attached to an atom depends not only upon the proximity of that electron to the atomic nucleus but also upon its motion about the nucleus and its own axial spin. Because energy appears to occur in increments known as *quanta* (photons are one example of this behavior), it is possible for an electron to gain or to lose energy only in discrete amounts. For this reason, each atom contains very distinct energy levels or shells which it is permissible for an electron to occupy. An electron, in moving from one shell to another, must exchange energy in the exact amount of the difference in level between the shells.

Atoms vary in complexity from the simple hydrogen atom, with a single electron revolving about a single nuclear proton, to extremely complicated systems consisting of scores of electrons moving about complex nuclei containing scores of protons and neutrons. If the shells about an atomic nucleus are numbered in the direction of increasing energy level, the maximum number of electrons which can occupy the nth shell is $2n^2$. Thus, the maximum number of electrons that can be contained in the first shell is 2; in the second, 8; in the third, 18; and so on. A given shell may contain less than the maximum number of electrons specified by this rule, but it may contain no more. Many of the more complex of the elements found in nature possess partially empty outer shells when in the normal state. The innermost shell represents the lowest energy level.

An atom from which one of the normal outershell electrons has been removed is said to be *ionized.* The restraining force that tends to prevent such removal is the electrostatic attraction between the departing electron and the rest of the atom. This attraction results from the fact that in the normal state, the number of negatively charged electrons of an atom exactly balances the number of positively charged nuclear protons, making the atom electrically neutral. Removal of one negative electronic charge in the form of the electron disturbs the balance and leaves a net of one positive electronic charge on the remainder of the atom. It is this positive charge which attracts the electron. For each element, a definite minimum amount of energy called the *ionizing potential* must be given to an outer-shell electron of an atom in the normal state to enable it to escape from the atom.

For a given type of electron arrangement, the attachment of a remote outer-shell electron is less than that of an inner-shell electron. An electron which normally occupies an entire outer shell by itself, or at most shares it with only one or two other electrons, is much less strongly attached than one which occupies an outer shell in company with many other electrons. Good electric conductors are made from the elements, scattered all through the periodic table, in which the outermost shell that is occupied contains only one, two, or—in a few elements—three electrons.

Atoms of these elements, which are known as *conducting materials,* are so closely packed together when in solid assembly that the outer-shell electrons are no more closely associated with one atom than with another. Such electrons, which are not bound to any one atom but are free to rove freely throughout the interior of the material, are known as *free electrons.*

It is of interest to note that since these free electrons continually interchange their kinetic energies of motion with vibrational kinetic energies of the atoms of the metal, the freedom of electron motion aids in the transfer of heat energy from one point to another, thus making materials which are good conductors of electricity also good conductors of heat.

Through the force of electrostatic attraction, atoms combine to form the molecules which make up our material world. Subdivided molecules or groups of molecules, and subdivided atoms or groups of atoms, if they bear charges, are called *ions*. Under the right conditions, these ions can move, carrying with them their charges. The movement of charges accompanying the flow of guided electromagnetic energy can consist either of moving free electrons or of moving ions.

It is observed that a given flow of guided electromagnetic energy will give rise to oppositely moving positive and negative ions. From this fact we can deduce that *movement of positive charges in a positive direction is equivalent to movement of negative charges in a negative direction.* As will be discussed later, either movement is said to constitute a positive flow of electricity, inasmuch as they both accompany the same direction of flow of electromagnetic energy.

Through metals and through a vacuum, the flow of electricity is achieved by the movement of electrons only, sometimes termed *electronic conduction.* Conduction in electrolytes, fused salts, and some solids is achieved by the movement of ions, sometimes termed *ionic conduction.* In gases, conduction can consist of a combination of the two. The translation of electrons attached to small particles of matter suspended in an inert medium can be termed *colloidal conduction;* an example is the electrical precipitation of smoke and dust.

2.02. Quantity of Electricity, or Charge, Q; the Coulomb

"*The quantity of electricity (or charge) on (or in) a body is the excess of one kind of electricity over the other kind. A plus sign indicates that the positive electricity is in excess, a minus sign indicates that the negative electricity is in excess.*"[1] As we have already seen, the smallest indivisible quantity of electricity is the charge of the electron. The proton carries a charge of equal magnitude but opposite sign. So small is this quantity of electricity that numbers of astronomical size would be required to specify a practical amount of charge. For this reason, a more convenient quantity of electricity equal in magnitude to approximately 6.25×10^{18} electronic charges has been adopted as the practical unit. This practical unit of electricity is called the *coulomb*. The symbol for charge is the letter Q.

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[1] *American Standard Definitions of Electrical Terms*, No. 05.10.035. New York: American Institute of Electrical Engineers, 1942.

2.03. Electric Current, I; the Ampere

We can measure the flow of fluid through a pipe in terms of the quantity of fluid moving past a given section of pipe in unit time. Thus, water current may be expressed in terms of *gallons per second.* In like manner, we can measure the flow of electricity through a conductor in terms of the quantity of electricity (or charge) moving past a given section of the conductor in unit of time. Thus, electric current may be expressed in terms of *coulombs per second.* For brevity, and to honor an outstanding pioneer in the study of electricity, the unit *coulomb per second* has been given the name *ampere.* The ampere is the practical unit of electric current and denotes a rate of flow of one coulomb—or 6.25×10^{18} electronic charges—per second.

Inasmuch as electric current can be supported by the movement of either positive or negative charges, it becomes important to define arbitrarily a mathematically "positive" direction of electric current. By international agreement, *the direction of current flow to be considered algebraically positive in a given physical situation is the direction in which positive charges would move if they were the predominating kind in motion.* Negative charges moving in the negative direction would be the equivalent of positive charges moving in this positive direction. If current is supported by both kinds of charges moving in opposite directions, then the resultant positive current will be equal in magnitude to the sum of the absolute values of the two components and will have the direction of the positive charges.

These ideas are embodied in the two following definitions adopted by the American Institute of Electrical Engineers: "*An electric current through a surface is the time rate at which positive or negative electricity passes through the surface. If both positive and negative electricity simultaneously pass through the surface, it is the time rate of passage of the algebraic sum of the two.*"[2] "*The direction of a current through a surface is, by agreement, taken as the direction of the movement of the positive electricity when it is the predominating component in motion and as the direction opposite to the direction of movement of*

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[2] *Ibid.*, No. 05.20.010.

the negative electricity when the latter is the predominating component in motion."[3]

By international agreement, the symbol I (or i), from the French word *intensité,* is used throughout the world to represent electric current. In circuit diagrams we shall follow the practice of indicating the direction of current which we call positive by means either of a closed arrowhead placed on the conductor or by an arrow with a closed head beside the conductor. The directions of particles of a given sign can be indicated by appropriately marked arrows.

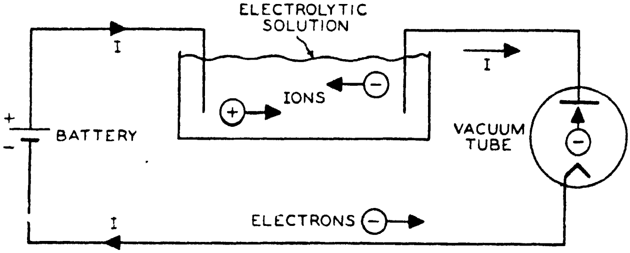

Fig. 2.03. Clockwise electric current.

Figure 2.03 illustrates these conventions as well as illustrating several types of conduction.

The foregoing discussion has been confined to the topic of *conduction current,* or current supported by moving charged particles. It will be necessary to present also, in a subsequent chapter, the concept of *displacement current,* or current supported by a changing electromagnetic field.

Example 2.03. A certain type of vacuum tube can be used to measure currents as low as 100 electrons per second. Express this current in amperes.

The current I = 100 electrons per second

$$= \frac{100}{6.25 \times 10^{18}} = 1.601 \times 10^{-17} \frac{\text{coulomb}}{\text{second}} \text{ or ampere.}$$

2.04. Electron Drift; Random Current Density

It is common knowledge that electric impulses travel along conductors at velocities approaching that of light. From this fact it is sometimes erroneously inferred that the moving charges constituting the flow of electricity also race along the conductors with a velocity approaching that of light. Such is not the case at all. Their velocity along the conductor is, in truth, amazingly low. The fact is that the guided electromagnetic waves do travel with velocities approaching that of light but that the accompanying movement of charges within the conductor is merely a slow drift.

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[3] *Ibid.*, No. 05.20.015.

It should not be supposed that the free electrons of a conductor are sluggish; they move very rapidly—in a random direction. Under the influence of thermal agitation they dart and bounce around inside the conductor at high speeds. The direction of travel of such a particle continually undergoes a series of abrupt changes as the result of multitudinous collisions with other particles. In terms of these electrons, under normal conditions electric current is no more than a drift superimposed on the chaotic motion of thermal agitation. It might be expected that an increase in temperature would increase the rate of thermal agitation and frequency of collision of the particles, thereby reducing their ability to support an electric current. Observation verifies this reasoning for many cases. The electric conductivity of metallic conductors undergoes marked decrease with increase of temperature.

The random velocities of the free electrons have values which spread over a wide range. On the average, these random velocities are very much greater than any velocities that may be imparted by an applied electric field. Computations show that the random current density of a metal is of the order of 10,000 billion amperes per square centimeter.[4] The random current density is the current crossing a unit surface because of random electron velocities, found by counting the electrons crossing in one direction but neglecting the equal number crossing in the reverse direction. The exact value depends upon the metal and, to a slight degree, upon its temperature.

EXAMPLE 2.04. An ordinary 100-watt, 125-volt incandescent lamp takes a current of 0.8 ampere under rated conditions. If this lamp is supplied by a copper wire having a diameter of 0.0641 inch and if copper contains about 1.39×10^{24} free electrons per cubic inch, what is the velocity, in inches per second, with which electrons drift along the wire supplying the lamp?

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[4] M.I.T. Staff, *Applied Electronics*, page 67. New York: John Wiley & Sons, Inc., 1943.

$$\text{The current } I = 0.8 \text{ ampere} = 0.8\,\frac{\text{coulomb}}{\text{second}}$$

$$= 0.8 \times 6.25 \times 10^{18} = 5.00 \times 10^{18}\,\frac{\text{electrons}}{\text{second}}.$$

The cross-sectional area of the wire is

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4} \times (0.0641)^2 = 0.00323 \text{ square inch,}$$

where: d = diameter of the wire.

The number of free electrons per inch of length, or the axial charge, is

$$\tau = 0.00323 \times 1.39 \times 10^{24} = 4.49 \times 10^{21}\,\frac{\text{electrons}}{\text{inch}}.$$

Then the velocity of electron drift is

$$v = \frac{I}{\tau} = \frac{5.00 \times 10^{18}}{4.49 \times 10^{21}} = 1.115 \times 10^{-3} = 0.001115\,\frac{\text{inch}}{\text{second}}.$$

For comparison, the velocity of light c = 186,300 miles per second!

2.05. Kirchhoff's Current Law, $\Sigma I = 0$

From the foregoing considerations, it is now possible to discuss an important principle known as *Kirchhoff's current law*, which can be stated: *The sum of all the currents approaching a given point in an electric circuit must equal the sum of all the currents leaving that point.* If this statement were not true, in the case of conduction current, electric charges would be either created or destroyed at the point in question. Such an occurrence would be an unwarranted violation of the conservation principle, and therefore the statement must be true of conduction current. Field theory can be employed to show its validity also in the case of displacement current. Figure 2.05(a) applies this principle to the junction of three current-carrying conductors. The arrowheads are drawn to indicate the actual current directions at the instant under consideration.

If we wish, we can arbitrarily decide to attach positive algebraic signs only to currents which are approaching a point. This convention would make all departing currents negative approaching currents. To accommodate this viewpoint, Kirchhoff's current law would read: *The algebraic sum of the currents approaching any point in an electric circuit is zero.* Figure 2.05(b) applies this form of the principle to the same junction. Here, the arrowheads are drawn to indicate the arbitrarily assumed (or nominal) positive current directions at the instant under consideration. Two of the currents, I_b and I_c, although indicated by arrowheads to be approaching the point, actually are departing currents. When evaluated numerically, I_b and I_c are therefore provided with negative values.

Although the two statements of Kirchhoff's current law are equivalent, the second form is the more generally used. This principle, which will prove to be of vital importance in our future work dealing with the solution of network problems, can be stated mathematically as

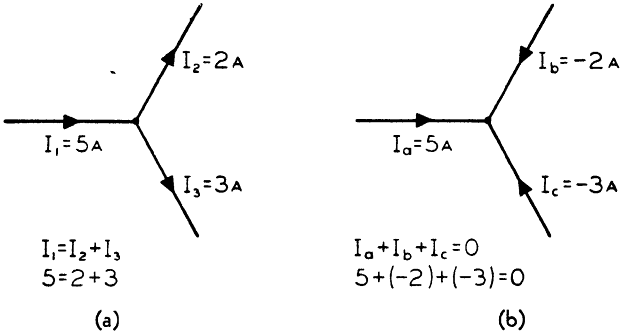

Fig. 2.05. Illustration of Kirchhoff's Law. (a) Arrows indicate actual positive current directions. (b) Arrows indicate nominal positive current directions.

▶ $$\Sigma I = 0, \quad \text{or} \quad \Sigma i = 0. \tag{2-1}$$

2.06. Current as a Function of Time

Electric currents may be classified according to their direction of flow and characteristic mode of variation with time. A summary of some of these classes follows:[5]

1. *Unidirectional Current.* A unidirectional current is a current which has either all positive or all negative values.

2. *Direct (or Continuous) Current.* A direct current is a unidirectional current in which the changes in value are either zero or so small that they may be neglected.

3. *Bidirectional Current.* A bidirectional current is a current which has both positive and negative values.

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[5] *American Standard Definitions of Electrical Terms*, Group 05, Section 20. New York: American Institute of Electrical Engineers, 1942.

4. *Oscillating Current.* An oscillating current is a current which alternately increases and decreases in magnitude with respect to time according to some definite law.

5. *Periodic Current.* A periodic current is an oscillating current the values of which recur at equal intervals of time.

6. *Alternating Current.* An alternating current is a periodic current the average value of which over a period is zero.

7. *Symmetrical Alternating Current.* A symmetrical alternating current is an alternating current of which all pairs of values that are separated by a half period have the same magnitude but the opposite sign.

8. *Pulsating Current.* A pulsating current is a periodic current which is the sum of a direct current and an alternating current.

9. *Ripple Current.* A ripple current is the alternating current component of a pulsating current when this component is small relative to the direct-current component.

These definitions will assume more significance as we progress through this book.

PROBLEMS

2-1. A certain storage battery has a rating of 300 *ampere-hours* (amperes × hours). Express this value in coulombs.

2-2. Express a current of 15×10^{20} electrons per second in terms of: (a) amperes; (b) microamperes; (c) milliamperes; (d) kiloamperes; (e) megamperes.

2-3. A certain capacitor is charged with 0.05 coulomb of electricity in 2 milliseconds. What is the average charging current?

2-4. Under a certain condition of operation, the rated plate current of a 6J7 detector-amplifier vacuum tube is 2 milliamperes. If all the current is due to electrons moving through the vacuum, how many electrons arrive at the plate each second?

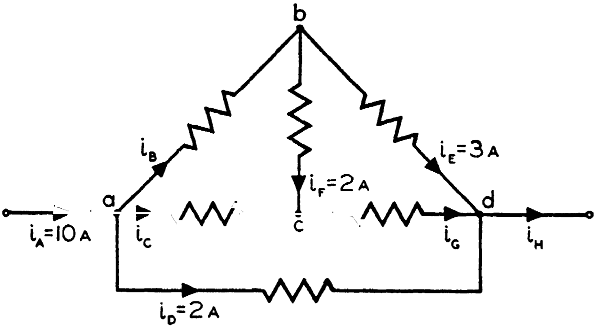
2-5. A tubular container one centimeter in diameter and 3 centimeters long contains a solution of sodium chloride in water. This particular solution contains 2×10^{19} sodium ions and an equal number of chlorine ions. Each sodium ion carries one positive electronic charge, and each chlorine ion carries one negative electronic charge. If, under certain conditions, the sodium ions move in one direction with a velocity of 2.5×10^{-4} centimeter per second and the chlorine ions move in the opposite direction with a velocity of 3.5×10^{-4} centimeter per second, what current is flowing?

2-6. The concentrations of mercury ions and of electrons in the plasma of a certain mercury-arc rectifier are equal to 17.2×10^{12} ions or electrons per cubic inch. If the current density through the plasma is 10 amperes per square inch and if 0.25 per cent of the current is due to mercury ions:

(a) What is the current-contributing component of velocity of the mercury ions?

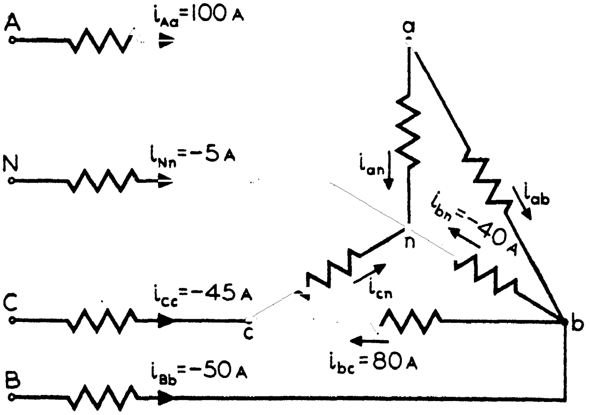
(b) What is the current-contributing component of velocity of the electrons?

2-7. Find all the current values for the accompanying circuit.



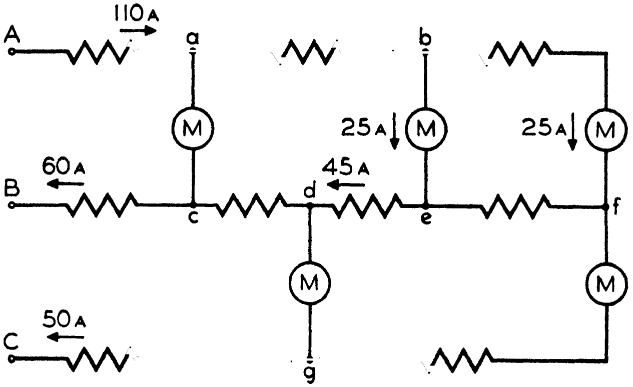
Prob. 2.7.

2-8. Find all the current values for the accompanying circuit. Note that the *algebraic* sum of the currents i_{Aa}, i_{Bb}, i_{Cc}, and i_{Nn} is zero. Must this always be so? Explain.



Prob. 2.8.

2-9. Find all the current values for the accompanying circuit.



Prob. 2.9.

2-10. Using current as the ordinate and time as the abscissa, sketch curves of currents as functions of time to illustrate the nine definitions of Article 2.06.

CHAPTER 3

POTENTIAL DIFFERENCE AND ELECTROMOTIVE FORCE

3.01. Electric Potential Difference, E or V; the Volt

The movement of an electric charge is always accompanied by either the translation or the transformation of energy. Whenever there is an electric current, a flow of energy is occurring. It now becomes important to define the quantity *electric potential difference,* which is symbolized by the letters E and V (or e and v).

"*The electric potential difference between two points is equal to the work associated with the transfer of unit quantity of positive electricity from one point to another. If outside energy is required to transfer unit positive quantity from a to b, b is at higher potential than a.*"[1] The electric potential difference between two points is, then, the difference of energy level experienced by a unit positive charge in moving between those points. In moving between two points of an electric circuit, if a unit positive charge *gains* energy (experiences a *rise* in potential), then the charge has passed through an energy *source:* work has been done upon the charge; energy has entered the circuit from outside. If, on the other hand, the unit positive charge in moving between two points of an electric circuit *loses* energy (experiences a *drop* in potential), then the charge has passed through an energy *sink:* work has been performed by the charge; energy has left the circuit.

It should be noted that the foregoing definition involves the energy interchanges of a unit positive charge. It often happens that electric current, defined as being positive in the direction of motion of positive charges, actually is supported by negative charges moving in the negative direction. Inasmuch as this situation is completely equivalent to the positive flow of positive charges as far as accompanying energy flow is concerned, we need feel no concern. All points of the circuit still possess the same relative energy levels, or potentials, as though the current actually consisted of positively directed positive charges.

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[1] *American Standard Definitions of Electrical Terms,* No. 05.15.025. New York: American Institute of Electrical Engineers, 1942.

We can now define the practical unit of potential difference: *One volt is the potential difference between two points of an electric circuit when one joule of energy is gained or lost by one coulomb of electricity in moving from one of these points to the other.*

Expressed as an equation, the foregoing definitions appear as follows:

▶ $$e_{ab} = \frac{dw_{ab}}{dq}, \qquad (3\text{-}1)$$

where:

e_{ab} = *potential difference* in **volts** between points a and b, a positive value indicating a potential rise,

dq = *increment of charge* in **coulombs** transferred from point a to point b, a positive value indicating a positive charge,

dw_{ab} = *increment of energy* in **joules** exchanged by dq in moving from point a to point b, a positive value indicating energy gained by the moving charge.

It is interesting to note that, to gain energy, a negative charge must go through a drop of potential!

EXAMPLE 3.01. A total of 20 joules of work is required to move a positive charge of 5 coulombs through a particular part of a circuit. (a) What is the potential difference across this part of the circuit? (b) Is this a potential drop or a potential rise in the direction of charge movement?

(a) The potential difference $e = \frac{dw}{dq} = \frac{20}{5} = 4$ volts.

(b) It is a potential rise in the direction of charge movement. Inasmuch as both the energy gained by the charge and the charge itself are positive, the resulting potential difference is positive in the positive current direction.

3.02. Electric Potential

"*The electric potential of a point is the potential difference between the point and some equipotential surface, usually the surface of the earth, which is arbitrarily chosen as having zero potential. A point which has a higher potential than a zero surface is said to have a posi-*

tive potential; one having a lower potential has a negative potential."[2] In an electric circuit it may be convenient to designate arbitrarily some point as a reference of zero potential. If the circuit is grounded, the grounded point is a convenient zero reference. If the circuit is ungrounded, any other point, such as one terminal of the source or one terminal of the load, may conveniently be so designated.

Unfortunately, the term "potential" has nearly lost the unique meaning defined above. By common usage, the word "potential" is now virtually synonymous with the term "potential difference," and may be so employed upon occasion throughout this text.

3.03. Instantaneous Power, p; the Watt

If we deal with the small amount of energy, dw, gained by the small amount of positive electricity, dq, in moving from point a to b in the small time interval, dt, equation (3-1) can be modified as follows:

$$e_{ab} = \frac{dw_{ab}}{dq} = \frac{dw_{ab}/dt}{dq/dt}. \tag{3-2}$$

The numerator of this equation, dw_{ab}/dt, is the instantaneous rate of energy transformation, or *instantaneous power* entering the circuit, between point a and point b. The denominator of this equation, dq/dt, is the instantaneous rate of movement of positive charges, or *instantaneous electric current,* from point a to point b. This can, therefore, be rewritten as

▶ $$e_{ab} = \frac{p_{ab}}{i_{ab}}, \qquad \text{or} \qquad p_{ab} = i_{ab}e_{ab}. \tag{3-3a}$$

This has been called the "pie" formula for instantaneous electric power in terms of instantaneous current and instantaneous potential difference.

Equation (3-3a) can be used to define the unit of electric power, the *watt.* This definition can be stated thus: *The watt is the power exchange occurring when a current of one ampere flows between two points having a potential difference of one volt.*

If the current and potential are unidirectional and do not vary with time, the power likewise is unvarying, and equation (3-3a) can be written with capital letters. This is the *direct-current* (*d-c*) *formula for power:*

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[2] *Ibid.,* No. 05.15.030.

▶ $$P_{ab} = I_{ab}E_{ab}. \tag{3-3b}$$

A modification of this formula is necessary before it can be applied to alternating-current circuits of greater than zero frequency.

Example 3.02. A certain direct-current generator is supplying unvarying current of 10 amperes to an electric circuit at an unvarying potential difference of 125 volts. What power is the generator supplying to the circuit?

$$P = IE = 10 \times 125 = 1250 \text{ watts.}$$

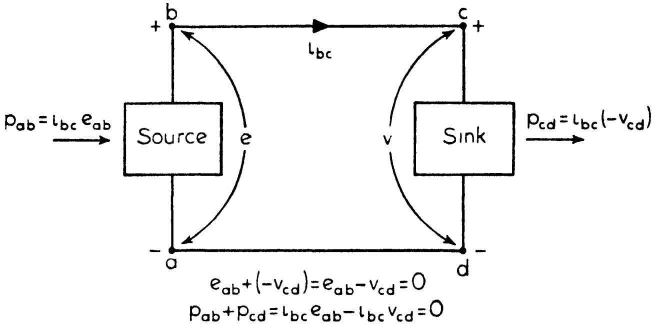
3.04. Algebraic Signs and Electric Polarity; Potential Drop, V.

At this point the student can avoid a great deal of needless future difficulty by taking a moment to review carefully the interrelationship of potential difference, current, and power, with particular reference to the implications of the algebraic signs employed in the power equation, $p = ie$. In this study reference will be made to Fig. 3.04.

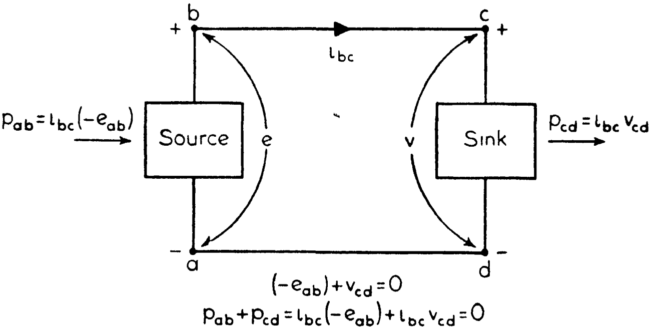
By definition, a *potential rise* is said to exist between points a and b if energy must be expended to transfer positive charge from a to b. Under this circumstance, b is at a higher potential than a and is said to be *positive* relative to a. Conversely, point a is at a lower potential than b and is said to be *negative* relative to b. There is a potential drop from b to a. Positive charge transferred from b to a will give up energy. The relative potentials, or *polarities*, of the two points can be designated by placing on the circuit diagram a plus (+) sign at the point of higher potential, and a minus (−) sign at the point of lower potential, as shown in part (a) of the figure. The same statements apply to negative charge transferred in the opposite directions.

Also, by definition, electric current flows from a to b either if positive charges are moving from a to b or if negative charges are moving from b to a. Either kind of charge movement constitutes *positive current* from a to b. The positive direction of current may be designated on a circuit diagram, as in part (a) of the figure, by means of an arrow associated with the symbol for current.

At a source, since energy is entering the circuit, there is always a rise of potential in the positive current direction. At a sink, since energy is leaving the circuit, there is always a drop of potential in the positive current direction. By equations (3-3), then, if we trace through a potential rise in the positive direction of current flow, the resulting positive power value will indicate that we are at a source, with energy flowing into the circuit from outside. On the other hand, if we trace through a potential drop in the positive current direction, the resulting negative power value will indicate that we are at a sink, with energy flowing away from the circuit. These facts are summarized in the first two rows of Table III.


b
c
+
+
i_{bc}
$P_{ab}=i_{bc}e_{ab}$
Source
e
v
Sink
$P_{cd}=i_{bc}(-v_{cd})$
−
−
a
d
$e_{ab}+(-v_{cd})=e_{ab}-v_{cd}=0$
$P_{ab}+P_{cd}=i_{bc}e_{ab}-i_{bc}v_{cd}=0$

(a) Potential rises considered positive.



(b) Potential drops considered positive.

Fig. 3.04. Polarities and current direction for a particular circuit.

The last two rows of Table III recognize the fact that in many circuit solutions, the direction of tracing through a given part of the circuit may be opposite to the indicated positive current direction. In such a case, we are tracing, in effect, in the negative current direction, with a consequent reversal of sign for current. Since potential difference is now encountered in a reversed sense, potential drops (negative rises) replace rises, and potential rises (negative drops) replace drops. Inasmuch as the algebraic signs of both current and potential are reversed, power is still (+) at sources and (−) at sinks.

TABLE III

Algebraic Signs of Potential Difference, Current, and Power, Potential Rises Considered Positive: $p = ie$

Direction of Tracing	Potential Difference	Power Direction	Circuit Element
In positive current direction (i is +)	Potential rise (e is +)	Energy entering circuit (p is +)	**Source**
	Potential drop (e is −)	Energy leaving circuit (p is −)	**Sink**
Opposite to positive current direction (i is −)	Potential drop (e is −)	Energy entering circuit (p is +)	**Source**
	Potential rise (e is +)	Energy leaving circuit (p is −)	**Sink**

The preceding principles can, with relative ease, be applied directly to simple circuits. In more complicated circuits, however, the indication of all polarities with (+) and (−) signs and of all positive currents with arrows might hopelessly clutter the circuit diagram with a confusion of marks and symbols. Such a situation may be handled more neatly by means of *double-subscript notation*, in which the order of the subscripts indicates the direction of traversal from terminal to terminal of a portion of the circuit. It is often helpful, further, to let the symbol e (or E) stand uniquely for a potential rise or emf of a source, and to let the symbol v (or V) stand uniquely for a potential drop in the direction of the appended subscripts. For currents, i (or I) always stands for positive current in the direction of the appended subscripts. As we shall see in the next chapter, double subscripts appended to circuit parameters have no directional sense but merely serve to identify the parameters with reference to terminals or to portions of the circuit with which they are associated.

Frequently the number of energy sinks to be handled in a given problem exceeds the number of energy sources. In such a case there is some advantage to denoting as "positive" all power leaving the circuit, merely by redesignating all potential drops as positive and all potential rises as negative. The designation of positive current is unchanged. By employing this notation and calling potential drops positive, we may rewrite the power equation as

▶
$$p_{ab} = i_{ab}v_{ab}, \tag{3-4a}$$

where:

p_{ab} = *instantaneous power* between points *a* and *b*, in **watts,** a positive value indicating energy flow away from the circuit,

i_{ab} = *instantaneous current* from *a* to *b*, in **amperes,** a positive value indicating the assumed direction of movement of positive charges,

v_{ab} = *instantaneous potential* drop, in **volts,** experienced by current in flowing from *a* to *b*.

The corresponding formula for *d-c power* at a sink, with potential drops considered positive, is

▶
$$P_{ab} = I_{ab}V_{ab}. \tag{3-4b}$$

If potential drops are to be considered positive, all of the algebraic signs of Table III except those of current must be reversed. Table IV summarizes the relations which then apply.

TABLE IV

ALGEBRAIC SIGNS OF POTENTIAL DIFFERENCE, CURRENT, AND POWER, POTENTIAL DROPS CONSIDERED POSITIVE:

$p = iv = i(-e)$

Direction of Tracing	Potential Difference	Power Direction	Circuit Element
In positive current direction (**i is +**)	Potential rise (**v is −**)	Energy entering circuit (**p is −**)	**Source**
	Potential drop (**v is +**)	Energy leaving circuit (**p is +**)	**Sink**
Opposite to positive current direction (**i is −**)	Potential drop (**v is +**)	Energy entering circuit (**p is −**)	**Source**
	Potential rise (**v is −**)	Energy leaving circuit (**p is +**)	**Sink**

We shall use the notational scheme of Table IV in most of the later developments of this book, although the student should be able to handle that of Table III with equal facility.

The foregoing discussion prepares us for what probably is the most important portion of this article: *the method of designating polarities and current directions for a circuit in which actual polarities and current directions are unknown.* In many electric network solutions, it happens that these actual potential polarities and current directions are unimportant. Only the magnitudes of the potentials and currents, and the polarities and current directions *relative to others within the circuit,* are of significance in such a case. Although the actual polarities and current directions may never be known, the circuit can be solved by using a procedure which keeps track of the relative current directions and polarities throughout the circuit. **This procedure requires that potential polarities and positive current directions be arbitrarily assigned, and that the circuit equations be written in consistent agreement with these assumptions. The solution of these equations gives the correct magnitudes for all currents and potentials. If negative values appear for certain currents and potentials, it simply means that the mathematics shows those particular assumed directions and polarities to be inconsistent with the rest.** To be consistent with the rest, those which turn out to be negative may be considered positive in the opposite direction. With appropriate interpretation for alternating-current circuits, this principle will be utilized throughout the rest of this book.

3.05. Kirchhoff's Potential Law, $\Sigma E = 0$ or $\Sigma V = 0$

Now we come to a principle which is of fundamental importance in the development of electric circuit theory. This principle, called *Kirchhoff's potential law,* can be stated thus: *If a closed path is traversed through an electric circuit, the sum of the potential rises which are encountered in the direction of traversal must equal the sum of the potential drops which are so encountered.* By means of appropriately developed electromagnetic field theory, this statement may be proved for either an open circuit, in which no current can flow, or for a closed circuit, in which current can flow.

Kirchhoff's potential law is a logical consequence of the principle of conservation of energy. We have chosen to conceive of the electric circuit as a guide for the flow of electromagnetic energy in which no net storage of energy can take place. If, now, we imagine a unit positive "test" charge to be placed at some point in the circuit which we can designate as having zero potential, we can state that the test charge possesses a relative energy content of zero. As in imagination we move this test charge through the circuit along some path which ultimately brings the charge back to its starting point, we realize that energy gains (potential rises) experienced by the charge must be exactly balanced by equal energy losses (potential drops), if the charge is to possess no net surplus or deficiency of energy upon its return to the starting point. This argument holds regardless of the exact path followed through the circuit by the "test" charge in making the traverse. It holds whether the path of traversal includes open switches or closed switches. It holds whether the path of traversal is followed in a clockwise direction or in a counterclockwise direction.

If we understand that, with a given path of traversal, *all potential drops can be treated mathematically as negative potential rises,* we can state Kirchhoff's potential law in this form: *If a closed path is traversed through an electric circuit, the algebraic sum of the potential rises across the individual circuit elements in the direction of traversal is zero.* Or, mathematically,

▶ $$\Sigma e = 0, \qquad \text{or} \qquad \Sigma E = 0. \tag{3-5a}$$

If we decide to call potential drops positive and treat potential rises as negative drops, it is convenient to state Kirchhoff's potential laws as follows: *If a closed path is traversed through an electric circuit, the algebraic sum of the potential drops across the individual circuit elements in the direction of traversal is zero.* Mathematically, this version of Kirchhoff's potential law is

▶ $$\Sigma v = 0, \qquad \text{or} \qquad \Sigma V = 0. \tag{3-5b}$$

Equation (3-5b) is equivalent to equation (3-5a) with all algebraic signs reversed. Figure 3.04 illustrates these ideas.

3.06. Electromotive Force, or EMF

"*Electromotive force is the property of a physical device which tends to produce an electric current in a circuit.*"[3] We realize that any device which produces an electric current in a circuit also introduces energy into the circuit, and thereby functions as a source. We might paraphrase the foregoing definition by writing: *Electromotive force (abbreviated emf) is the property of a physical device which tends to make that device function as a source of energy.* The word "tends" must not be overlooked, because it implies that some emf's do not succeed in producing electric current. An open switch may, for example, prevent current flow. It is even possible for an emf to be bucking the flow of current produced by another and greater emf. In such a case, the bucking emf does not serve as a source but as a receiver of energy. A chemical battery undergoing charging or an electric motor (a reversed generator) under load are examples of this function. Such a bucking electromotive force is sometimes called a *counter-electromotive force*, or *counter-emf.*

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[3] *Ibid.*, No. 05.20.055.

When we stop to realize that charges gain energy in moving through sources, we also realize that an emf must produce a potential difference across the terminals of that source. As a matter of fact, emf is measured in units of potential, such as volts, and might be classified as a special kind of potential difference which is inherently capable of serving as a source of energy for an electric circuit. This property makes an emf uniquely capable of separating electric charges. Inherently, all emf's can function reversibly as either energy sources or energy sinks; those potential differences which are not emf's function irreversibly as energy sinks. The potential difference across the terminals of a storage battery is, basically, an emf: the battery can be either charged or discharged; and even without current flow, the potential difference is observable because of the chemically produced separation of charges. The potential difference across the terminals of an incandescent lamp is not an emf: regardless of direction of current flow, the lamp never supplies energy to the circuit; and without current flow, the potential difference vanishes, demonstrating the absence of charge separation.

As electric circuit quantities, emf's are handled by the rules already developed for potential differences in general.

Example 3.06. Figure 3.06 shows in block form the diagram of a particular electric circuit, with energy flows and positive current direction indicated. (a) Indicate, by means of (+) and (−) signs, the polarities of all elements. (b) Find the values of all potential differences and tabulate them as emf's, irreversible drops, and net potential differences, in such a way as to demonstrate the validity of Kirchhoff's potential law.

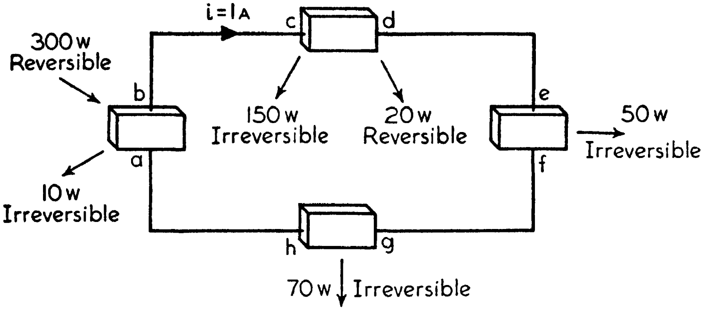

Fig. 3.06. Block diagram of a particular circuit.

Using a clockwise direction of tracing, utilizing the relation $e = p/i$, and recalling that potential rises in the direction of tracing are to be considered positive, we arrive at the following results:

Circuit Element	Polarity	Emf's, in Volts	Irreversible Drops, in Volts	Net Potential Differences, in Volts
ab	− +	300	−10	300 + (−10) = 290
cd	+ −	−20	−150	−20 + (−150) = −170
ef	+ −	0	−50	0 + (−50) = −50
gh	+ −	0	−70	0 + (−70) = −70
Totals		280 v	−280 v	$\Sigma E = 0$ v

3.07. Sources of EMF

Our discussion of the atomic theory showed that energy must be supplied from some external source whenever an electron is to be separated from its nucleus. The process can be regarded as one of raising the energy level of the electrons in the outermost normal orbit to the point where these electrons leave their atom and can be regarded as free electrons. Furthermore, these electrons can rejoin more or less permanently the family of electrons surrounding a nucleus only after the excess of energy which has been acquired from the external source has been released in some form to a receiver. The form or manner in which energy must be supplied to an electron to make it leave its nucleus, thereby creating a separation of charges and a resultant electromotive force, was not specified. As a matter of fact, with different materials and under different conditions, this result can be accomplished in different ways.

It is convenient to classify and discuss the principal sources of emf according to the forms of energy involved. The major classifications can be outlined as follows:

(1) Electromagnetic radiation (photovoltaic effect)
(2) Thermal energy
  (a) Thermoelectric (Seebeck) effect
  (b) Thomson effect
  (c) Pyroelectric effect
(3) Chemical energy (electrochemical action)
(4) Electrostatic energy (electrostatic induction through electrostatic coupling)
(5) Magnetic energy
  (a) Electromagnetic induction through magnetic coupling
  (b) Hall effect
(6) Mechanical Energy
  (a) Electromagnetic induction through motion
  (b) Piezoelectric effect (pronounced "pie-ee′-zo")

The next several articles are devoted to brief discussions of these major sources of electromotive force. For a more complete treatment of these topics, the reader is referred to the articles listed in the footnote below.[4]

3.08. Electromagnetic Radiation as a Source of EMF

When light falls upon the surface of contact between certain combinations of liquid and solid conductors, a small emf is developed. This phenomenon, known as the *photovoltaic effect*, produces an emf which is approximately in proportion to the amount of light reaching the surface of contact but is independent of the area of the contact surface. The emf appears to be a function, not only of amount of light but of wave length, angle of incidence, time rate of change of light, temperature, and humidity.

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[4] The following articles on electric power sources constitute a series appearing in *Electrical Engineering:* Trump, J. G., "Electrostatic Sources of Electric Power," June 1947, Vol. 66, No. 6, page 525; Cady, W. G., "Nature and Use of Piezoelectricity," August 1947, Vol. 66, No. 8, page 758; Matsch, L. W., and Brown, W. C., "Electric Power Sources," September 1947, Vol. 66, No. 9, page 880; Vinal, G. W., "Electrochemical Sources of Electric Power," part 1, April 1948, Vol. 67, No. 4, page 354, and part 2, May 1948, Vol. 67, No. 5, page 456; Osborne, J. A., "Magnetostriction Generators," June 1948, Vol. 67, No. 6, page 571; Ellis, G. B., "Thermoelectric Generator Design," July 1948, Vol. 67, No. 7, page 657.

The dry-type photovoltaic cell employing selenium and iron as the active pair of materials forms the basis of many modern control and light-measuring devices.

3.09. Thermal Energy as a Source of EMF

In the *thermoelectric, or Seebeck, effect,* an electromotive force results from a difference of temperature between two junctions of dissimilar metals in the same circuit. Thus, if one junction of a copper-iron circuit is kept in melting ice and the other in boiling water, it will be found that a current passes from copper to iron across the hot junction. If both junctions are at the same temperature, the emf's at those junctions exactly balance out. The difference of potential which exists between the two metals in contact depends solely on the metals used, the temperature, and the intimacy of contact between the two metals. Provided that no electrolyte is employed at the junction, the potential difference is independent of the area and shape of the junction, as well as of the sizes or shapes of the two pieces of metal. *Thermocouples,* as such two-junction combinations of dissimilar metals are called, find many uses in the measurement of temperature. They are especially valuable in applications where ordinary thermometers are too sluggish or where temperatures are too high for the ordinary thermometer. In "heat runs" on electric devices under test, sometimes hundreds of thermocouples are embedded in strategic locations. Thermocouples have proved especially useful in keeping a check on the temperatures developed in reacting atomic piles in the generation of nuclear energy, where ordinary thermometers cannot be read with safety.

The *Thomson effect* is a phenomenon by which an electric potential difference appears along a current-carrying conductor whose ends are at different temperatures. The value of the emf developed is a function of several factors, including the material, the temperature gradient, and the current density. In most engineering work this effect is of little significance.

An emf is produced in certain crystals, notably of quartz and tourmaline, under the influence of heating. This phenomenon, known as the *pyroelectric effect*, has been of little practical value so far.

3.10. Chemical Energy as a Source of EMF

If two plates made of different metals are immersed in an electrolyte (a conducting medium in which the flow of electric current is accompanied by the movement of matter), a difference of electric potential can be observed between the two plates. This phenomenon is known as *electrochemical action*. The magnitude of the emf will depend solely upon the kinds of metals and the kind of electrolyte employed. The *electrodes*, as the metal plates are termed, are the terminals of what is called an *electrolytic cell*. A group of such cells serving in electrical combination is termed a *battery*.

Electrolytic cells can be classified as follows:

(1) Primary cells
    (a) Wet cells
    (b) Dry cells
(2) Secondary (or storage) cells

A *primary cell* is a cell designed to produce electric current through an electrochemical reaction which is not efficiently reversible; and hence the cell, when discharged, cannot be efficiently recharged by an electric current. A *wet cell* is one whose electrolyte is in liquid form and free to flow and move. A *dry cell* is one in which the electrolyte exists in the form of a jelly or is absorbed in a porous medium, or is otherwise restrained from flowing from its intended position. Such a cell is completely portable and the electrolyte nonspillable.

A *secondary, or storage, cell* is an electrolytic cell for the generation of electric energy which, after being discharged, may be restored to a charged condition by an electric current flowing in a direction opposite to that previously followed by the discharge current.

Although electrolytic cells represent one of the oldest sources of electromotive force, they are not a practical source of commercial power for general use. They are highly important, however, where energy must be stored for intermittent use or where portability is required. Batteries are used extensively in automobiles, trains, emergency lighting systems in hospitals, portable radios, laboratory standards, hearing aids, and so on.

3.11. Electrostatic Energy through Coupling as a Source of EMF

The phenomenon of electrostatic induction of emf through electrostatic coupling is treated in Chapter 13, "Coupled Circuits." It is found in practice both as a tool and as a nuisance, both in power and communication equipment.

3.12. Magnetic Energy as a Source of EMF

Electromagnetic induction through magnetic coupling as a source of emf underlies the operation of such apparatus as transformers and induction coils, and therefore constitutes a phenomenon of great importance in electrical engineering. It is discussed in Article 3.14, "Faraday's and Lenz's Laws."

The *Hall effect* is the production of an emf in a long, thin current-carrying conductor oriented at right angles relative to a magnetic field to which it is subjected. The emf is developed at right angles to both the direction of the current and to the magnetic field. So far, this effect has had little practical significance in engineering work.

3.13. Mechanical Energy as a Source of EMF

Practically all electric energy for power purposes is obtained from mechanical energy by electromagnetic induction. This phenomenon is discussed in detail in Article 3.14.

Another means of deriving an emf from mechanical energy is through a phenomenon which receives its name from the Greek *piezein*, "to press." Certain crystalline substances such as quartz, tourmaline, and Rochelle salts develop an electromotive force between opposite sides when mechanically strained. The direction and magnitude of this effect, known as the *piezoelectric effect*, depends on the nature and amount of the strain, and on the direction of the strain with reference to the axes of the crystal. On such crystals, the converse effect is also observed: a strain, or mechanical deformation of the crystal, results from the application of an electric field. Although relatively high values of emf can be produced by piezoelectric action, the power available from a given crystal is so minute as to make amplification necessary in many applications. The most active commercially available piezoelectric substance is Rochelle salts. This material finds wide application in piezoelectric microphones, loudspeakers, and phonograph pickups. Quartz, because of its permanence, its low temperature coefficient, and its low frictional losses, is employed to advantage in controlling the frequency of electronic oscillators. In recent years, substitute materials such as crystals of potassium hydrophosphate and lithium potassium tartrate have been introduced.

3.14. Faraday's and Lenz's Laws

In 1830, Michael Faraday discovered that a change in the amount of magnetic flux linking with an electric circuit induces an electromotive force in that circuit. Later it was noted that this induced emf is proportional to the time rate of change of the flux linkages. The following *law of electromagnetic induction*, sometimes referred to as *Faraday's law*, incorporates both principles: "*The electromotive force induced in a circuit is proportional to the time rate of change of the flux of magnetic induction linked with the circuit.*"[5]

In 1834, Heinrich Lenz observed further that the induced emf is of such a polarity as to tend to produce a current opposing the change of flux linkages. Because energy is tending to move into the electric circuit containing the coil, the induced emf tends to circulate current in the direction of higher induced potential, the entire active length of the coil serving as an electric circuit source. If a complete circuit exists, such a current may actually flow, producing flux of its own in the proper direction to compensate for the change in the original flux linkages. Faraday's law is merely one form of the general physical law of equal and opposite action and reaction, and can be expressed mathematically as

▶ $$e = -N \frac{d\varphi}{dt}, \tag{3-6}$$

where:

e = *induced electromotive force*, in **volts,**
N = *number of turns* of electric conductor linked by flux,
φ = *flux* in **webers** (1 weber = 10^8 maxwells, or lines of flux),
t = *time*, in **seconds.**

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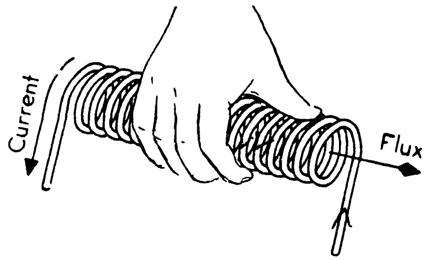
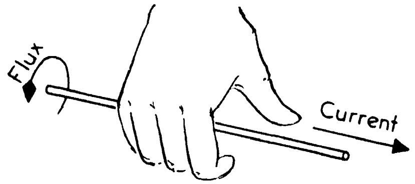
[5] *American Standard Definitions of Electrical Terms*, No. 05.40.010. New York: American Institute of Electrical Engineers, 1942.

The negative sign is used to indicate that the induced emf is a counter-emf, offering opposition to the very action from which it arises! The similarity between this effect in an electric circuit and the phenomenon of inertia in a mechanical system will be emphasized in the appropriate place.



(a) Right-hand rule for coil.



(b) Right-hand rule for straight conductor.

Fig. 3.14A. Rule-of-thumb for relative current-flux relations.

A handy and literal rule of thumb to be employed in determining relative positive space directions of associated electric and magnetic quantities can take either of the following forms: (*a*) *If a current-carrying coil is grasped with the right hand so that the fingers point the positive current direction, the extended right thumb points the direction of resultant magnetic flux linking through the coil.* (*b*) *If a current-carrying conductor is grasped with the right hand so that the extended thumb points the positive current direction, the fingers point the direction of the resultant encircling magnetic flux.* Figure 3.14A illustrates this rule.

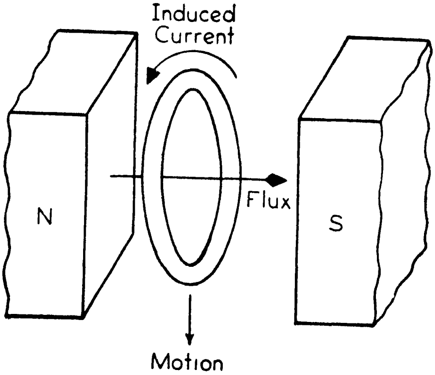


Fig. 3.14B. Current resulting from emf induced by decreasing flux linkage.

We shall now employ this rule of thumb in discussing the operation of Faraday's law. If a conducting ring linked by the flux of a permanent magnet, as shown in Fig. 3.14B, is being snatched from between the poles of the magnet, the linkage of flux with the ring is rapidly decreasing in value. If current were to flow around the ring in the indicated direction, it would establish flux of its own toward the right—in a direction such as to help sustain the original value of flux linkage. Therefore the induced emf has the polarity indicated, tending to give rise to an induced current in the indicated direction. Similarly, thrusting the ring back into a position between the poles reestablishes the original flux linkage. This increase of flux linkage now is opposed, this time by emf of opposite polarity (and resulting current in the opposite direction), tending to establish opposing flux toward the left.

At this point it must be noted that if an induced current actually flows as the result of an induced emf, there is an actual mechanical force on the coil. This fact is expressed in the following verbal statement of the principle sometimes known as Lenz's law: "*The current induced in a circuit as the result of its motion in a magnetic field is in such a direction as to exert a mechanical force opposing the motion.*"[6] A moment of reflection will reveal that it is the relative motion between coil and magnetic field which gives rise to the mechanical force, and that also a force will be experienced by a stationary coil past which a field is moving. This statement has some interesting implications, usually discussed under the headings "generator action," and "motor action."

3.15. The Elementary Synchronous Generator

The *synchronous generator*, or *alternator*, is, perhaps, fundamentally the simplest rotating machine used to transform mechanical power to electric power. The operation of this machine is based on the laws of electromagnetic induction presented in the preceding article.

Figure 3.15A shows in elementary form one type of synchronous generator, consisting of a single-turn coil rotated at a constant angular velocity about an axis perpendicular to a uniform rectilinear magnetic flux field. We observe that a maximum flux linkage occurs when the coil is in a vertical plane, perpendicular to the flux lines. From the simple geometry involved, it should be apparent that the variation of flux linkage can be expressed as a cosine function of the angle of displacement, α, between the plane of the coil and the vertical. A moment of reflection reveals that, on the other hand, the rate of change of flux linkage is zero when the coil is in a vertical position and is a maximum as the coil passes through the horizontal. The maximum rate of change of linkage occurs when there is a reversal of direction of linking.

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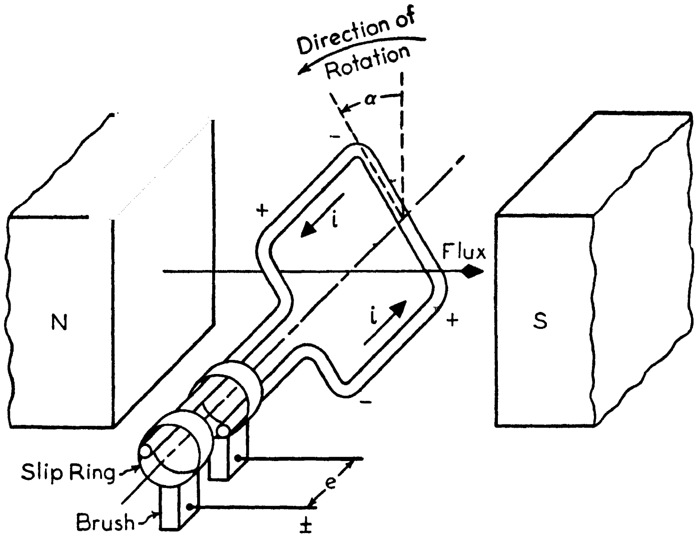
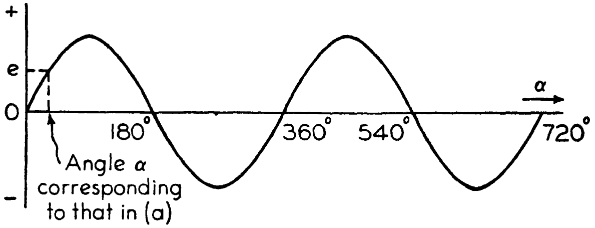
[6] *Ibid.*, No. 05.40.020.



(a) Schematic diagram of generator.



(b) Wave of generated emf.

Fig. 3.15A. One type of elementary synchronous generator.

Let us denote by the symbol Φ_m the maximum value of flux to link the one turn of the coil. We can now write the following expression for the value of flux, φ, effectively linking the coil at any angular displacement, α, of the coil from the position of maximum linkage:

$$\varphi = \Phi_m \cos \alpha. \tag{3-7}$$

Inasmuch as the coil is assumed to be rotating at a uniform angular velocity, ω, if we let t denote the value of elapsed time since the instant of maximum flux linkage, equation (3-7) may be rewritten as follows:

$$\varphi = \Phi_m \cos \omega t. \tag{3-8}$$

We recall the mathematical statement of Lenz's law given by equation (3-6), where N symbolizes the number of turns linked by the flux. Substituting (3-8) in (3-6) and performing the indicated operations, we obtain

$$\begin{aligned} e &= -N \frac{d\varphi}{dt} \\ &= -N \frac{d}{dt} \Phi_m \cos \omega t && \text{(3-9)} \\ &= \omega N \Phi_m \sin \omega t. && \text{(3-10)} \end{aligned}$$

The coefficient of the sine term of this equation is the maximum value which this electromagnetically induced (or generated) emf can have. Denoting

$$E_m = \omega N \Phi_m = \text{maximum value of } e, \tag{3-11}$$

we can write

▶ $$e = E_m \sin \omega t = \omega N \Phi_m \sin \omega t, \tag{3-12}$$

where:

e = *instantaneous generated emf*, in **volts,**

ω = *angular velocity* of coil, in **radians per second,**

Φ_m = *maximum flux* linking coil, in **webers,**

N = *number of turns* in the coil,

t = *time elapsed* since instant of initial maximum linkage, in **seconds,**

E_m = *maximum value* (*or "amplitude"*) of generated emf, in **volts.**

In the language of the mathematician, both the generated emf of equation (3-12) and the flux linkage of equation (3-8) are *sinusoidal functions of time.* This form is very convenient from the standpoint of electric circuit theory, and designers of synchronous generators strive to produce machines capable of producing an emf which is closely sinusoidal in form under all rated load conditions.

Figure 3.15B shows in elementary form another type of alternator construction, in which the conductors remain at rest in the frame (*stator*), and the poles constitute the rotating element (*rotor*). This is basically the type of design usually employed in practice in all but the smallest synchronous generators. A moment of reflection should reveal that so long as relative motion exists, it does not matter whether the coils or the magnetic poles move relative to the frame. The foregoing equations apply equally well to this type of synchronous generator.

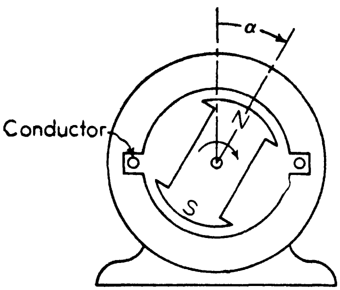


Fig. 3.15B. Another type of elementary synchronous generator.

Ignoring losses, when any type of generator is supplying power to an electric circuit, an equal value of power must be supplied in mechanical form to the shaft of that generator. Lenz's law defines the principle under which the mechanical torque of the prime mover meets a mechanical countertorque produced by the generator. In tending to oppose the continual changing of flux linkages, the current flowing through the armature windings of the generator develops the necessary countertorque to balance the mechanical torque supplied by the prime mover.

3.16. Generator Action from the Flux-cutting Viewpoint

The principle of electromagnetic induction discussed in Article 3.14 can be given a special interpretation which sometimes may conveniently be employed to describe the mechanical generation of emf. From the viewpoint of this special interpretation, emf is generated whenever a single straight conductor is so moved as to "cut" magnetic flux lines.

From Fig. 3.16A we see that, from the principle of equal and opposite action and reaction, an emf must be generated with the indicated polarity to oppose the forced downward movement of the conductor. This requirement can be easily "explained." The simple rule of thumb calls for our grasping the conductor, in imagination, with the right hand in such a manner that the fingers encircle the moving conductor and point the direction of the magnetic field on the leading edge. The extended thumb will then point the direction in which induced current tends to flow. This direction is such that, were current actually to flow, it would establish encircling flux so directed as to result in a piling up of flux on the leading edge of the conductor and a canceling of flux on the trailing edge. If we imagine a sort of magnetic aerodynamics to function, we can perceive that a drag will then oppose continued motion of the conductor.

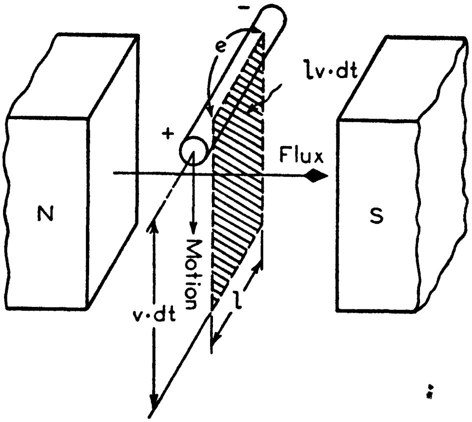


Fig. 3.16A. Generation of emf in a "flux-cutting" conductor.

A mathematical analysis based on this flux-cutting viewpoint leads to some relations which are useful in electric machinery design. Referring to Fig. 3.16A, we can express the generated emf as follows:

$$e = +\frac{d\varphi}{dt} \tag{3-13}$$

where:

$\frac{d\varphi}{dt}$ = *"rate of cutting" of flux*, in **webers per second.**

The positive sign is employed inasmuch as we wish to follow the convention which we followed in the previous development, denoting potential rises as positive in the positive current direction.

If we now denote:

v_n = *component of conductor velocity normal to flux*, in **meters per second,**

l = *active length* of conductor, in **meters,**

B = φ/A_n = *flux density*, in **webers per square meter, and**

dA_n = *increment of normal area*, A_n, of uniform flux field cut through by conductor in time dt, in **square meters,** we can write

$$e = \frac{d\varphi}{dt} = B\frac{dA_n}{dt}. \tag{3-14}$$

However, in time *dt* the conductor cuts through an area

$$dA_n = lv_n\,dt. \tag{3-15}$$

By substitution, (3-14) becomes

▶
$$e = Blv_n. \tag{3-16}$$

This "*Blv* formula" may be applied to the analysis of an elementary synchronous generator. Referring to Fig. 3.16B, we can resolve into components the tangential velocity, v, of the active ("flux-cutting") length of one side of a one-turn coil. Let us denote the normal component as v_n. If the coil rotates at a uniform angular velocity and if the field is uniform and rectilinear, the normal velocity component, v_n, will vary sinusoidally with the angle α, and equation (3-16) can be written

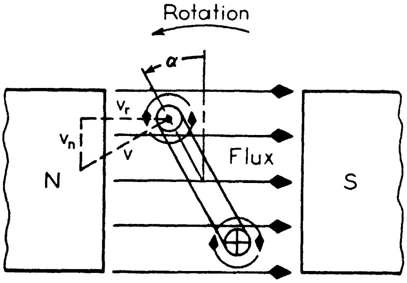


Fig. 3.16B. "Flux-cutting" analysis of elementary generator.

$$e = Blv_n = Blv \sin \alpha. \tag{3-17}$$

Let us now denote r = *radius of coil,* in **meters.** Then $v = \omega r$. Since, also, $\alpha = \omega t$, equation (3-17) now becomes:

$$e = Blr\omega \sin \omega t. \tag{3-18}$$

This is the emf generated in a single conductor. The total emf generated by the two active sides of a one-turn coil will be

$$e_{\text{turn}} = 2e = 2Blr\omega \sin \omega t. \tag{3-19}$$

If the coil has N turns,

$$e_{\text{coil}} = \omega N(2Blr) \sin \omega t. \tag{3-20}$$

It is realized that the maximum flux that can be enclosed by the coil has an absolute value

$$\Phi_m = BA_m = B(2lr). \tag{3-21}$$

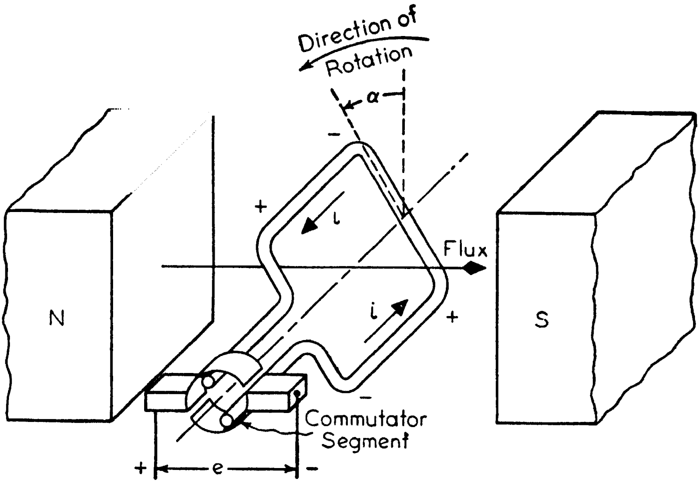
Equation (3-20) can then be written

$$e_{\text{coil}} = \omega N \Phi_m \sin \omega t = E_m \sin \omega t, \tag{3-22}$$

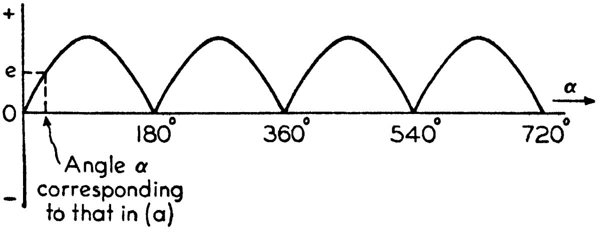
which is seen to be identical with equation (3-12) developed from the "change-of-linkage" viewpoint.

3.17. The Elementary D-C Generator

The elementary d-c generator is like the elementary synchronous generator of Fig. 3.15A, with the exception that the slip rings of the synchronous generator are replaced by a commutator, as indicated by part (a) of Fig. 3.17A. The commutator performs the function



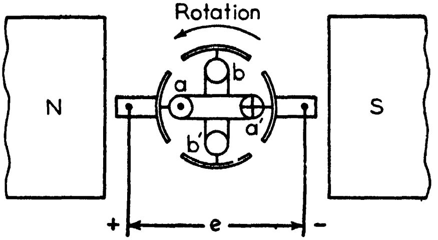
(a) Schematic diagram of generator.



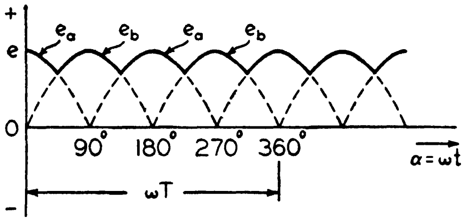
(b) Commutated wave of emf.

Fig. 3.17A. The elementary single-coil d-c generator.

of converting the alternating emf developed in the rotating armature coil into a unidirectional emf like that shown in part (b) of Fig. 3.17A. From the sketch we can see that the left-hand brush (marked +) is always connected, through the proper commutator segment, to the positive terminal of the rotating coil. The right-hand brush (marked −) is always connected, through the proper commutator segment, to the negative terminal of the rotating coil.



(a) Schematic diagram of generator.



(b) Wave of emf at brushes.

Fig. 3.17B. The elementary two-coil d-c generator.

By the addition of more coils uniformly disposed around the shaft and provided with their own pairs of commutator segments, a "smoother" unidirectional emf is obtained. In this case, the brushes pass from contact with one coil to contact with the next while the emf's are still near their peak values. The output emf of the two-coil d-c generator of Fig. 3.17B is appreciably smoother than that of the single-coil machine. By having many coils and pairs of commutator segments, modern d-c generators provide at their terminals unidirectional emf's of amazingly uniform value.

3.18. Dynamic Equilibrium of an Electromagnetic System

One of the fundamental principles observed in nature deals with the transformation of energy from the electric to the magnetic form. This process is the inverse of that by which emf's are induced mechanically through operation of the principles of electromagnetic induction. This *principle of dynamic equilibrium of an electromagnetic system* can be stated in either of the following two ways:

(*a*) "*Any two circuits carrying current tend so to dispose themselves that the flux of magnetic induction linking the two will be a maximum.*"

(b) "*Every electromagnetic system tends to change its configuration so that the flux of electromagnetic induction will be a maximum.*"[7]

It is apparent that these two statements describe, in broad terms, the principle of *motor action* which is so important to electrical engineering. The first form of the law, for example, describes the action of electric measuring instruments of the electrodynamometer type. The second form accounts for the force that tends to separate two parallel conductors carrying oppositely directed currents.

3.19. Reversed Generator Action

In Articles 3.14 and 3.15 we saw the result of utilizing a prime mover to rotate a coil of wire in a uniform magnetic field. An induced emf capable of establishing a flow of current, with the attendant transformation of power from the mechanical form to the electrical form, was this result. We realize that a significant aspect of power transformation by this generator action was the fact that current flowed through the generator winding in the direction of increasing generated potential.

We may now speculate on the result of our employing suitable means to force current to flow "backward" through the generator winding, in the direction of decreasing generated potential. Fundamental theory tells us that the machine will no longer be serving as a source but as a sink. Power transformation from the electric to the mechanical form will be taking place; in short, we shall be operating an *electric motor*.

3.20. The Elementary Synchronous Motor

In construction, the elementary synchronous motor is identical with the elementary synchronous generator discussed in Article 3.15. One form of elementary synchronous motor is shown in Fig. 3.20(a). The proper direction of current relative to polarity of induced emf is indicated for the case of motor action.

A brief qualitative discussion of the operation of this machine follows. Let us suppose that at the instant pictured, a value of current, i, is made to flow in the indicated direction by the application of an emf which gives the terminals the indicated polarity. This current establishes a flux of its own linking through the coil.

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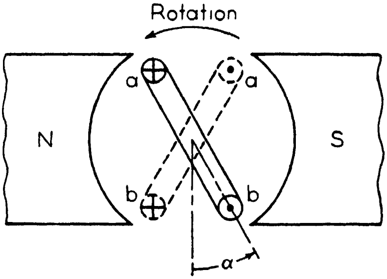
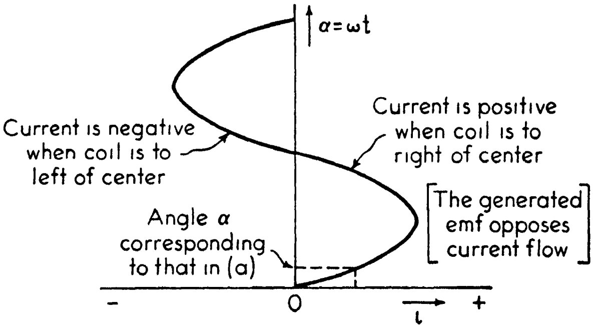
[7] *Ibid.*, No. 05.40.015.

The coil was already being linked, however, by part of the flux established by the magnet. Under the dynamic equilibrium principle, the coil will tend to rotate in a counterclockwise direction until the two components of flux pass through the coil in the same direction and in maximum amount.

Under the influence of mechanical momentum, the coil will tend to rotate past the vertical position of maximum linkage. If the current, i, alternates in proper synchronism with the rotation of the coil, each successive reversal of current imparts an additional impulse to the rotation. Figure 3.20(b) illustrates this effect.



(a) Schematic diagram of motor.



(b) Alternating armature current.

Fig. 3.20. One type of elementary synchronous motor.

The value attained by the current in this machine is limited largely by the alternating counter-emf generated in the rotating coil. The generated counter-emf provides the potential drop which the current must experience if it is to deliver electric energy to be converted to the mechanical form. If more mechanical output is required, physical conditions alter the counter-emf in such a way as to permit a current change such as to increase the electric power input to the machine.

Synchronous motors are usually built in the form of the elementary synchronous generator pictured in Fig. 3.15B.

3.21. Motor Action from the Flux-cutting Viewpoint

Just as we approached generator action, in Article 3.16, from the flux-cutting viewpoint, so shall we approach motor action from the flux-cutting viewpoint.

Figure 3.16A and the rule of thumb enable us to picture this physical occurrence. If current is forced "backward" through the conductor from the end of higher potential to the end of lower potential, the flux lines established by that current will encircle the conductor in a clockwise direction. These lines will combine with those of the original field to produce a "piling up" of flux above the conductor and a canceling of flux below the conductor. The net result is a force tending to move the conductor downward. However, motion downward produces, by the flux-cutting principle, a counter-emf through which the flow of current and the rate of transformation of energy from the electric to the mechanical form are controlled.

The *Blv* formula, equation (3-16), may be utilized in a quantitative analysis of the motor action just described. Neglecting the conductor resistance and assuming a uniform downward velocity through a uniform magnetic field, we find that the potential drop experienced by the current in passing through the conductor is:

$$v = -e = -Blv_n. \tag{3-23}$$

This emf has the same polarity as before, relative to the conductor. Because the current is reversed, from the viewpoint of the current, this emf is a potential drop and accordingly is given the negative sign.

If, now, an external emf is applied to the conductor in sufficient amount to send a current, *i*, through the conductor in a direction opposite to the generated emf, then power leaves the electric system in the amount

$$p = vi = -ei = -Blv_n i. \tag{3-24}$$

Because it is e which is converting energy from the electric form, and since e varies with the value of mechanical velocity, it is logical to assume that the energy conversion is to the mechanical form. But we also know that mechanical power can be expressed in terms of mechanical force, F, times mechanical velocity, v_n. Utilizing the principle of conservation of energy, we can write

$$p = -Fv_n = -ei = -Blv_ni. \tag{3-25}$$

Dividing this through by $(-v_n)$ gives us

$$\blacktriangleright \qquad F = Bli, \tag{3-26}$$

where:

F = *mechanical force* on conductor, in **newtons,**
B = *flux density*, in **webers per square meter,**
l = *active length* of conductor in magnetic field, in **meters,** and
i = *current* through conductor, in **amperes.**

This "*Bli*" formula may be employed in the computation of torque acting on a one-turn coil situated in a uniform magnetic field, as in Fig. 3.16B. Here, if current is forced to flow counter to the generated emf's, the left-hand conductor will be forced downward, and the right-hand conductor will be forced upward. Each of these forces is computed by equation (3-26). The component of each of these forces acting normal to the plane of the coil is

$$F_n = F \sin \alpha = Bli \sin \alpha, \tag{3-27}$$

and the torque due to one conductor acting at radius r is

$$T_{\text{cond}} = Blir \sin \alpha. \tag{3-28}$$

The total torque acting on the one-turn coil is

$$T_{\text{turn}} = 2Blir \sin \alpha. \tag{3-29}$$

It is instructive to note that the torque is maximum when the plane of the coil is parallel to the flux lines and that it is zero when the plane of the coil is perpendicular to the flux lines. It will be noted, further, that the torque direction is such as to tend always to move the coil into a position of maximum flux linkage. This tendency, of course, is in accord with the principle of dynamic equilibrium of an electromagnetic system.

3.22. The Elementary D-C Motor

The elementary d-c motor is like the elementary synchronous motor pictured in Fig. 3.20(a), except that the slip rings of the synchronous motor are replaced by a commutator, as indicated by Fig. 3.22. The function of the commutator is to maintain a given current direction in that armature conductor which is closer to a given magnetic pole and to maintain the opposite current direction in the other conductor, which is closer to the other pole. As the coil rotates, the two conductors alternate in being closer to a given magnetic pole. For this reason, although direct current is supplied to the terminals of the machine, the commutated current actually received by a given armature conductor is alternating. Hence the operation of the elementary d-c motor is similar to the operation of the elementary synchronous motor described in Article 3.20.

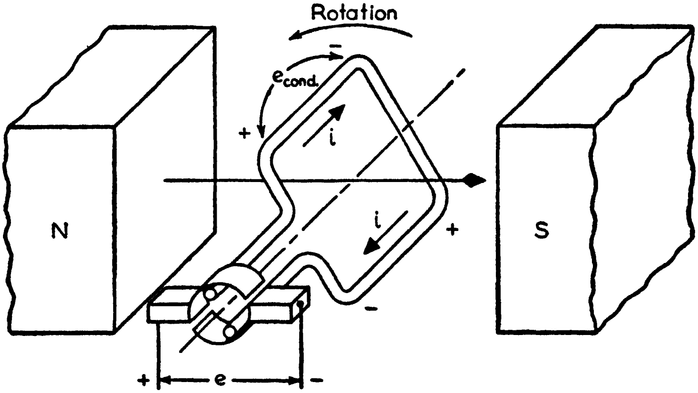


Fig. 3.22. The elementary single-coil d-c motor.

The speed at which the d-c motor operates is not governed by the frequency of alternation of the armature current, since this is not an "independent" quantity. Analysis reveals that the speed of a d-c motor is governed by the counter-emf developed in the armature conductors. Neglecting losses, the condition of equilibrium occurs at that speed for which the electric power input exactly balances the mechanical power output.

3.23. The Elementary Induction Motor

Another important application of the principle of dynamic equilibrium of electromagnetic systems is the induction motor, which operates on alternating-current systems. Figure 3.23 illustrates the principle of the elementary induction motor, showing simply a cylindrical metal rotor in a rotating magnetic field. It is beyond the scope of this article to discuss the origin and nature of the rotating magnetic field beyond stating the fact that this field can be established relatively easily by a set of stationary armature windings of the proper type. Assuming the existence of such a field, whose rate of rotation is proportional to the frequency of the current supplied to the armature, we can easily understand the principle under which a torque is developed tending to cause the rotor to turn, following the rotating field.

Let us consider one tiny element of the solid metal rotor, such as that indicated by the broken line in Fig. 3.23. This can be considered a one-turn coil. As the rotating magnetic field moves clockwise from a horizontal position, flux linkage is established with this "coil." By Lenz's law, an emf is induced, causing current to circulate as indicated. This induced current establishes its own component of magnetic flux field. Clockwise rotation of the cylinder results as the induced flux attempts to align with the rotating flux, according to the principle of dynamic equilibrium. If opposing torque exists and work must actually be done to maintain the rotation of the cylinder, it is evident that the cylinder cannot rotate precisely in synchronism with the rotating field. If it did so, no emf would be induced in the cylindrical rotor, no rotor current would flow, and no motor torque would be developed. A certain amount of *slip* of the rotor relative to the rotating field is necessary if torque is to be developed.

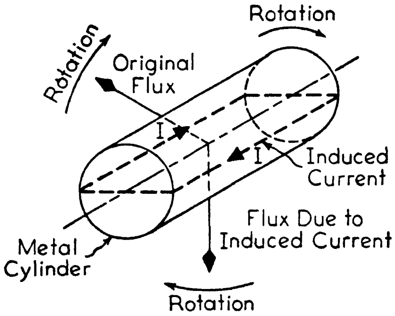


Fig. 3.23. Principle of the induction motor.

With slight modification, this effect can be used, in conjunction with stationary magnetic fields, to produce braking of moving metallic bodies!

3.24. The Elementary Transformer

Another fundamental device that is most important in electrical engineering is the transformer, which consists essentially of two coils linked by a common magnetic circuit. If two separate electric circuits are connected to these two coils, the transformer serves as a device for coupling the two circuits through the medium of the interlinking magnetic circuit. Energy supplied by the primary electric circuit is transferred through the magnetic circuit to the secondary electric circuit. This process makes use of the phenomenon of electromagnetic induction and is completely reversible, either coil being capable of serving as the *primary winding* of the transformer, the other serving as the *secondary winding.*

Consider the elementary transformer depicted by Fig. 3.24. The primary winding is connected to a source of alternating emf, e_p. The ($\pm$) sign indicates those terminals which are always of like polarity. Let us imagine that the wave of current through the primary, i_p, has just passed through zero and is flowing in a positive direction, as indicated. It is apparent that this current, as it increases, establishes an increasing magnetic flux, φ, in the magnetic circuit. By Lenz's law, an increase of flux linkage with any coil will be opposed by an induced emf. The induced emf's in both primary and secondary coils for the condition of increasing flux in the magnetic circuit are such as to make the top terminals of both windings positive. Once the current and flux waves pass their peak and the flux is decreasing in value, the polarity of the induced emf's will reverse, although the current has not yet reversed direction. The alternating emf, e_s, of the secondary winding can be used as a source for introducing energy into the secondary circuit and circulating current through it.

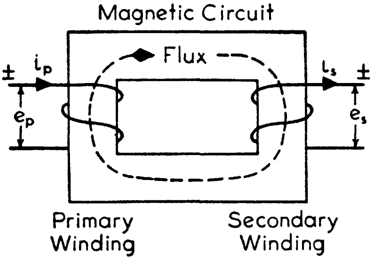


Fig. 3.24. The elementary transformer.

Denoting the positive current direction to be clockwise in both windings, we see from the polarity markings that the primary current experiences a potential drop in flowing through the primary winding. Meanwhile, the secondary current experiences a potential rise in flowing through the secondary winding. Thus, we see that at the instant considered, the primary winding functions as a sink in the primary circuit. At the same instant, we see that the secondary winding functions as a source in the secondary circuit.

Inasmuch as a detailed analysis of transformer action appears in Chapter 13, "Coupled Circuits," we shall not pursue the present discussion further.

3.25. Potential Difference as a Function of Time

The definitions for potential differences (or emf's) which are functions of time can be obtained from the definitions of Article 2.06 by substituting the term "potential difference" (or "emf") for the word "current" wherever the latter occurs in those definitions.

3.26. Waves in Electric Circuits

In the general sense, the term "wave" can be defined as follows: "*A wave is a disturbance which is propagated in a medium in such a manner that at any point in the medium the displacement is a function of the time, while at any instant the displacement at a point is a function of the position of the point.*"[8] This definition applies, of course, to physical phenomena which occur in three-dimensional space. Sound waves and radio waves are examples. Analyzing the behavior of such waves is primarily a field problem.

In this text, however, we are concerned with the specialized analysis of circuits. Accordingly, we may expect to find a simpler, more specialized definition of the term "wave" as it applies to the study of electric circuits. "*A wave (in an electric circuit) is the variation of current and/or potential in the electric circuit.*"[9] This definition, as distinguished from the more general definition, is implied whenever the term "wave" is applied to potential or current in this book. It suggests *a repetitive variation in time* of the given quantity.

The waves of potential or current with which we shall deal may be considered primarily as functions of time. Because the angular displacement of an armature conductor in a generator also is a function of time, it follows that waves of potential and current may, if so desired, be expressed as functions of angular displacement of a generator armature conductor. Thus, from equations (3-7) and (3-12), we may write the expression for generated emf, e, as either a function of time, t, or as a function of angular displacement, $\alpha = \omega t$:

$$e = E_m \sin \alpha = E_m \sin \omega t. \tag{3-30}$$

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[8] *Ibid.*, No. 05.05.405.

[9] *Ibid.*, No. 35.90.050.

The *form* (or shape) of this wave of emf produced by the elementary synchronous generator can be described as sinusoidal. This particular wave is a sine wave. We may obtain the form of a wave by plotting instantaneous values of the quantity as ordinates against values of time (or of angular displacement) as abscissas. Figure 3.27A shows the plot of equation (3-30). Cosine waves likewise are sinusoidal, since they have the same form when plotted.

A major portion of the remainder of this text deals with sinusoidal waves of potential, current, and power. Sinusoidal waves are highly desirable in much of electrical engineering work because sinusoidal quantities are very easy to manipulate mathematically. They may be subjected to virtually any type of standard mathematical operation—addition, subtraction, multiplication, division, integration, differentiation—and the result is always another sinusoidal quantity. This advantage leads electrical designers to strive for alternating-current generators, oscillators, and impedance devices which will produce sinusoidal potential and current waves.

Many factors arise in practice to render waves of potential, current, and power nonsinusoidal. This fact need not prevent our becoming thoroughly versed in the techniques of handling sinusoidal waves, because later we shall learn how to extend these techniques even to the handling of nonsinusoidal quantities. Figure 3.27A shows also one type of nonsinusoidal wave which may be encountered in practice.

3.27. Definitions Relating to Alternating-current Circuit Quantities

We shall now define for future use a number of terms relating to the waves of electric quantities encountered in alternating-current (a-c) circuits. These definitions are patterned after those appearing in Group 05, Section 05, of *American Standard Definitions of Electrical Terms* published by the American Institute of Electrical Engineers in 1942. Whereas the original definitions, in most cases, apply to any periodic quantity, some are defined here in terms which apply particularly to the periodic quantities encountered in a-c circuits. Equation (3-30) and equation (3-35), accompanying the last of the definitions, will be used to illustrate most of these terms.

1. *Sinusoidal Quantity.* A sinusoidal (or simple harmonic) quantity is one which is the product of a constant and either the sine or cosine of an angle and which has values that vary linearly

with the values of the independent variable. Such a quantity is both alternating and periodic, and is exemplified by equation (3-30), in which E_m is the constant and $(\alpha = \omega t)$ is the angle which varies linearly with time, t.

2. *Cycle.* A cycle (symbol ∿) is the complete series of values of a periodic quantity which occur during a period. In the case of a sinusoidal wave, a cycle would be one complete set of positive and negative values, corresponding to the total angular variation of 2π radians, or 360°. Fig. 3.27A illustrates this definition.

3. *Period.* The period of a periodic quantity is the smallest value of the increment of the independent variable which separates recurring values of the quantity. In equation (3-30) the period, T, is an increment of time such that

$$t = \frac{2\pi}{\omega} \text{ seconds} = T. \tag{3-31}$$

The period is the time required for one complete cycle.

4. *Frequency.* The frequency, f, of a time-varying periodic quantity is the number of periods occurring in unit time. It is customarily expressed in cycles per second (frequently abbreviated to "cycles"). Thus,

$$f = \frac{1}{T} \text{ cycles per second.} \tag{3-32}$$

If a certain event, such as a one-cycle alternation of emf, required $\frac{1}{60}$ second, then the same event could occur repetitively 60 times in one second. The frequency would be 60 cycles per second.

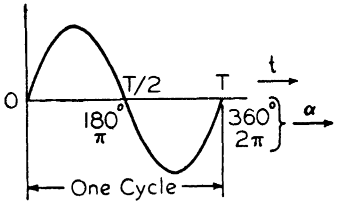
We shall now relate the frequency of circuit emf to the speed of rotation of a synchronous motor or generator operating in the circuit. Referring to the four-pole synchronous generator of Fig. 3.27B, we see that in one half revolution, one *pair* of poles will have gone past a given conductor, generating one cycle of emf. One complete revolution will generate two cycles of emf in the conductor. A bit of reflection reveals that, in general, *in one complete revolution of the generator shaft, one cycle of emf is generated per pair of poles.* This statement leads us to write that the frequency, f, is equal to the number of pairs of poles multiplied by the number of revolutions per second. Mathematically,

$$f = \frac{p}{2} \times \text{(rev. per sec.)} = \frac{p}{120} \times \text{(rev. per min.)}, \tag{3-33}$$

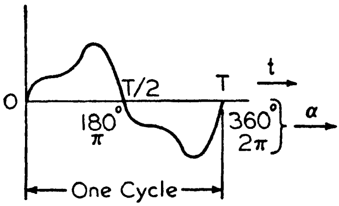
where:
f = *frequency*, in **cycles per second,** and
p = *number of poles.*

The same relation holds for a synchronous motor.

5. *Angular Velocity.* The angular velocity, ω, of a periodic quantity is the frequency multiplied by 2π. It is the number of radians per second, where 2π radians correspond to one cycle. From equations (3-31) and (3-32), we see that

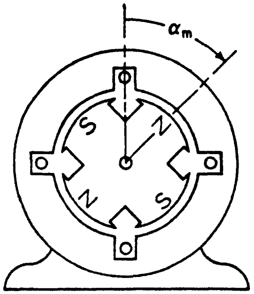


(a) Sine wave.

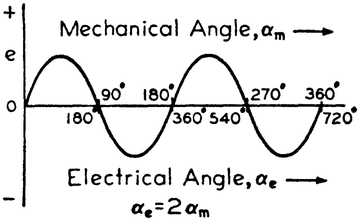


(b) Non-sinusoidal wave.

Fig. 3.27A. One cycle of periodic quantities.



(a) Schematic diagram of generator.



(b) Emf wave in single conductor.

Fig. 3.27B. Elementary four-pole synchronous generator.

$$\omega = 2\pi f = \frac{2\pi}{T} \text{ radians per second.} \qquad (3\text{-}34)$$

By further reference to Fig. 3.27B, we see that an *electrical angle,* in either radians or degrees, is equal to the corresponding *mechanical angle* multiplied by the number of pairs of poles. We have already checked this identity in pointing out that, for a four-pole machine, two cycles (4π electrical radians) correspond to one revolution (2π mechanical radians) multiplied by two pairs of poles. It is easy to see that *electrical angular velocity is equal to the product of the mechanical angular velocity multiplied by the number of pairs of poles.*

6. *Amplitude.* The amplitude of a sinusoidal quantity is the largest value that the quantity attains. Thus, E_m is the amplitude of the emf wave described by equation (3-35).

7. *Phase.* The phase of a periodic quantity, for a particular value of the independent variable, is the fractional part of a period through which the independent variable has advanced, measured from an arbitrary point of reference. In the case of a simple sinusoidal quantity, this point of reference is usually taken as the last previous passage through zero from the negative to the positive direction.

8. *Phase Angle.* The phase angle of a periodic quantity, for a particular value of the independent variable, is the angle obtained by multiplying the phase by 2π if the angle is to be expressed in radians, or by 360° if the angle is to be expressed in degrees. Thus, θ_e is the phase angle of the emf wave of equation (3-35).

9. *Initial Phase Angle.* The initial phase angle may be defined as the phase angle of a periodic quantity at the instant measurement of the independent variable is begun. In equation (3-35), the initial phase angle is θ_e. This might be described as the angle by which the wave "had a head start" on the measurement of time. An integral multiple of 2π radians (or 360°) may be added to or subtracted from the initial phase angle without affecting the value of a simple sinusoidal quantity. This fact is sometimes utilized to keep the absolute value of the initial phase angle as small as possible. Thus, an initial phase angle of 240° might be expressed as

$$\theta_e = 240° = (240° - 360°) = -120°.$$

10. *Phase Difference.* The phase difference between two sinusoidal quantities which have the same period is the fractional part of a period by which corresponding values of the two quantities are separated.

11. *Angular Phase Difference.* The angular phase difference between two periodic quantities which have the same period is 2π radians (or 360°) times the phase difference. Thus, if θ_e is the initial phase angle of wave, e, and θ_i is the initial phase angle of wave, i, the angular phase difference is $\theta = (\theta_e - \theta_i)$. (See Fig. 3.27D.)

12. *Angle of Lead.* The angle of lead of one sinusoidal quantity with respect to a second sinusoidal quantity having the same period

is the angular phase difference by which the second quantity must be advanced if it is to coincide with the first. Thus, in Fig. 3.27D, e leads i by an angle of $|\theta| = |\theta_e - \theta_i|$.

13. *Angle of Lag.* The angle of lag of one sinusoidal quantity with respect to a second sinusoidal quantity having the same period is the angular phase difference by which the second quantity must be retarded if it is to coincide with the first. In Fig. 3.27D, i lags e by an angle of $|\theta| = |\theta_e - \theta_i|$.

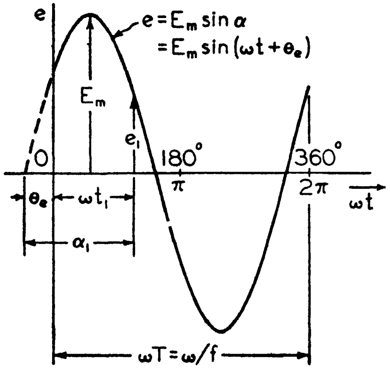


Fig. 3.27C. A sinusoidal emf.

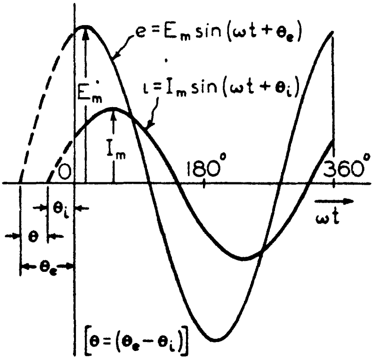


Fig. 3.27D. Two sinusoidal waves.

The equation for the emf wave of Fig. 3.27C is:

▶
$$\begin{aligned} e &= E_m \sin \alpha \\ &= E_m \sin (\omega t + \theta_e) \\ &= E_m \sin (2\pi f t + \theta_e) \\ &= E_m \sin \left(2\pi \frac{t}{T} + \theta_e\right), \end{aligned} \qquad (3\text{-}35)$$

where:

e = *instantaneous value of potential difference,* in **volts,**
E_m = *amplitude of potential difference,* in **volts,**
α = *phase angle,* in **radians (or degrees),**
ω = *angular velocity,* in **radians (or degrees) per second,**
t = *time* (the independent variable), in **seconds,**
θ_e = *initial phase angle,* in **radians (or degrees),**
f = *frequency,* in **cycles per second,**
T = *period,* in **seconds per cycle.**

Angular measure may be expressed in degrees instead of radians if one is careful to note that 2π radians = 360°.

Figure 3.27D pictures sinusoidal waves of potential and current for a particular case in which the potential wave leads the current wave.

3.28. The Frequency Spectrum

The values of frequency encountered in natural electromagnetic phenomena range over a virtually limitless band. On every hand, we find that the particles and the physical bodies of the universe are in various modes and states of vibration. Many of these vibrations give rise to electromagnetic waves which appear to penetrate and fill even the outermost reaches of space. We live, quite literally, in a universe of vibration.

The apparently limitless total range of electromagnetic frequencies which fill the universe is termed the *electromagnetic spectrum.* Like inefficient radio receivers capable of tuning to but one or two out of a multitude of transmitting stations, the sensory organs of our bodies are capable of detecting only certain restricted portions of this spectrum. A small band of the spectrum which we call *visible light* can be perceived by our eyes. In response to light waves of different frequencies, our eyes send to our brains signals which our brains interpret as colors. Another spectral region which we call the *infrared* causes certain nerve endings in our bodies to signal an impression of high or low temperature. The response of our skins to the *ultraviolet* region of the electromagnetic spectrum takes the form of pigmentation, which we call tanning. We have extended our knowledge of the electromagnetic spectrum through utilizing the physical properties of special materials and devices which enables them to respond to still other regions of this vast spectrum. Photosensitive and heat-sensitive chemicals, and radio receivers are examples. We have even learned how to generate electromagnetic waves in many regions of the spectrum. A great deal of this activity comes within the province of the electrical engineer.

The electromagnetic frequencies for which the electrical engineer must design energy-guiding and energy-transforming devices range from that of *direct current,* which may be treated mathematically as *zero-frequency alternating current,* to the frequencies of microwave radio, which are of the order of 10^{10} cycles per second. Energy-guiding devices may range from a two-conductor open-wire circuit to the completely closed wave guides of ultra-high-frequency systems. The mathematical techniques of handling the phenomena vary as the frequencies and the nature of the guiding devices vary.

Electric power frequencies in common use in the United States today are *zero cycles per second* (direct current), 25, 50, and 60 *cycles per second*, and, recently, 400 and 800 *cycles per second.* Because of certain advantages in generation and transmission, alternating currents are at present preferred to direct currents in the electric-power industry. The most prevalent power frequency is 60 cycles per second. This is preferred to lower frequencies because, in general, the higher the frequency, the lower the size and cost of basic machinery and the less noticeable the flicker of electric lamps.

Audio frequencies, or frequencies audible to the human ear when transformed into sound by loudspeakers, range from approximately 20 cycles per second to approximately 20,000 cycles per second. The sound systems used in many modern theaters are designed to accommodate frequencies ranging from 30 to 12,000 cycles per second, whereas the average radio receiver is more restricted in its audio-frequency range.

Radio frequencies in use today, including television and radar, range from 20,000 cycles per second to more than 10 billion cycles per second!

PROBLEMS

3-1. (a) How much energy, in watt-seconds (or joules), does an electron exchange in going through a potential difference of one volt? This amount of energy is one *electron-volt.*

(b) Express the answer to (a) in kilowatt-hours.

(c) How much energy, in electron-volts, does a proton exchange in going through a potential difference of one volt?

3-2. The potential of the plate of a certain vacuum tube is +200 volts relative to the potential of the cathode. An initially stationary electron at the cathode is accelerated toward the plate by this potential difference. Assuming the field to be uniform and rectilinear:

(a) What is the velocity of the electron, in centimeters per second, as it reaches the plate?

(b) How much kinetic energy, in joules, does the electron possess as it reaches the plate?

3-3. An electron starts from rest at the negative plate of a plane parallel-plate capacitor whose plates are 3 centimeters apart. If the potential between the plates is 2000 volts:

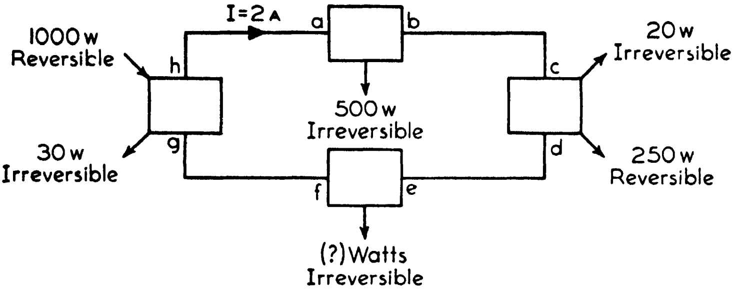
(a) How far must the electron travel before it acquires a speed of 10^9 centimeters per second?

(b) Through what potential difference has the electron fallen in acquiring this speed?

3-4. A certain incandescent lamp operating at 120 volts takes 60 watts. What current does the lamp take?

3-5. (a) For the circuit of the accompanying figure, compute the following potential differences, and express them as potential rises: E_{ab}, E_{cd}, E_{ef}, E_{gh}.

(b) Repeat (a), expressing the potential differences as potential drops.



Prob. 3.5.

3-6. Redraw the circuit of Prob. 3-5, using the symbol $\underset{+}{\dashv}\underset{-}{\vdash}$ for emf's and the symbol $-\!\wedge\!\!\wedge\!\!\wedge\!-$ for resistances. Indicate with polarity marks (+, −) the polarity of each element of the circuit.

3-7. An emf of 10 millivolts is produced by a certain thermocouple. The resulting current through the connected circuit is 1000 microamperes.

(a) At what rate, in electrical units, is heat converted into electric power?

(b) Does the thermocouple tend to be at a temperature other than that of its surroundings? Explain.

3-8. The armature of a certain d-c generator develops an emf of 500 volts. What armature current is flowing if 100 kilowatts of mechanical power is being converted into electric power?

3-9. A certain electric motor is converting 10,000 watts of electric power into mechanical power. What is the counter-emf developed by this machine, if the armature current is 50 amperes?

3-10. A double-blade knife switch is suddenly opened to interrupt the flow of current through a 500-turn coil. If the flux linking the coil just previous to the opening of the switch was 10 webers and if the current drops to zero in 0.1 second following the opening of the switch, what average potential difference appears between the poles of the switch during the period of current decay? Does this value suggest a possible hazard? Explain.

3-11. At a certain point on the surface of the earth, the magnetic field of the earth has a strength of 0.5 gauss and makes an angle of 67° with respect to the horizontal, in a north-south vertical plane. (Magnetic north is assumed to coincide with geographic north; declination is zero.) At this point, a coil of 500 turns and 10 square centimeters cross section is held in a vertical plane perpendicular to the north-south line.

(a) Write the equation for emf induced in the coil by rotation at a uniform angular velocity about a vertical axis.

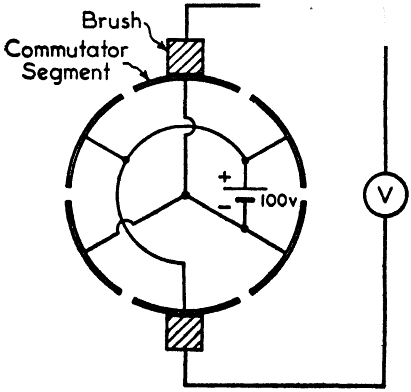
(b) What average emf would be induced in the coil if it is rotated about its vertical axis through 90° in 0.01 second?

(c) What average emf would be induced in the coil if it is rotated about its vertical axis through only 30° at the same uniform angular velocity?

3-12. The maximum flux density in the gap of the elementary synchronous generator of Fig. 3.15B is one weber per square meter. The effective length of the conductor in each slot is 0.15 meter. If the space distribution of flux density is sinusoidal and if the armature turns at 3600 rpm, what is the equation of the emf generated in the one-turn coil shown?

3-13. At a certain point on the surface of the earth, the magnetic field of the earth has a strength of 0.5 gauss and makes an angle of 67° relative to the horizontal, in a north-south vertical plane. A railroad car with axles measuring 4 feet 8 inches between wheels is moving at a rate of 60 miles per hour along a horizontal track at this point. Assuming magnetic north and geographic north to coincide, what value of emf is generated in each axle: (a) if the train is traveling due north? (b) if the train is traveling due east?

3-14. In the accompanying figure, each break between the commutator segments is one per cent of the total circumference of the commutator. All segments are the same size. Plot a curve of the voltmeter reading as a function of angular displacement for one revolution of the commutator.



Prob. 3.14.

3-15. The battery emf of Prob. 3-14 is replaced by a sinusoidal alternating emf. This emf has an amplitude of 100 volts and goes through zero at the moments when the centers of the breaks between segments are aligned with the brush centers, and there is no brush-to-segment contact. Plot a curve of the potential across the voltmeter as a function of angular displacement for one revolution of the commutator.

3-16. A conductor makes an angle of 60° with respect to a magnetic field whose strength is 0.05 weber per square meter. What is the force per foot, expressed in pounds, acting on the conductor when 900 amperes is flowing?

3-17. A current of 40 amperes is sent through the single-turn coil of Fig. 3.22. If the coil is one inch in radius and 4 inches long, and if the flux density is 5500 gausses, what is the torque acting on the coil: (a) if the angle $\alpha = 0°$; (b) if the angle $\alpha = 30°$; (c) if the angle $\alpha = 60°$; (d) if the angle $\alpha = 90°$? α = displacement of coil from the vertical.

3-18. Using numerical coefficients, write the equation for torque as a function of angular displacement for the case where a current of 10 amperes is sent through the coil of Prob. 3-11.

3-19. (a) What is the frequency of a 24-pole synchronous generator running at 6000 rpm? (b) What is the maximum speed for which a synchronous generator can be designed to yield a frequency of 60 cycles per second?

3-20. In interconnecting two power systems of different frequencies, it is necessary to employ a frequency changer. A frequency changer is essentially two a-c rotating machines directly coupled to the same shaft but provided with different numbers of poles, in the ratio of the frequencies of the interconnected systems. Depending on which system supplies power to the other, one of these machines serves as a motor to drive the other as a generator. Find the highest possible speed and the numbers of poles of the two machines for a frequency changer designed to interconnect: (a) a 60-cycle system and a 25-cycle system; (b) a 60-cycle system and a 50-cycle system.

3-21. The time variation of emf produced by a given synchronous generator is represented as:

$$e = 100 \cos 377t \text{ volts.}$$

(a) What is the amplitude of the generated emf?

(b) What is the frequency of the generated emf?

(c) Re-express the generated emf as a sine wave having an appropriate initial phase angle, and make a sketch to show this wave.

3-22. Express the following sinusoidal waves as positive sine waves, and sketch each wave: (a) $e = -10 \sin 157t$; (b) $e = 10 \cos (157t + 30°)$; (c) $e = -10 \cos (157t + 30°)$; (d) $e = -10 \sin (157t - 30°)$; (e) $e = -10 \cos (157t - 30°)$.

CHAPTER 4

CIRCUIT PARAMETERS

4.01. Types of Physical Circuit Parameters

The energy transformations and translations that occur in an electric circuit are affected by certain properties of the circuit which may be termed *circuit parameters.* In many cases these parameters are very nearly constant in value; in other cases they may noticeably be functions of some other quantity, such as frequency or current density. In the cases where the parameters are substantially constant in value, they are sometimes termed *circuit constants.*

There are three types of circuit parameters, *resistance, inductance,* and *capacitance,* for which the symbols are R, L, and C. From the viewpoint of energy transformation, they correspond in the rectilineal mechanical system to, respectively, friction, mass, and compliance.[1] Both resistance and friction dissipate energy as heat, both inductance and mass function to store kinetic energy, and both capacitance and compliance function to store potential energy.

The physical nature of these three circuit parameters must be well understood before we undertake the study of electric circuits. We shall therefore study each in turn, in some detail.

A. RESISTANCE, R

4.02. Definition and Nature of Resistance, R

"*Resistance is the* (*scalar*) *property of an electric circuit or of any body that may be used as part of an electric circuit which determines for a given current the rate at which electric energy is converted into heat or radiant energy and which has a value such that the product of the*

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[1] Olson, H. F., *Dynamical Analogies.* New York: D. Van Nostrand Company, Inc., 1943.

resistance and the square of the current gives the rate of energy conversion."[2] In mathematical form, this definition states

▶ $$r = \frac{p}{i^2}, \tag{4-1}$$

where:
r = *instantaneous value of resistance,* in **ohms,**
p = *instantaneous absolute value of power,* in **watts,**
i = *instantaneous value of current,* in **amperes.**

The absolute value of power is employed, inasmuch as resistance is always algebraically positive. This equation is very general, since it applies to instantaneous values. A more specific equation can be written if resistance is constant for the operating conditions being considered:

$$R = \frac{p}{i^2}, \tag{4-2}$$

where: R = *resistance* (assumed constant), in **ohms.**

Resistance, in offering opposition to the flow of electricity, may be compared dynamically to friction in mechanical and hydraulic circuits. Friction converts into heat some of the energy associated with the mechanical movement of matter, whereas resistance converts into heat (or some other form of radiant energy) some of the energy associated with the movement of charged particles through an electric circuit. In both processes this transformation of energy is irreversible. The resulting potential drop across the resistance in the positive current direction is always just that: a drop. Because of the fact that the energy transfer and transformation is not reversible, the potential difference across a current-carrying resistance is not an emf.

Effective resistance is defined by equation (4-3):

▶ $$R = \frac{P}{I^2}, \tag{4-3}$$

where:
R = *effective resistance* (at a given frequency), in **ohms,**
P = absolute value of *average power* (the wattmeter indication), in **watts,** and
I = *effective current* (the ammeter indication), in **amperes.**

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[2] *American Standard Definitions of Electrical Terms,* No. 05.20.150. New York: American Institute of Electrical Engineers, 1942.

This is the equation which we shall employ in our unified approach to the study of electric circuits. It may be employed for any given frequency, including zero cycles per second—the direct-current case.

Effective resistance, which is defined by equation (4-3), varies considerably over the broad frequency range encountered in practice. The lowest value which the effective resistance may have, so far as frequency is concerned, is the value which occurs at zero frequency: the so-called *d-c resistance.* It is the d-c resistance which is sometimes implied in practice by the simple term "resistance." Effective resistance, then, may be regarded as composed of the basic d-c resistance (sometimes called *ohmic* resistance) plus certain "fictitious" resistance components arising from secondary effects related to the frequency of the system. We shall consider first the various aspects of d-c resistance and shall then investigate the fictitious resistance components which combine with this to form the higher resistance values encountered at higher frequencies.

4.03. D-C Resistance, $R_{\text{d-c}}$.

The definition of resistance given in the preceding article and represented by equation (4-3) is valid for any frequency, including zero cycles per second. If we assume the zero-frequency (direct-current) case, we can rewrite this equation as

$$R_{\text{d-c}} = \frac{P_{\text{d-c}}}{I^2_{\text{d-c}}}, \tag{4-4}$$

where the subscript "d-c" denotes the "direct-current value."

By equation (3-4), we can write, for a direct-current circuit,

$$P_{\text{d-c}} = V_{\text{d-c}} I_{\text{d-c}}. \tag{4-5}$$

In utilizing this equation, we must remember that, if we call *drops* positive, when we trace in the positive current direction through a sink, both P and V will be numerically positive. Substituting (4-5) in (4-4) gives

$$R_{\text{d-c}} = \frac{V_{\text{d-c}} I_{\text{d-c}}}{I^2_{\text{d-c}}}, \tag{4-6}$$

or

▶ $$R_{\text{d-c}} = \frac{V_{\text{d-c}}}{I_{\text{d-c}}}, \tag{4-7}$$

where:

$R_{\text{d-c}}$ = *d-c resistance,* in **ohms,**
$V_{\text{d-c}}$ = *d-c potential drop* across resistance, in positive current direction, in **volts,**
$I_{\text{d-c}}$ = *direct current* through resistance, in **amperes.**

The practical unit of resistance, the *ohm,* can be defined by equation (4-7). The unit of resistance in any other self-consistent system of units can also be defined by equation (4-7) by insertion of unit values of $V_{\text{d-c}}$ and $I_{\text{d-c}}$ expressed in that system.

Equation (4-7) is called *Ohm's law,* in honor of G. S. Ohm, who first published his discovery of it in 1826.

4.04. Resistivity, ρ

The specific resistance, or resistivity, ρ, of a material is the direct-current resistance between the opposite, parallel faces of a portion of the material having unit length and unit cross section. In scientific work, resistivity is usually specified in terms of the d-c resistance of a material between opposite faces of a one-centimeter cube of the material. The unit of resistivity thus specified consists dimensionally of $\frac{(ohms)(centimeter)^2}{(centimeter)}$. This unit of resistivity is, logically, called the *ohm-centimeter.*

Much engineering work in the United States has been done in English units. For this reason, plus the fact that many conductors are circular in cross section, resistivity is often expressed in terms of the resistance between opposite ends of a one-foot length of the material, one *circular mil* in cross-sectional area. The true dimension of this unit of resistivity is *circular-mil ohms per foot* (or *ohm-circular mils per foot*), but through common usage the dimensionally erroneous term "ohms per circular mil foot" has often been used as the standard name for this unit. The unit of area "circular mil" is defined and discussed in Article 4.06.

4.05. Davy's Law

The definition of d-c resistance expressed by equation (4-4) might be termed a "functional" definition. In 1821 Sir Humphry Davy observed what we may term the "structural" aspect of d-c resistance. In symbolic form, Davy's law states

▶
$$R_{\text{d-c}} = \rho \frac{l}{A}, \tag{4-8}$$

where:

$R_{d\text{-}c}$ = *d-c resistance* of the conductor, in **ohms,**
ρ = *resistivity* of conductor material, in ohmic units,
l = *length* of conductor, in units consistent with ρ, and
A = *cross-sectional area* of conductor, in units consistent with ρ.

Through use of this relation, the resistance of a conductor having any combinations of length and cross section may be determined, provided that the resistivity of the conducting material is known. For electrical properties, including resistivities, of commonly used materials, refer to Table A4 of the Appendix.

Example 4.05. What is the d-c resistance between ends of a rod 1 meter long having a cross section of 2 square centimeters and made of copper having a resistivity of 1.72×10^{-6} ohm-centimeters?

$$R_{d\text{-}c} = \rho \frac{l}{A} = 1.72 \times 10^{-6} \times \frac{100}{2} = 86 \times 10^{-6} \text{ ohm} = 86 \text{ microhms.}$$

4.06. Units of Area: the Circular Mil

Dimensionally, *area* is composed of the square of linear distance. Thus, an area of one *square meter* equals the product of one meter multiplied by one meter. Although the unit of area, the square meter, can have the shape of a square, it is not necessarily square in form. Thus, a circle or a triangle also could have an area of one square meter. A "square" unit, when applied to the measure of area, evidently does not necessarily describe the form, or shape, of the area whose size is being expressed.

Much more convenient than either the square meter or the square centimeter in the measurement of practical wire cross sections is the *circular mil,* which is equivalent in size to the area of a circle one *mil* (one-thousandth inch) in diameter. Like the term "square meter," the term "circular mil" is a unit of size only and does not necessarily describe the shape of the area under consideration.

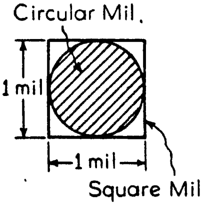

Fig. 4.06. The circular mil and the square mil.

From Fig. 4.06 it is apparent that the circular mil is a smaller unit than the square mil, being only $\pi/4$ times (about three-quarters) the size of the square mil. A given area expressed in square mils would require a greater number of circular mils ($4/\pi$ times as many) for its evaluation in these smaller units.

Since the area of a circle varies as the square of its diameter, and since the area of a circle one mil in diameter is one circular mil, the area of any circle in circular mils equals the square of its diameter in mils, or

$$A' = d^2, \tag{4-9}$$

where:

A' = *cross-sectional area of circle,* in **circular mils,**
d = *diameter of circle,* in **mils.**

Example 4.06A. What is the area of one square meter, expressed in circular mils?

$$\begin{aligned} 1 \text{ sq. meter} &= 10^4 \text{ sq. cm.} = \frac{10^4}{(2.54)^2} \text{ sq. in.} \\ &= (0.155 \times 10^4) \times 10^6 \text{ sq. mils} \\ &= (0.155 \times 10^{10}) \times \frac{4}{\pi} \text{ cir. mils} \\ &= 1.975 \times 10^9 \text{ cir. mils.} \end{aligned}$$

Example 4.06B. What is the cross-sectional area, in circular mils, of a $\frac{1}{8}$-inch-diameter conductor?

$$\begin{aligned} d &= \tfrac{1}{8} \text{ inch} = 0.125 \times 10^3 \text{ mils} = 125 \text{ mils.} \\ A' &= d^2 = (125)^2 = 15{,}625 \text{ cir. mils.} \end{aligned}$$

4.07. Resistance and Circular-mil-foot Resistivity

In the United States the length, l, of a conductor is customarily expressed in feet. Because conductors are circular rather than square in cross section, it is convenient to express the cross-sectional area in circular, rather than in square, units, thus eliminating the necessity for employing the conversion factor $\pi/4$. Most conductors are less than one inch in diameter, and therefore their cross-sectional areas may conveniently be expressed in circular mils, rather than in some larger circular unit such as circular feet or circular inches.

For circular conductors, equation (4-9) may be written

▶ $$R_{d\text{-}c} = \rho \frac{l}{A'} = \rho \frac{l}{d^2}, \tag{4-10}$$

where:

$R_{d\text{-}c}$ = *d-c resistance* of conductor, in **ohms,**
ρ = *resistivity* of material, in **circular-mil ohms per foot** (or "ohms per circular mil foot," through popular—but erroneous—usage),

l = *length* of conductor, in **feet,**
A' = *cross-sectional area* of conductor, in **circular mils,** and
d = *diameter* of conductor, in **mils.**

The resistivity of annealed copper at 20°C is 10.376 absolute circular-mil ohms per foot. The resistivities of other materials encountered frequently in practice appear in the Table A4, Appendix. It will be noted that copper has differing properties, depending upon the manner in which it is prepared. Alloys have resistivities which are nearly always higher than those of the constituent metals and always higher than that of the constituent metal of lowest resistivity.

For a more complete compilation of material properties, attention is called to the *Smithsonian Physical Tables*, published by the Smithsonian Institution, Washington, D. C.

EXAMPLE 4.07. What is the d-c resistance of one mile of $\frac{1}{4}$-inch-diameter copper wire at 20° centigrade?

$$d = \tfrac{1}{4}\text{ in.} = 0.250\text{ in.} = 250\text{ mils.}$$
$$\rho = 10.376\text{ cir.-mil ohms per foot at }20°\text{C.}$$
$$l = 1\text{ mile} = 5{,}280\text{ ft.}$$
$$R_{\text{d-c}} = \rho\frac{l}{d^2} = 10.376 \times \frac{5280}{250 \times 250} = 0.875\text{ ohm.}$$

4.08. D-C Conductance, $G_{\text{d-c}}$

Direct-current conductance, $G_{\text{d-c}}$ (as distinguished from alternating-current conductance, which will be defined later), is the reciprocal of d-c resistance. Mathematically,

$$G_{\text{d-c}} = \frac{1}{R_{\text{d-c}}}, \tag{4-11}$$

where:
$G_{\text{d-c}}$ = *d-c conductance*, in **mhos** (reciprocal ohms), and
$R_{\text{d-c}}$ = *d-c resistance*, in **ohms.**

The d-c resistance of a conductor may be viewed as opposition offered to the flow of direct current by the conductor material. As the inverse of d-c resistance, d-c conductance may be viewed as the direct-current conducting ability of the conductor, as a function of the material.

4.09. Conductivity, γ

Conductivity, γ, is the reciprocal of resistivity, ρ. Mathematically,

$$\blacktriangleright \qquad \gamma = \frac{1}{\rho}, \tag{4-12}$$

where:

γ = *conductivity* of conductor material in any system of units,
ρ = *resistivity* of conductor material in the same system of units.

When resistivity is measured in circular-mil ohms per foot, conductivity is measured in *foot-ohms per circular mil.*

By inversion of the equation for Davy's law and substitution of equation (4-12), we obtain the "structural definition" for d-c conductance:

$$\blacktriangleright \qquad G_{d\text{-}c} = \gamma \frac{A}{l}, \tag{4-13}$$

where all units are as previously defined.

4.10. Per Cent Conductivity

The per cent conductivity of a given material is the ratio of conductivity of that material to the conductivity of standard annealed copper, expressed as a percentage. This ratio is convenient in discussing the relative merits of various possible materials or combinations of materials. Standard annealed copper with a 20°C resistivity of 1.724 microhm-centimeters is said to have a conductivity of 100 per cent. Ordinary copper runs between 98 and 100 per cent conductivity, but it can be purified to the point of having more than 100 per cent conductivity!

4.11. Copper-wire Tables

For many reasons it is advantageous to limit the number of wire sizes normally used in practice to a few standard diameters. Accordingly, the Bureau of Standards has prepared tables showing the resistance per 1000 feet of standard annealed copper wire at standard temperatures for a series of standard sizes. These sizes are designated by gauge numbers and are accompanied by related data such as diameter in mils, cross-sectional areas in circular mils, and weight in pounds per 1000 feet.

The standard wire gauge in general use in the United States is the Brown and Sharpe (B. and S.), commonly called the American wire gauge (A.W.G.). The Birmingham wire gauge (B.W.G.) is in general use in Great Britain.

The B. and S. gauge numbers are based on a geometric progression of wire diameters such that as the wires decrease in size, every third gauge number halves the cross-sectional area and doubles the resistance. It is suggested that the student verify these statements by reference to typical gauge numbers in Table A1 of the Appendix.

4.12. Stranded Cables

Solid wires, particularly in the larger sizes, are hard to pull into conduits and are liable to break when bent at sharp angles. To overcome these difficulties, the much more flexible stranded cables are often used. It will be noted that No. 0000 solid conductor, having a diameter of approximately one-half inch, is the largest size shown in Table A1 of the Appendix. When a larger size of conductor is required, it is practically always made in the stranded form, smaller sizes, too, are often stranded.

Obviously, the total cross-sectional area of a stranded cable is the sum of the areas of the individual strands of which it is composed. Because of inevitable small amounts of "waste space" between strands, the outside diameter of a stranded cable will be slightly greater than that of a solid conductor of the same total cross-sectional area.

Because of the spiral arrangement of the strands, the effective length, as well as the mass, of a stranded cable is greater than it would be for a solid conductor of the same cross section covering the same distance. As a consequence, the resistance likewise is greater for stranded cable than for solid conductor. The amount by which the resistance is increased depends upon the "lay," or degree of spiral, of the strands. For one typical lay of strands it is customary to compute the resistance of stranded cable by adding 2 per cent to the resistance of a rod of the same cross section.

4.13. Copper-clad Steel Wire

Copper-clad steel wire consists of a steel core to which has been welded a covering of copper. This type of wire has a lower cost and a higher tensile strength than solid copper wire of the same resistance per foot.

The circular-mil foot resistance of such wire depends on the relative cross-sectional area of copper and steel and on the resistivities of the copper and steel employed.

Two standard grades of copper-clad wire which are commercially available are the so-called "30 per cent conductivity" and "40 per cent conductivity" wires. Just as the names imply, these conductors possess conductivities which are, respectively, 30 per cent and 40 per cent that of standard annealed copper wire of the same size. In the 30 per cent conductivity wire, the cross-sectional area of the steel core is 79.5 per cent of the over-all cross section, whereas in the 40 per cent conductivity wire, the steel section is 68.2 per cent of the total.

4.14. Aluminum Wire and A.C.S R.

At 20°C, commercially available aluminum wire has a resistivity of 17.01 circular-mil ohms per foot—1.64 times that of copper—but a density only 0.345 that of copper. Thus, although the volume resistivity of aluminum wire is greater than that of copper wire, the mass resistivity, or resistance per unit mass of a given length, is less. This is another way of saying that an aluminum conductor of a certain resistance per unit length will possess a greater diameter, but will weigh less, than a copper conductor of the same resistance per unit length.

On high-voltage transmission lines, corona loss is reduced by an increased diameter, thus providing an additional argument in favor of aluminum wire. On the other hand, an increased cross section results in greater loading due to winds or to the accumulation of ice during a sleet storm. Such extreme weather conditions might break an aluminum conductor, with its relatively low tensile strength, where one of copper would have survived.

The above-mentioned disadvantage of aluminum conductors is met by the use of aluminum cable steel reinforced (A.C.S.R.) consisting of a core of one or more steel strands surrounded by aluminum strands. A.C.S.R. has proved capable of sustaining thick layers of ice and high winds, and is a real competitor to copper wire for the construction of electric-power transmission lines.

4.15. Maximum Allowable Current

An electric current develops heat in any conductor through which it passes. If heat is generated within the wire at a rate greater than that at which it is dissipated by the surface of the wire, the temperature of the wire will rise until the rate of dissipation balances that of generation. Wires confined within conduits and within the walls of buildings or within the windings of machines will develop a higher temperature with a given current flow than will the same size of conductor freely suspended between transmission-line towers out in the open. So that the temperatures attained by confined conductors will never rise high enough to cause the insulation to deteriorate, the National Board of Fire Underwriters has issued a table of the safe current-carrying capacities of copper wire of the sizes used in house wiring. Local regulations may be even more stringent in limiting current capacities. In any event, the currents carried by any interior wiring should not exceed the values given in this table (Table A3, Appendix).

The determination of the correct sizes of wires for the different parts of an interior distribution system therefore involves not one but two considerations:

(a) Current in any conductor must never exceed the safe maximum specified by the Underwriters' table. Tentative selection of sizes can be made by computing carefully the maximum current values to be expected in the various parts of the system and then referring to this table.

(b) The potential drop throughout the system must not exceed certain limits. Five per cent in illumination systems and 10 per cent in other types of loads are allowable limits.

4.16. Theories of Conduction

In previous articles we have discussed the atomic theory of matter as it relates to the availability of free electrons in individual atoms of the various elements. We saw that the electrons surrounding the nucleus of a given atom can occupy only certain restricted "shells" about the nucleus, each such shell representing a particular energy level. Energy must be supplied to an electron to move it from a shell of one level to a shell of a higher energy level.

These permissible energy levels can be represented by graphs such as those of Fig. 4.16A. In these graphs, there is but the one variable: energy, W, which is plotted vertically. Each horizontal line represents a permissible energy level. An electron is permitted to have the energy corresponding to any one of these lines, but it cannot have a value of energy between them. The *electron-volt* is usually employed as the unit of energy in quantitative plots of this sort, although the accompanying sketches are to no particular scale. The electron-volt is the energy required to move the charge of one electron through a potential difference of one volt.

The normal condition of equilibrium of an atom is the condition of lowest energy level—in the case of hydrogen, the innermost permissible shell. The atom can be "excited" by the addition of extra energy which raises the electron to a higher energy level. The electron soon returns to its original state of lower energy, however, by giving up the extra energy in the form of light or of electromagnetic radiation of another frequency band. Since the wave length (or color) of the emitted light depends on the amount of energy emitted and since this amount is equal to the drop in energy level, the atoms of each particular element can emit only certain wave lengths of light. This theory has been verified experimentally by studies of the spectra of many different elements in the excited state. When excitation is carried to its limit, the electron has been completely removed from the atom, which is then said to be *ionized.*

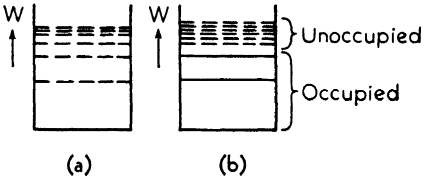

Fig. 4.16A. Permissible atomic energy levels. (a) Hydrogen atom. (b) More complex atom.

Because the analysis of more complex atoms becomes impossibly difficult in terms of the theory of orbits, modern treatment is almost entirely in terms of wave equations based on the principles of the quantum theory. By this method, permissible energy levels of the type thus far discussed can be worked out for the more complex atoms. The (a) part of Fig. 4.16B represents such an atom. As was pointed out in Article 2.01, there is a maximum number of electrons which may occupy a given energy level. Figure 4.16B is for the case of only one electron removed from the uppermost level.

A different set of permissible energy levels would have to be drawn for each different case, such as the simultaneous raising of two electrons or the raising of one electron from a lower level. In a complicated atom, then, the number of possible energy levels for the atom as a whole is rather great.

When two atoms having the same number of electrons are brought close together, the electric fields surrounding their individual particles combine to form a net field which is different from the field of either atom individually. The result is that the permissible energy levels for the pair of atoms now appear in slightly separated pairs, as indicated in part (b) of the figure. Consequently, for a cubic centimeter of a solid, which contains on the order of 10^{23} atoms, the permissible energy levels appear in groups of millions of barely separated distinct levels. Each such group, in effect, thus represents a *band* of permissible energy, as indicated by part (c) of Fig. 4.16B. The complexity of the bands increases with the number of kinds of atoms and with the complexity of the atoms.

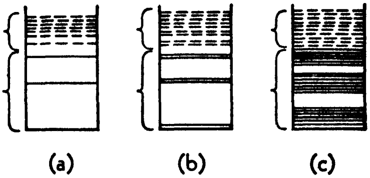

Fig. 4.16B. Permissible energy levels for groups of atoms. (a) Single atom. (b) Two adjacent atoms. (c) A group of atoms. (Note emergence of "band" pattern.)

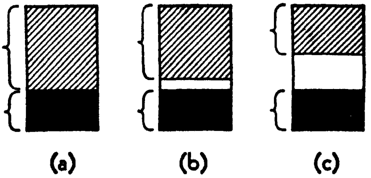

Fig. 4.16C. Energy-level diagrams for solids. (a) One type of conductor. (b) A semiconductor. (c) An insulator. (Black bands are occupied. Cross-hatched bands are permissible but unoccupied. White bands are forbidden.)

On the basis of the foregoing discussion, we can now give an explanation of the differing d-c conductivities found in various types of solid materials. Referring to Fig. 4.16C, we see the band theory of conduction in solids[4] applied, respectively, to conductors, semiconductors, and insulators. The three are distinguished from each other solely by the manner in which the fields of individual atoms combine to broaden permissible atomic energy levels into energy bands, by the process described above.

In conductors, even the most minute amounts of energy can be accepted by the electrons, which begin to drift in the direction dictated by the applied emf. In semiconductors, a similar situation prevails, since the lowest band of permissible energies is separated from the next by only a narrow forbidden band. In an insulator, electrons can accept energy only if it is sufficient to raise them above the wide forbidden band into the unoccupied permissible band. This statement means that an increase of potential difference applied across a perfect insulator will result in exactly no current flow until a certain value is reached, beyond which the material suddenly "breaks down" and becomes a conductor.

An additional factor in the conducting property of a solid material is the spacing between the atoms of that material. Thus, carbon in the form of graphite is a conductor, whereas carbon in the form of a diamond is an insulator. Of course this discrepancy is readily explained in terms of the foregoing theory.

Another physical theory which is useful in picturing some of the factors underlying the phenomenon of electrical conduction in solids has to do with the manner in which solids are built up of atoms. This theory is based on evidence that solid materials in their pure form are composed of crystals, each of which is an orderly arrangement in definite geometrical form of atomic nuclei, together with their electrons of low energy level. If these patterns are absolutely regular, the resistivity of the material is zero, which is to say that the conductivity is infinite. At least two factors are usually present to upset this regularity of pattern. One is the presence of impurities. The other is the thermal agitation of the atoms which exists at any temperature above absolute zero. This agitation causes the atoms to vibrate around an average position within the crystal. In support of this theory, interesting experiments have been conducted to demonstrate the "superconductivity" of certain materials at temperatures in the vicinity of absolute zero.[5,6] The study of alloys also has contributed a great deal to the support of this theory.[7]

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[4] Pearson, G. L., "The Physics of Electronic Semiconductors," *Electrical Engineering*, July 1947, Vol. 66, No. 7, page 638.

4.17. Temperature Variation of Resistance

We have seen that electric current takes the form, in solids, of a drift superimposed upon the random motion of free electrons within the solid. The greater the thermal agitation of the particles within the solid, the greater the chance that a given particle will collide with other particles. Assuming a fixed number of free electrons, a lower average rate of drift—a lower current—will then prevail under the influence of an applied potential difference of given value. This is another way of stating that, for a given number of free electrons within a solid conductor, the higher the temperature, the higher the resistance of the conductor. This is the characteristic exhibited by most conducting materials, including pure metals.

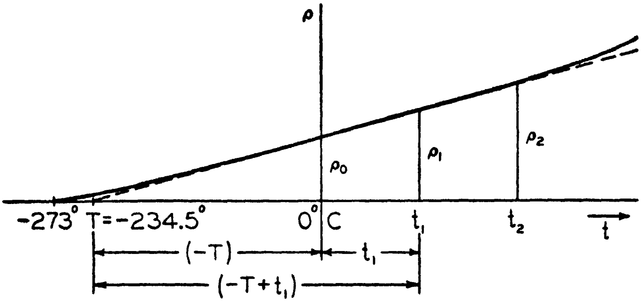

Fig. 4.17A. Relation of resistivity to temperature for copper.

Although, in general, the variation of resistance with temperature is not linear, for many materials it closely approaches linearity throughout the temperature range normally encountered in electrical engineering practice. Figure 4.17A shows approximately the relation of resistivity to temperature for standard annealed copper.

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[5] Burton, E. F., Smith, H. G., and Wilhelm, J. O., *Phenomena at the Temperature of Liquid Helium.* New York: Reinhold Publishing Corporation, 1940.

[6] Collins, S. C., "Low Temperature and Some of Its Effects upon the Behavior of Matter," *Science,* April 2, 1948, Vol. 107, No. 2779, page 329.

[7] Bardeen, J., "Electrical Conductivity of Metals," *Journal of Applied Physics,* February 1940, Vol. 2, No. 2.

The broken straight line coincides closely with the actual curve over this normal range of temperatures.

The straight line which coincides closely with the actual curve throughout the region of usual temperature variation provides us with a convenient means of computing the resistivity of a material at any temperature, t_2, if that resistivity is known at some other temperature, t_1. We note, first of all, that for standard annealed copper, the intersection of the extended straight portion of the temperature-resistance curve with the temperature axis occurs at -234.5°C. Let us symbolize the distance from this point of intersection to the origin, at 0°C, by the letter T. Thus, for standard annealed copper, $T = -234.5$°C. For other materials it has other values, many of which are given in Table A4 of the Appendix.

Referring again to the figure, we note that, from similar right triangles, we may write, *for copper*,

$$\frac{\rho_2}{\rho_1} = \frac{234.5 + t_2}{234.5 + t_1}. \tag{4-14}$$

In general, *for any material*, we may write this as

▶ $$\frac{\rho_2}{\rho_1} = \frac{-T + t_2}{-T + t_1} = \frac{t_2 - T}{t_1 - T} = \frac{T - t_2}{T - t_1}, \tag{4-15}$$

where:

ρ_1 = *resistivity of conductor* (in any units) *at temperature* t_1,
ρ_2 = *resistivity of conductor* (in same units) *at temperature* t_2,
T = *temperature intercept of extended straight portion of the temperature-resistance characteristic of conductor material in the same units as* t_1 *and* t_2 (normally, centigrade degrees).

The ability of certain solid materials, such as carbon, to act as conductors of electricity depends on temperature in an inverse manner. This statement may be "explained" somewhat sketchily as follows. At absolute zero (-273°C), such materials as carbon would possess no free electrons, and the "occupied" energy band would be adjacent to a "forbidden" band. At this temperature, these materials would be excellent insulators. As the temperature of these materials is raised above absolute zero, however, some of the electrons in the outermost energy levels absorb thermal energy in the amounts required to lift them through the forbidden band, and in some cases to remove them completely from the atom. These electrons then are free electrons capable of sustaining electric current flow, and the conductivity of the material has increased. Thus, the conductivity of carbon increases with temperature—another way of saying that carbon possesses an inverse temperature-resistance characteristic: the resistivity of carbon decreases with an increase of temperature. Figure 4.17B shows approximately the relation of resistivity to temperature for carbon.

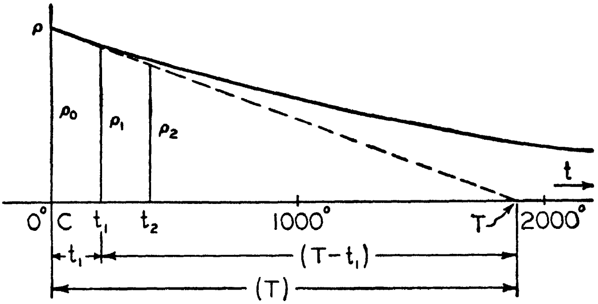


Fig. 4.17B. Relation of resistivity to temperature for carbon.

Here, again, the "straight" portion of the characteristic curve may be extended until it intersects the temperature axis. In this case, the T intercept is positive, inasmuch as the slope of the curve is negative and the curve lies within the first quadrant. By the method of similar triangles, we obtain the following relation for materials having an inverse temperature-resistivity characteristic, with negative slope:

$$\frac{\rho_2}{\rho_1} = \frac{T - t_2}{T - t_1}. \tag{4-16}$$

This is the same as the equation that was derived for materials having a temperature-resistivity characteristic with positive slope!

Example 4.17. What will be the resistance of a copper wire at 10°C if the resistance at 60°C is 5 ohms?

Inasmuch as both length and cross-sectional area are effectively constant, we can write

$$\frac{R_2}{R_1} = \frac{\rho_2}{\rho_1} = \frac{T - t_2}{T - t_1}, \qquad \text{or} \qquad R_2 = R_1 \frac{T - t_2}{T - t_1}.$$

Then $R_2 = 5 \times \dfrac{-234.5 - 10}{-234.5 - 60} = \dfrac{5 \times 244.5}{294.5} = 4.15$ ohms at 10°C.

4.18. Temperature Coefficient of Resistance, α_n

The equation of the extended straight portion of a temperature-resistivity characteristic curve derived in the preceding article may be modified to give another form which is usually employed in practice. Referring to Fig. 4.18, we see that an increment of temperature increase $\Delta t = t_2 - t_1$ is accompanied by an increment of resistance increase $\Delta R = R_2 - R_1$ for a conductor of fixed dimensions. Throughout the straight portion of the curve, the ratio of such increments is, of course, constant.

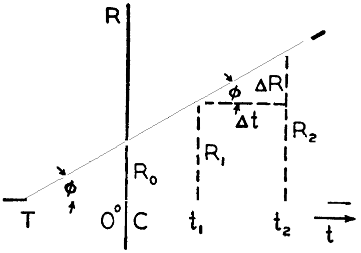


Fig. 4.18. Basic diagram for defining temperature coefficient of resistance.

It proves convenient to express the incremental increase of resistance as a decimal fraction of the resistance at an arbitrary reference temperature. This expedient would give us the per unit change of resistance brought about by a particular temperature change. The particular per unit value of a given ohmic value of resistance change depends upon the particular temperature employed as reference. For example, in the hypothetical case shown in Fig. 4.18, ΔR is 0.5 of R but only 0.333 of R_2. If we now arbitrarily restrict the value of ΔR to that occurring in a temperature change $\Delta T = 1°\text{C}$, we are ready to make the following definition: *The temperature coefficient of resistance, α_n, is, for a conductor of fixed dimensions, the per unit change of resistance per centigrade degree change of temperature, referred to an arbitrary point (t_n, R_n) on the temperature-resistance characteristic.* It is expressed in **ohms per ohm per centigrade degree.** In terms of the figure,

$$\alpha_1 = \frac{\Delta R}{\Delta t} \cdot \frac{1}{R_1}, \tag{4-17}$$

or

$$\frac{\Delta R}{\Delta t} = \alpha_1 R_1 = \alpha_n R_n. \tag{4-18}$$

We note also that

$$\tan \phi = \frac{R_1}{-T + t_1} = \frac{\Delta R}{\Delta t} = \alpha_1 R_1 = \alpha_n R_n. \tag{4-19}$$

Then

▶ $$\alpha_n = \frac{1}{t_n - T}, \tag{4-20}$$

where:

α_n = *temperature coefficient of resistance at temperature t_n*, in **ohms per ohm per degree centigrade,**

T = *intercept with t axis,* in **degrees centigrade,**

t_n = *reference temperature,* in **degrees centigrade.**

Example 4.18. What is the temperature coefficient of resistance of standard annealed copper at a temperature of 20°C?

$$\alpha_{20°\text{C}} = \frac{1}{20 - (-234.5)} = \frac{1}{254.5} = 0.00393 \text{ for standard copper.}$$

4.19. Temperature-Resistance Computations Employing α_n

By further reference to Fig. 4.18, we can write

$$R_2 = R_1 + \Delta R, \tag{4-21}$$

which, upon substitution of (4-18), becomes

$$R_2 = R_1 + \alpha_1 R_1 \Delta t, \tag{4-22}$$

or

▶ $$R_2 = R_1[1 + \alpha_1(t_2 - t_1)], \tag{4-23}$$

where:

R_2 = *resistance* (in any units) *at temperature t_2,*

R_1 = *resistance* (in same units) *at temperature t_1,*

α_1 = *temperature coefficient of resistance* (in appropriate units) *at temperature t_1.*

This is the equation normally employed in temperature-resistance computations, since tables of α for different materials at a standard temperature are readily available in handbooks (see Table A4, Appendix). It must be emphasized that the value of α must be consistent with the reference temperature. If the value of α given by the tables applies to a reference temperature different from any of the temperatures given in the problem, the computations must take this fact into account. The following example shows one method of computation.

Example 4.19. Solve Example 4.17 through the use of equation (4-23) and the answer to Example 4.18.

Inasmuch as we must use $\alpha_{20°C} = 0.00393$, the reference temperature which we use must be 20°C. Neither the initial temperature nor the final temperature of the problem is 20°C; therefore the solution must be performed in two temperature steps, with 20°C as the intermediate temperature.

$$
\begin{aligned}
R_{60°C} &= R_{20°C}[1 + \alpha_{20C}(60° - 20°)]. \\
5 &= R_{20°C}[1 + 0.00393(40)] = R_{20°C} \times 1.1572. \\
R_{20°C} &= \frac{5}{1.1572} = 4.33 \text{ ohms.} \\
R_{10°C} &= R_{20°C}[1 + \alpha_{20°C}(10° - 20°)] \\
&= 4.33[1 + 0.00393(-10)] \\
&= 4.15 \text{ ohms.}
\end{aligned}
$$

This answer, of course, checks the one obtained by the other method.

4.20. Temperature Coefficients of Alloys and Semiconductors

As a study of Table A4 in the Appendix will reveal, the temperature coefficients of resistance for most *pure* metals are all nearly the same, with values in the neighborhood of 0.004. Alloys with much lower coefficients—some reaching zero, and even negative values—have been developed.

Wherever instruments or laboratory standards having fixed resistance values are required, it is necessary to make windings of an alloy possessing a temperature coefficient such as to yield the desired characteristic. The last column of Table A4 in the Appendix gives the *t*-intercepts of linearly extended temperature-resistance curves. The more nearly constant the resistance, the larger this intercept value. Thus, "Ia1a," with an intercept of −200,000°C, is seen to possess a highly satisfactory temperature-resistance characteristic. Other physical properties, as well as such factors as cost and availability, would help decide whether this material is to be specified for a given application.

A great many instruments are wound with Manganin, an alloy of copper, nickel, iron, and manganese. With a *t*-intercept of about −33,000°C this material possesses a temperature coefficient so low as to be negligible. The circular-mil foot resistivity of this material varies from 250 to 450 ohms, depending upon the proportions of the different metals used.

Certain substances, such as carbon (which has already been discussed), porcelain, and glass, possess large negative temperature coefficients. Thus, the hot resistance of a carbon-filament lamp is only about one-half the cold resistance. Contrasted with this is the tungsten lamp filament, for which a positive temperature coefficient serves to produce a hot resistance nearly 16 times the cold resistance.

Table 491, "Resistivities at High and Low Temperatures," on page 417 of the *Smithsonian Physical Tables*, 8th edition, contains a brief discussion of the temperature variation of resistivity for the several classifications of conducting materials.

4.21. Resistivities of Insulators

The band theory of conduction shows that certain materials, when in pure form and in a state of extremely low atomic excitation, could serve as perfect insulators, or nonconductors of electricity. Such "perfect" insulators are largely hypothetical, however, since even the best of practical insulators will conduct electricity to some extent. This limitation is probably due in part to unavoidable impurities. Nevertheless, insulators are distinguished by resistivities which are very high relative to those of metals. Thus, where the resistivity of standard annealed copper is 1.724×10^{-6} ohm-centimeter, the resistivity of glass is about 5×10^{13} ohm-centimeters —approximately 30,000,000,000,000,000,000 times as much! (See Table A7 of the Appendix for the resistivities of other materials.)

So high are the values of resistivity of many insulating materials that often the leakage of current over the surface of insulators exceeds that through the material. To decrease such leakage of current over the surface, insulators frequently are designed to provide a maximum length of surface path. The familiar "petticoats" of high-voltage power insulators serve to increase this path as well as to minimize the effects of dust and of rain on the surface resistance. In most practical work, the leakage of current through or around insulators is so low as to be considered negligible, although in some special applications it must be taken into account.

Most insulating materials have very large negative temperature coefficients of resistance. At a red heat, most of them are fairly good conductors.

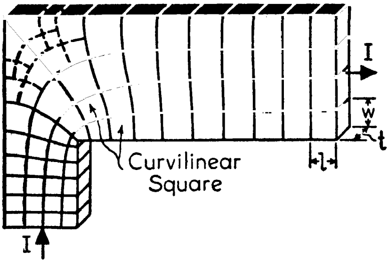
An interesting fact about most insulating materials is that their resistivities at a given temperature are not constant but vary widely with the "potential gradient," or potential per unit thickness. Ohm's law therefore is inapplicable. In general, the higher the potential gradient, the lower the resistivity.

4.22. D-C Resistance of Odd Shapes

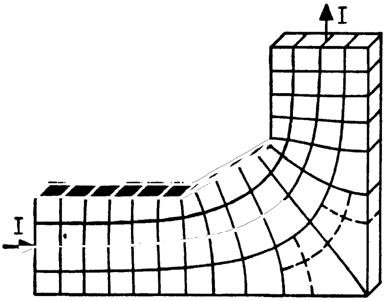
Occasionally, the d-c resistance of an odd-shaped piece of metal must be determined. Two general approaches are possible: the piece itself (or a scale model) may be tested, or graphical field mapping may be employed. The first method requires no further discussion. The second method is so useful in a variety of engineering problems that the reader is urged to consult one or more of the references listed in the footnotes[8,9,10] and to master the techniques presented therein. A brief discussion of the basic idea will be presented here.

The discussion which follows assumes the odd-shaped conductor to be of uniform thickness. In practice, if this is not the case, the method may still be applied in modified form.

Figure 4.22 depicts two odd-shaped pieces of metal of the same uniform thickness. The resistances of the two are equal, and both are equal to four times the resistance of one of the small *curvilinear squares* into which the pieces have been "mapped." Since the ratio of effective length to effective width for all curvilinear squares is equal to unity and since the thickness is everywhere the same, we can easily see from equation (4-8) that all these curvilinear squares must have the same resistance:



(a) Unsymmetrical 90° angle.



(b) Modified angle.

Fig. 4.22. Determination of resistance by free-hand graphical field mapping.

[8] Attwood, S. S., *Electric and Magnetic Fields*, 2d ed., Chapters 7 and 15. New York: John Wiley & Sons, Inc., 1941.

[9] Bewley, L. V., *Two-Dimensional Fields in Electrical Engineering*, Chapter 6. New York: The Macmillan Company, 1948.

[10] Boast, W. B., *Principles of Electric and Magnetic Fields*, Chapter 14. New York: Harper & Bros., 1948.

$$\blacktriangleright \quad R_{\text{square}} = \rho \frac{l}{A} = \rho \frac{l}{w \times t} = \rho \frac{l}{l \times t} = \frac{\rho}{t}, \tag{4-24}$$

where:

R_{square} = *d-c resistance of curvilinear square*, in **ohms,**
ρ = *resistivity of material*, in ohmic units,
l = *effective length of curvilinear square*, in any units,
$w = l$ = *effective width of curvilinear square*, in the same units, and
t = *thickness of conductor*, in the same units.

Letting m = length of the metal piece, in curvilinear squares, and n = width of the metal piece, in curvilinear squares, we may again utilize equation (4-8) to express the resistance of the entire metal piece:

$$\blacktriangleright \quad R_{\text{piece}} = \rho \frac{l'}{w't} = \rho \frac{ml}{nw \times t} = \rho \frac{ml}{nlt} = \frac{m}{n}\frac{\rho}{t} = \frac{m}{n} R_{\text{square}}, \tag{4-25}$$

where:

l' = *length of the metal piece*, in units employed in (4-24),
w' = *width of metal piece*, in same units,
m = *length of metal piece*, in **curvilinear squares,** and
n = *width of metal piece*, in **curvilinear squares.**

In both of cases shown in Fig. 4.22

$$R_{\text{piece}} = \frac{m}{n} R_{\text{square}} = \frac{16}{4} R_{\text{square}} = 4R_{\text{square}} = 4\frac{\rho}{t}.$$

Despite their greatly different shapes, the two odd-shaped metal pieces have the same resistance between opposite ends.

The rules for successful two-dimensional graphical field mapping include the following:

1. *Sketch smooth flow lines, dividing the current into equal parts. Note that the sides of the piece are flow lines.*

2. *Sketch smooth equipotential lines which divide the total potential drop into equal parts. This is done by keeping the equipotential lines everywhere orthogonal to flow lines, and by maintaining curvilinear squares. Note that the ends of the piece are equipotential lines. Curvilinear squares can best be drawn "by eye"; the ratio of effective length to effective width is unity for a curvilinear square. Successive subdividing of an odd-shaped "square" will soon show whether it is, in truth, a square, as indicated by the dotted lines of Fig. 4.22.*

The major problem in graphical field mapping is to analyze the flow problem correctly in the first place, so that the originally sketched flow lines are in approximately the correct location. With complicated shapes this is easy only after one has had considerable experience.

4.23. A-C, or Effective, Resistance

The foregoing series of articles has dealt with various detailed aspects of the broad topic of d-c resistance. As was pointed out in the early part of this chapter, at frequencies other than zero certain secondary effects which depend on frequency function to make the resistance appear to be higher than the d-c resistance value. In the case of high-frequency communications circuits these effects are of the utmost significance, and they cannot be ignored even at 60 cycles. It is the purpose of this article to explore some of these secondary effects so that the true nature of effective resistance may be understood.

Equation (4-3) defined effective resistance in terms of the rate at which electric energy is converted into heat (or some other form of radiant energy) with a given value of current through the conductor:

$$R = \frac{P}{I^2}. \tag{4-26a}$$

Davy's law provided us with a means of describing d-c resistance in terms of conductor geometry:

$$R = \rho\frac{l}{A}. \tag{4-26b}$$

We are now prepared to list six secondary effects which can function to make a-c, or effective, resistance greater than d-c, or "ohmic," resistance:

1. *Skin effect,* which causes alternating current to crowd toward the outside of conductors, reduces the effective cross-sectional area of the conductor and hence, by equation (4-26b), increases the effective resistance. At power frequencies, skin effect is appreciable only in conductors of large cross-sectional area (250,000 circular mils or greater).

2. *Proximity effect* in parallel cylindrical wires tends to make currents concentrate in the parts of the conductors which are

nearest to each other, thus reducing the effective cross-sectional area. Therefore, by equation (4-26b), effective resistance is increased. This effect is of significance only at high frequencies.

3. *Eddy currents* induced in any conducting material (including the conductor itself!) located within the varying magnetic field established by the current increase the time rate of electric energy dissipation as heat, for a given effective value of current. By equation (4-26a), the effective resistance is increased by this effect, which is important even at power frequencies.

4. *Hysteresis* in any magnetic materials located within the varying magnetic field increases the time rate of electric energy dissipation as heat, thereby causing an increase in resistance, according to equation (4-26a). This effect is important even at power frequencies.

5. *Dielectric loss* due to "dielectric hysteresis" in any capacitances associated with the circuit increases the time rate of energy dissipation as heat, thereby causing an increase in resistance, according to equation (4-26a). This effect is important largely at high frequencies.

6. *Electromagnetic radiation* from the circuit through antenna action increases the time rate of electric energy dissipation and, by equation (4-26a), causes an increase in resistance. This effect is negligible below about 15,000 cycles, and, in any case, depends on the physical dimensions of the circuit relative to the wave length of the electromagnetic wave.

All six of these secondary effects increase with frequency. General quantitative predetermination of these effects involves mathematical functions of types which are not generally within the experience of undergraduates and so will not be taken up here.

In some practical applications it is desirable to make one or more of the above six factors prominent, whereas under other conditions it is essential to minimize some or all of them. Industrial induction heating, for example, requires the conversion of as much electric energy as possible into thermal energy by means of eddy currents. Medical and industrial dielectric heating require a maximum conversion of electric energy into thermal energy by means of dielectric loss in the material being treated. A radiobroadcast transmitter is designed to utilize to maximum advantage the phenomenon of electromagnetic radiation.

Example 4.23. A certain coil has an air core. Laboratory tests yield the following data:

Test	*Applied Potential*	*Current*	*Power*
D-C	8 v	10 A	80 W
A-C	100 v	10 A	85 W

Determine both the d-c resistance and the a-c resistance of the coil.

$$R_{\text{d-c}} = \frac{P_{\text{d-c}}}{I^2_{\text{d-c}}} = \frac{80}{10^2} = 0.8 \text{ ohm},$$

or

$$R_{\text{d-c}} = \frac{V_{\text{d-c}}}{I_{\text{d-c}}} = \frac{8}{10} = 0.8 \text{ ohm}.$$

$$R_{\text{a-c}} = \frac{P_{\text{a-c}}}{I^2_{\text{a-c}}} = \frac{85}{10^2} = 0.85 \text{ ohm}.$$

4.24. Sinusoidal Waves at Sources and at Sinks

A slight digression is necessary at this point in order to develop a new concept which is basic to the analysis of alternating-current circuits by the methods of electrical engineering. This is the concept of *complex impedance*—that property of a circuit element which relates mathematically the amplitude and initial phase angle of a sinusoidal wave through the element to the amplitude and initial phase angle of the resulting sinusoidal potential drop across the element. First we must discuss the waves of current and potential drop.

In Chapter 3, the following general equation for a sinusoidal potential wave was developed:

$$e = E_m \sin(\omega t + \theta_e), \tag{4-27}$$

where:

e = *instantaneous value of potential at time* t, in **volts,** a positive value indicating a potential rise in the direction of tracing,
E_m = *amplitude of wave,* in **volts,**
$\omega = 2\pi f$ = *angular velocity of wave,* in **radians per second,**
t = *time,* in **seconds,** and
θ_e = *initial phase angle of wave,* in **radians.**

Inasmuch as we are dealing with an energy sink, we expect to encounter a potential drop in the positive current direction. Calling drops positive and recalling that $v = -e$, we may write the following equation for a sinusoidal potential drop:

$$v = V_m \sin(\omega t + \theta_v) = -e = -E_m \sin(\omega t + \theta_e), \tag{4-28a}$$

where V_m and E_m are equal in value and are the amplitudes, respectively, of a wave of potential drop and of the corresponding wave of potential rise. Figure 4.24 shows a plot of equations (4-27) and (4-28a).

A study of the figure reveals that equation (4-28a) could have been written

$$\blacktriangleright \quad v = V_m \sin(\omega t + \theta_v) = -E_m \sin(\omega t + \theta_e)$$
$$= E_m \sin(\omega t + \theta_e \pm 180°). \quad \text{(4-28b)}$$

It is clear that if the equation for a sinusoidal wave contains a negative amplitude, it may be rewritten with a positive amplitude of the same absolute value but with a phase angle which has been altered in size by π radians, or 180°. For example:

$$v = -100 \sin(377t + 30°)$$
$$= +100 \sin(377t - 150°).$$

The student may satisfy himself that this is true by plotting the two expressions. We conclude that, if V_m and E_m are both written as (equal) positive quantities, then

$$\theta_v = (\theta_e \pm 180°). \quad \text{(4-29)}$$

This is another way of saying that a wave of sinusoidal positive potential drop, v, is 180° out of phase with the wave of sinusoidal positive potential rise, e, of which it is the negative.

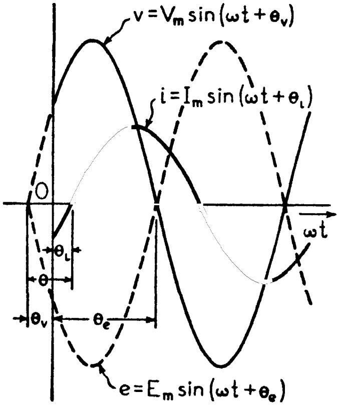


Fig. 4.24. Waves of current, potential rise, and potential drop at a sink.

In addition to the potential waves, Fig. 4.24 shows the wave of current which produces the potential drop of equations (4-28). This current can be described by the equation

$$\blacktriangleright \quad i = I_m \sin(\omega t + \theta_i). \quad \text{(4-30)}$$

As we consider the wave of potential rise, e, of equation (4-27) we are prone to wonder just what constitutes a "positive" value for e, assuming E_m to be positive. The answer lies in the value of the initial phase angle of e relative to that of the associated current, i. If the angular phase difference, $(\theta_e - \theta_i)$ is no greater than 90° in absolute value, then the wave e is considered to be "positive." That is to say, e is considered to be a potential rise in the positive current direction. The circuit element across which such a value of e appears must be serving as a source.

On the other hand, if the angular phase difference, $(\theta_e - \theta_i)$, has an absolute value equal to 90° or 180° or to any angle between 90° and 180°, e is considered to be a "*negative rise*," or a *potential drop*, in the positive current direction. The circuit element across which such a value of e appears is serving as a sink. Figure 4.24 portrays such a case.

Figure 4.24 also shows another way of representing the situation at the sink. Instead of using a wave of e (which will be negative), we may use a wave of v (which will be positive). By so denoting potential drops as positive, we shall find, by equation (4-29), that at a sink the angular phase difference between current and potential drop is no greater than 90° in absolute value. This result can be seen by reference to the figure.

4.25. Complex Impedance and Sinusoidal Potential and Current Waves

Relating mathematically a sinusoidal potential wave and a sinusoidal current wave of the same frequency requires the specification of two values: *the ratio of amplitudes* and *the angular phase difference*. This operation requires the use of a *complex number*, a mathematical quantity containing two component parts of different nature. The particular complex quantity which we are about to define is called *complex impedance.*

Complex impedance performs two functions:

1. *It expresses the ratio of* V_m *to* I_m.
2. *It expresses the phase angle by which* v *leads* i.

Several different schemes of notation are employed to represent in abbreviated form these two properties of complex impedance.[11] One such scheme appears like that used to represent vectors in polar coordinates, since it shows both a magnitude and an angle. Under this *polar form of complex notation,* complex impedance is

▶

$$\mathbf{Z} = \frac{V_m}{I_m} \underline{/(\theta_v - \theta_i)} = Z\underline{/\theta}, \tag{4-31}$$

---

[11] See Chapter 5, "Vectors and Complex Algebra."

where:

$\mathbf{Z}$ = *complex impedance*, in **complex ohms**,

$Z = \dfrac{V_m}{I_m}$ = *magnitude of impedance*, in **ohms**, and

$\theta = (\theta_v - \theta_i)$ = *phase angle of the impedance*, in **radians** (or **degrees**).

We note that, dimensionally, impedance is measured in ohms as a consequence of the definition of Ohm's law.

One of the first steps in general circuit analysis often is the evaluation of all complex impedances in the circuit.

EXAMPLE 4.25. A current wave $i = 3 \sin (377t + 30°)$ amperes produces a potential drop $v = 120 \sin (377t + 75°)$ volts through a particular circuit element. What is the complex impedance of this circuit element?

$$Z = \frac{V_m}{I_m} = \frac{120}{3} = 40 \text{ ohms},$$
$$\theta = (\theta_v - \theta_i) = (75° - 30°) = 45°,$$
$$\mathbf{Z} = Z\underline{/\theta} = 40\underline{/45°} \text{ complex ohms.}$$

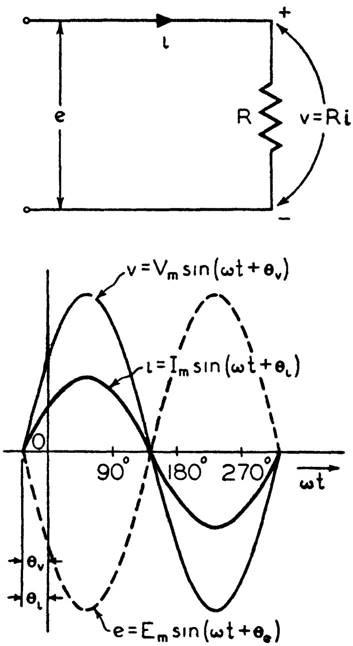


Fig. 4.26. Current and potential waves for pure resistance.

4.26. Complex Impedance for Pure Resistance

Inasmuch as all circuits inevitably possess to some degree all three circuit parameters R, L, and C, the case of a circuit that contains only one of these parameters in "pure" form is somewhat hypothetical. Nevertheless, within engineering accuracy it is often possible to treat certain circuits or circuit elements as though they did consist of only one parameter. The present discussion involving pure resistance is not so nearly hypothetical as might be supposed. It must be noted that the term "resistance" here means *effective* resistance, with all that the term implies.

Referring to Fig. 4.26 and to equations (4-28) and (4-30), we can write

$$\begin{aligned} v_R = (-e_R) &= V_{m_R} \sin(\omega t + \theta_v) \\ &= Ri = RI_m \sin(\omega t + \theta_i). \end{aligned} \tag{4-32}$$

Solving for i,

$$i = \frac{V_m}{R} \sin(\omega t + \theta_v) = I_m \sin(\omega t + \theta_i). \tag{4-33}$$

From this equation we may conclude that

$$I_m = \frac{V_m}{R} \quad \text{and} \quad \theta_v = \theta_i, \tag{4-34}$$

or

$$R = \frac{V_m}{I_m} \quad \text{and} \quad \theta_R = (\theta_v - \theta_i) = 0. \tag{4-35}$$

We may now write the expression for complex impedance for pure resistance as

$$\mathbf{Z}_R = Z_R\underline{/\theta_R} = R\underline{/0^\circ}, \tag{4-36}$$

where:

$\mathbf{Z}_R$ = *complex impedance for pure resistance,* in **complex ohms,**
Z_R = *pure resistance,* in **ohms,** and
θ_R = 0 = *angular phase difference for pure resistance.* (Current through resistance and resulting potential drop are in phase.)

Example 4.26. A current wave $i = 5 \sin(377t - 10^\circ)$ flows through a pure resistance of 100 ohms. What is the equation of the resulting potential drop across the resistance?

$$\begin{aligned} V_m &= I_m R = 5 \times 100 = 500 \text{ volts}, \\ \theta_v &= \theta_i = -10^\circ, \quad \text{and} \quad \omega t = 377t. \\ v_R &= -e_R = V_m \sin(\omega t + \theta_v) = 500 \sin(377t - 10^\circ) \text{ volts}. \end{aligned}$$

4.27. Power in a Purely Resistive A-C Circuit

Inasmuch as energy is a primary "commodity" of the electrical engineer,[12] the determination of time rate of energy translation or transformation, or power, associated with a given circuit is an important undertaking. In Chapter 3 we saw that instantaneous power associated with current between points a and b of an electric circuit is

$$p_{ab} = i_{ab} e_{ab}. \tag{4-37a}$$

---

[12] *Wave form* may be regarded as an additional "commodity" which has special importance in communications work.

Inasmuch as this relation was developed on the basis that potential rises are positive, a positive value for p indicates that energy is entering the circuit, whereas a negative value for p indicates that energy is leaving the circuit.

Since we are dealing here with an energy sink, let us call potential drops positive. Then energy leaving the circuit will be positive, and (4-37a) will become

$$p_{ab} = i_{ab}v_{ab}. \tag{4-37b}$$

In the case of pure resistance through which a sinusoidal current is flowing, we may write the following equation for instantaneous power:

$$\begin{aligned} p_R = iv_R &= I_m \sin(\omega t + \theta_i) V_m \sin(\omega t + \theta_v) \\ &= V_m I_m \sin(\omega t + \theta_v) \sin(\omega t + \theta_i), \end{aligned} \tag{4-38}$$

where potential drop, v, is positive, and power leaving the circuit is positive. Recalling that, for pure resistance, $\theta_v = \theta_i$, we may write

$$\begin{aligned} p_R &= V_m I_m \sin^2(\omega t + \theta_i) \\ &= \frac{V_m I_m}{2}[1 - \cos 2(\omega t + \theta_i)], \end{aligned}$$

or

▶ $$p_R = \frac{V_m I_m}{2} - \frac{V_m I_m}{2} \cos(2\omega t + 2\theta_i), \tag{4-39}$$

where: p_R = *instantaneous power at resistance*, in **watts**, and all other quantities are as previously defined.

Figure 4.27A shows the graphical interpretation of equation (4-39) for a particular case. The two components that make up the total wave are indicated by dashed lines. The first component, $V_m I_m/2$, does not vary with time and is positive in value. The second component is a sinusoidal wave of double frequency having an initial phase angle, *when measured on the double-frequency scale*, twice that of the current. The resultant wave, which is the sum of these two components, is a unidirectional wave of power with sinusoidal pulsations. *The fact that it is always positive indicates that the resistance is always acting to drain energy from the circuit.* The production of heat by the resistance is, by its very nature, an irreversible process.

The average value of purely resistive power is obtained by integration of equation (4-39). This operation yields

▶ $$P_R = \frac{V_m I_m}{2}, \tag{4-40}$$

where: P_R = *average power at pure resistance*, in **watts**.

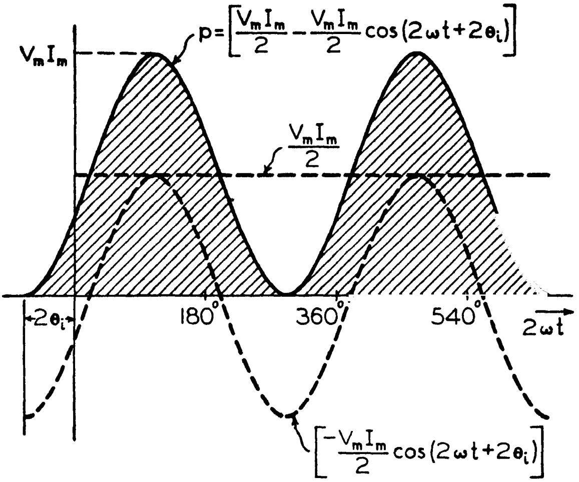


Fig. 4.27A. Instantaneous power and its mathematical components for pure resistance.

This is seen to be equal to the constant term of equation (4-39) and is, of course, positive. A wattmeter properly connected to indicate power handled by the resistance would indicate this value of average power.

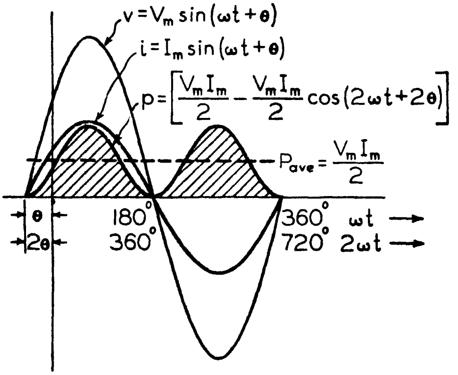


Fig. 4.27B. Instantaneous potential, current, and power for pure resistance.

The same power wave shown in Fig. 4.27A is shown again in Fig. 4.27B, together with the waves of potential drop and current. It is interesting to note, again, that energy is received by the resistance in pulses, one for each half cycle of current. Thus, in a 60-cycle circuit an incandescent lamp receives 120 energy pulses per second. At this frequency, persistence of vision of the human eye and thermal lag within the lamp render this pulsing unnoticeable. If the frequency were much lower, however, noticeable flicker would result.

4.28. Nonlinear Resistance and Wave Form

Earlier in this chapter it was shown that, in general, resistance is not constant but is a function of temperature. We have just seen that the production of heat by resistance occurs in the form of pulses which correspond to the alternating pulses of current. Although thermal lag would tend to modify the effect, it is easy to imagine that pulsations of temperature would tend to accompany the pulsations of current and energy. This effect, in turn, could be expected to cause pulsations in the values of a nonlinear resistance, which, in turn, would affect—in a pulsating manner—the current wave form. In addition to thermal lag, this effect depends on such factors as α, the temperature coefficient of resistance of the material. For some materials such effects are negligible under normal conditions, whereas for others they may be quite pronounced.

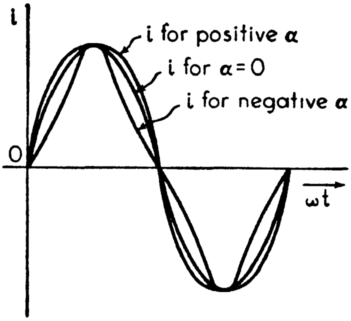


Fig. 4.28. Effect of α on current wave form with sinusoidal applied emf.

Figure 4.28 shows the effect of α on the wave form of current sent through a nonlinear resistance by a sinusoidal applied emf. If α is positive and large, current fails to increase in the same proportion as the wave of potential, and the current wave appears flattened. A negative value of α produces a current wave which is peaked, since current increases more rapidly than potential.

A variety of conducting and semiconducting materials whose nonlinearity (lack of straight-line relation between v and i) is due to thermal effects has been developed for special applications. Such materials are called *thermistors*.

Varistor is the term applied to a material exhibiting nonlinearity which is not due to thermal effects. An outstanding example of varistors is *Thyrite*, a silicon carbide compound of a ceramic nature whose resistance is a function of potential. The resistance decreases approximately 13 times as the potential is doubled. Consequently, the material is an excellent insulator at low potentials but an excellent conductor at high potentials. These properties suggest such applications as lightning arresters and field-discharge circuits. Another type of varistor is exemplified by the action of dry-type (copper oxide and selenium) rectifiers.

B. INDUCTANCE, L OR M

4.29. Definition and Nature of Inductance

"*Inductance is the (scalar) property of an electric circuit, or of two neighboring circuits, which determines the electromotive force induced in one of the circuits by a change of current in either of them.*"[18]

We saw in Chapter 3 that whenever the flux linking an electric circuit changes in value, an electromotive force is induced tending to oppose that change. Mathematically,

$$e = -N\frac{d\varphi}{dt}, \tag{4-41}$$

where the negative sign indicates dynamical opposition to the change of linkage. If this flux is established by electric current, two possibilities exist: the flux linking a given circuit may be due to current in that circuit, or the flux linking the circuit may be due to current in a neighboring circuit. *Self inductance* arises from the first situation; *mutual inductance*, from the second. Regardless of the source of the flux, an emf is induced in every circuit with which flux linkage is changing. In each such circuit, the induced emf possesses the proper polarity to tend to oppose the change of linkage.

In the circuit carrying the flux-establishing current, the induced emf has a polarity which is opposite to that of the current-producing emf of the source. With respect to the emf of the source, this induced emf is a counter-emf. That portion of the circuit in which the counter-emf is induced may be regarded as a sink, or receiver of energy, in contrast to the source. This viewpoint will be taken even though under some conditions the circuit operation may be such that the net energy taken from the circuit by this counter-emf may be zero. Tracing in the positive current direction requires that we regard this induced potential difference as a drop. Mathematically, this drop will require a sign opposite to that of the rise expressed by equation (4-41). This condition will be discussed further in the next article.

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[18] *American Standard Definitions of Electrical Terms*, No. 05.20.165. New York: American Institute of Electrical Engineers, 1942.

In a circuit which is neighboring to that carrying the flux-establishing current, the induced emf has a polarity which, again, tends to oppose the change of linkage. In this case, no effect is noticeable in the current-carrying circuit unless this neighboring circuit is closed, so that the emf induced in it may produce a current. Such an induced current in this neighboring circuit tends to establish "counterflux" which will neutralize the change in original flux linkage. The effect exerted by this induced current on the original current in the other circuit is discussed in Chapter 13, "Coupled Circuits."

Now that we have studied the operational mechanisms of self inductance and mutual inductance, the two forms of the circuit parameter, inductance, we shall treat them in more detail.

4.30. Self Inductance, L; the Henry

"*Self inductance is the property of an electric circuit which determines, for a given rate of change of current in the circuit, the electromotive force induced in the same circuit. Thus*

▶ $$e_1 = -L\frac{di_1}{dt}, \qquad (4\text{-}42)$$

where e_1 and i_1 *are in the same circuit and* L *is the coefficient of self inductance.*"[14] This equation may be used to define the unit of inductance, the *henry: If, by equation* (4-42), *one volt of emf is induced by current change at the rate of one ampere per second, the circuit is said to possess one henry of inductance.*

Equation (4-42) defines self inductance from the viewpoint of its function as a circuit parameter. In an energy sense, self inductance bears the same relation to the electric circuit that mass bears to the mechanical rectilineal system. Self inductance opposes a change in current, or rate of motion of electric charges; mass, the "degree of inertia" of a physical body, opposes a change in the rate of motion of that body. Both self inductance and mass can receive kinetic energy, and both can return that energy completely: both handle energy reversibly. Self inductance handles reversibly kinetic

[14] *Ibid.*, No. 05.20.170.

electric energy, or *magnetic energy;* mass handles reversibly kinetic mechanical energy.

It is enlightening to combine equations (4-41) and (4-42) to obtain a slightly different concept of self inductance from that gained from the formal definition. Setting these equations equal to each other, we have

$$e_1 = -N_1 \frac{d\varphi_1}{dt} = -L_1 \frac{di_1}{dt}, \tag{4-43}$$

from which it is seen that

▶ $$L_1 = N_1 \frac{d\varphi_1}{di_1} = \frac{d(N_1\varphi_1)}{di_1}, \tag{4-44a}$$

where:

L_1 = *coefficient of self inductance of circuit one,* in **henrys,**
N_1 = *number of turns of circuit* 1 *linked by its own flux,*
φ_1 = *flux established by current* i_1, in **webers,**
i_1 = *current of circuit* 1, in **amperes.**

This equation shows the self inductance of a given circuit to be the *change of flux linkage per unit change of current* of the circuit.

The value of flux linkage per unit current is a function of the geometrical configuration of the conductor and of the *magnetic permeability,* μ, of the surrounding medium. This fact makes it possible to compute the inductance of simple circuit configurations by methods developed in standard textbooks on magnetic fields.[15] Inductance formulas are available for the precise calculation of both self and mutual inductance in coils of many forms.[16,17]

In many practical cases—notably those in which ferromagnetic materials are absent—the magnetic permeability, μ, of the medium is effectively constant. In such cases, a linear relation exists between magnetic flux and establishing current; L_1 is constant, and equation (4-44a) can be written

▶ $$L_1 = \frac{N_1\Phi_1}{I_1}, \tag{4-44b}$$

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[15] Attwood, S. S., *Electric and Magnetic Fields.* New York: John Wiley & Sons, Inc., 1941.

[16] Terman, F. E., *Radio Engineers' Handbook,* page 47. New York: McGraw-Hill Book Company, Inc., 1943.

[17] Rosa, E. B., and Grover, F. W., "Formulas and Tables for the Calculation of Mutual and Self Inductance," *National Bureau of Standards Scientific Paper No.* 169, Washington, Government Printing Office, 1916.

where: L_1 = *coefficient of self inductance for constant permeability*, and the capital letters imply that Φ_1 and I_1 are linearly related.

Devices known as *inductors* are built for the primary purpose of introducing inductance into electric circuits.

The foregoing equations appear to have assumed that all of the flux, φ, links *every* turn of the N-turn coil. The study of magnetic fields shows that in a practical coil, all turns are not linked by the same value of flux. A detailed mathematical treatment of flux linkage, to be wholly correct, would have to sum up the values of flux linking with individual turns to obtain the total flux linkage. Using the symbol λ to represent this total flux linkage, and dividing it by the number of turns, N, we obtain the following expression for average flux per turn:

$$\blacktriangleright \qquad \varphi_{\text{ave}} = \frac{\lambda}{N} = \frac{\sum_{n=1}^{n=N} \varphi_n}{N}, \tag{4-45}$$

where:

φ_{ave} = *average flux per turn*, in **webers,**
N = *number of turns,*
λ = *total flux linkage*, in **weber-turns,**
φ_n = *flux through the nth turn*, in **webers.**

It is clear that the average flux per turn multiplied by the N turns yields the same total linkage, λ, as does the sum of the values of flux linking individual turns. In all of the flux-linkage equations of this book, the letter φ (or Φ), when symbolizing flux, symbolizes *average flux per turn.*

Example 4.30A. What is the self inductance of a coil in which an emf of 300 volts is induced when current is changing at the rate of 10×10^4 amperes per second?

From (4-42),

$$L = \frac{e}{di/dt} = \frac{300}{10 \times 10^4} = 3 \times 10^{-3} \text{ henry} = 3 \text{ millihenrys}.$$

Example 4.30B. Fluxmeter measurements on a coil having 100 concentrated turns reveal the coil to be linked by 1.5×10^{-3} weber when a current of 3 amperes is flowing. What is the self inductance of the coil?

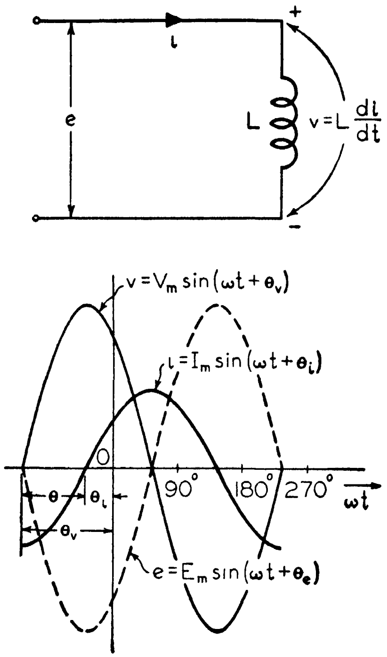
$$L = \frac{N\Phi}{I} = \frac{100 \times 1.5 \times 10^{-3}}{3} = 0.05 \text{ henry} = 50 \text{ millihenrys}.$$

4.31. Mutual Inductance, M

"*Mutual inductance is the common property of two associated electric circuits which determines, for a given rate of change of current in one of the circuits, the electromotive force induced in the other. Thus*

$$\blacktriangleright \qquad e_1 = M\frac{di_2}{dt} \quad \text{and} \quad e_2 = M\frac{di_1}{dt}, \qquad (4\text{-}46)$$

where e_1 *and* i_1 *are in circuit* 1, e_2 *and* i_2 *are in circuit* 2, *and* M *is the coefficient of mutual inductance.*"[18]



+
i
e
L
$v = L\frac{di}{dt}$
−
$v = V_m \sin(\omega t + \theta_v)$
$i = I_m \sin(\omega t + \theta_i)$
0
θ
θ_i
θ_v
90°
180°
270°
ωt
$e = E_m \sin(\omega t + \theta_e)$

Fig. 4.32. Current and potential waves for pure self inductance.

Comparison of equations (4-46) with equation (4-42) shows that mutual inductance is the same, dimensionally, as self inductance, and therefore is measureable in the same unit, the henry.

Mutual inductance is of major importance in engineering applications. However, to benefit fully from a discussion of this circuit property, we must first acquire an additional background of circuit concepts and techniques for handling circuit problems. Accordingly, all further discussion of mutual inductance is deferred until we reach Chapter 13, "Coupled Circuits."

4.32. Complex Impedance for Pure Self Inductance

We shall now investigate the result of sending a sinusoidal alternating current through a circuit element assumed to possess only the self-inductance form of the circuit parameter (see Fig. 4.32). The equation of the current is

$$i = I_m \sin(\omega t + \theta_i). \qquad (4\text{-}47)$$

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[18] *American Standard Definitions of Electrical Terms*, No. 05.20.175. New York: American Institute of Electrical Engineers, 1942.

Assuming the self inductance to be a true constant, we know that the resulting potential drop must be of the form

$$v = -e = V_m \sin(\omega t + \theta_v). \tag{4-48}$$

Upon substitution of the expression for self-induced emf expressed as a drop in the positive current direction, we have

$$v = V_m \sin(\omega t + \theta_v) = +L\frac{di}{dt}. \tag{4-49}$$

From this we may write

$$di = \frac{V_m}{L}\sin(\omega t + \theta_v)dt. \tag{4-50}$$

Integrating this expression yields

$$i = \frac{-V_m}{\omega L}\cos(\omega t + \theta_v) + K_1. \tag{4-51}$$

In Chapter 17, "Transients," we shall discuss the boundary conditions by means of which the constant of integration, K_1, may be evaluated. Except for a relatively short transient period extending over only a few cycles following the application of emf to the circuit, the value of K_1 is zero. In our present analysis we shall be concerned only with the steady-state condition in which K_1 is zero.

Since a cosine wave may be treated as a sine wave which has an initial phase angle of $\pi/2$ radians, a *negative* cosine wave may be treated as a sine wave which has an initial phase angle of $-\pi/2$ radians. Giving K_1 the value zero and re-expressing the cosine function as a sine function, we have

$$i = \frac{V_m}{\omega L}\sin\left(\omega t + \theta_v - \frac{\pi}{2}\right). \tag{4-52}$$

Comparing (4-52) with (4-47), we see that

$$I_m = \frac{V_m}{\omega L} \quad \text{and} \quad \theta_i = \left(\theta_v - \frac{\pi}{2}\right). \tag{4-53}$$

This equation tells us that in an a-c circuit energized by a sinusoidal emf, a pure self inductance of constant value plays a dual role:

1. *It determines the amplitude of the current wave relative to the amplitude of the wave of potential drop.*

2. *It causes the wave of current to lag the wave of potential drop by* $\pi/2$ *radians, or* 90°.

Following a procedure similar to that already followed with resistance, we may now write the following expression for complex impedance for pure self inductance:

$$\mathbf{Z}_L = \frac{V_m}{I_m}\underline{/(\theta_v - \theta_i)} = Z_L\underline{/\theta_L} = \omega L\underline{/\pi/2}. \tag{4-54}$$

Let us now denote $X_L = \omega L$, the *inductive reactance* of the circuit. Then the complex impedance is

$$\blacktriangleright \qquad \mathbf{Z}_L = Z_L\underline{/\theta_L} = X_L\underline{/\pi/2} = \omega L\underline{/90^\circ}, \tag{4-55}$$

where:

$\mathbf{Z}_L$ = *complex impedance for pure self inductance,* in **complex ohms,**
$Z_L = X_L = \omega L$ = *inductive reactance,* in **ohms,** and
$\theta_L = \pi/2$ radians = 90° = *angular phase difference for pure self inductance.* (Current through self inductance lags resulting potential drop by 90°.)

Example 4.32. A current wave $i = 5 \sin (337t - 10^\circ)$ flows through a pure inductance of 10 millihenrys. What is the equation of the resulting potential drop across the inductive element?

$$\begin{aligned} V_m &= I_m Z_L = I_m \omega L = 5 \times 377 \times 10 \times 10^{-3} = 18.85 \text{ volts,} \\ \theta_v &= (\theta_i + \theta_L) = (-10^\circ + 90^\circ) = 80^\circ. \\ v_L &= -e_L = V_m \sin (\omega t + \theta_v) = 18.85 \sin (377t + 80^\circ) \text{ volts.} \end{aligned}$$

4.33. Power in a Purely Inductive A-C Circuit

The following equation may be written for instantaneous power associated with a pure self inductance through which a sinusoidal current is flowing:

$$\begin{aligned} p_L = v_L i &= V_m \sin (\omega t + \theta_v) I_m \sin (\omega t + \theta_i) \\ &= V_m \sin (\omega t + \theta_v) I_m \sin (\omega t + \theta_v - 90^\circ) \\ &= -V_m I_m \sin (\omega t + \theta_v) \cos (\omega t + \theta_v) \\ &= \frac{-V_m I_m}{2} \sin 2(\omega t + \theta_v), \end{aligned} \tag{4-56}$$

or

$$\blacktriangleright \qquad p_L = \frac{-V_m I_m}{2} \sin (2\omega t + 2\theta_v), \tag{4-57}$$

where: p_L = *instantaneous power at self inductance,* in **watts,** and all other quantities are as previously defined.

The graphical interpretation of equation (4-57) appears in Fig. 4.33. We can see that here, too, instantaneous power varies sinusoidally with a double frequency. Most important is the further fact that this sinusoidal term oscillates about the zero axis, rather than about an axis displaced from zero, as it did for resistance. Integration of equation (4-57) verifies what we can see from the figure: that for pure self inductance, the average power is zero. Mathematically,

$$\blacktriangleright \qquad P_L = 0, \qquad (4\text{-}58)$$

where: P_L = *average power at pure self inductance,* in **watts.**

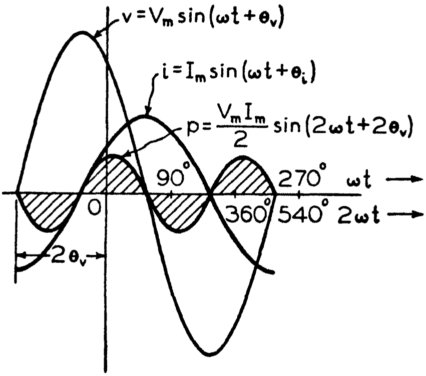

Fig. 4.33. Instantaneous potential, current, and power for pure self inductance.

This condition means that a wattmeter connected to measure average power taken by a pure self inductance in an a-c circuit would indicate zero, although an a-c ammeter would indicate current through the self inductance, and an a-c voltmeter would indicate potential difference across the self inductance!

We can understand this seeming anomaly of current and potential yet zero average power if we will stop to make a physical interpretation of Fig. 4.33. The fact that for every negative loop on the power wave there is an equal positive loop means that energy is alternately received from the circuit by the self inductance and then returned to the circuit. We note that, logically, the positive power loops, representing energy flow from the circuit into the magnetic field of the self inductance, coincide with the increase of absolute current value. The negative power loops, representing energy returning to the circuit from the magnetic field of the self inductance, coincide with the decrease of absolute current value. This result is in accord with the concept of magnetic energy as kinetic electric energy, associated only with moving charges.

The figure also enables us to verify the fact that the more rapid the change of current, the greater the value of induced emf. We can now appreciate better, perhaps, the implications of the statement that sinusoidal current through constant pure self inductance lags 90° behind the potential drop across that self inductance.

4.34. Determination of Energy Storage in Self Inductance

At a given instant, the value of magnetic energy stored in a given self inductance depends solely upon the value of current at that instant. This fact may be shown quite easily. Given the self inductance L, we may write

$$p_L = iv_L = i\left(+L\frac{di}{dt}\right) = Li\frac{di}{dt}. \tag{4-59}$$

This equation represents the rate of energy flow at the inductance. The positive sign shows it to be energy leaving the circuit and being stored in the magnetic field. In time dt the increment, dw_L, of energy thus stored is

$$dw_L = p_L\,dt = Li\,di. \tag{4-60}$$

As the current varies in value from zero to the value i, regardless of the manner of variation, the net energy storage is

▶ $$w_L = L\int_0^i i\,di = \frac{Li^2}{2}, \tag{4-61}$$

where:

w_L = *instantaneous net energy storage in self inductance,* in **joules,**

i = *instantaneous value of current,* in **amperes,** and

L = *constant self inductance,* in **henrys.**

EXAMPLE 4.34. What is the expression for maximum energy storage in a self inductance L placed in a circuit containing a current $i = I_m \sin(\omega t + \theta_i)$?

Obviously, maximum energy storage will occur at the current maximum. Then

$$W_{L\max} = \frac{LI_m^2}{2}.$$

This value corresponds to the total area under one positive loop of the power wave of Fig. 4.33.

4.35. Effective Inductance and Reactance

From equation (4-61), we may write

$$L = \frac{2w_L}{i^2}, \tag{4-62}$$

or, in terms of maximum values,

$$L = \frac{2W_{L\max}}{I_m^2}. \tag{4-63}$$

In Article 4.23 we observed a number of factors which could, for the same effective current, increase the rate of energy conversion to heat, thereby increasing the effective resistance of a circuit. Among these factors were magnetic hysteresis and eddy currents, both of which serve to drain off as heat some of the energy which otherwise would be stored in the magnetic field of self inductance of the circuit.

In an inductive circuit containing either or both conducting materials and ferromagnetic materials in the presence of the magnetic field, an increase of frequency, for the same value of current, brings about an increase of effective resistance at the expense of magnetic energy storage. From equation (4-63), we see that a reduction of magnetic energy storage for the same current results in an apparent decrease in self inductance. The resulting *effective inductance* tends to decrease with an increase of frequency. This decrease of inductance, in turn, gives rise, by equation (4-55), to an *effective reactance* which is smaller than the "true" reactance value that would correspond to a constant value of L. Mathematically, the effective reactance is related to effective inductance by the relation

$$X_{L\mathrm{eff}} = \omega L_{\mathrm{eff}}. \tag{4-64}$$

It is important to note that, in general, simple computations of circuit parameters based on alternating-current circuit tests are strictly valid only for the frequency at which the tests were made. Effective resistance and effective inductance may vary widely in value as frequency is varied, for example, from zero cycles per second (d-c) through power frequencies to radio frequencies.

C. CAPACITANCE, C

4.36. Definition and Nature of Capacitance; the Farad

"*The capacitance between two conductors is equal to the ratio of the charge placed on either conductor to the resulting change in potential difference, provided the conductors have received equal and opposite charges.*"[19] Mathematically,

▶

$$C = \frac{dq}{de}, \tag{4-65}$$

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[19] *Ibid.*, No. 05.15.060.

where:

C = *capacitance,* in **farads** (always a positive quantity),
q = *charge placed on each conductor,* in **coulombs,** and
e = *resulting potential difference between conductors,* in **volts.**

A device which is utilized for the primary purpose of introducing capacitance into an electric circuit is known as a *capacitor.* The property of capacitance, as the definition implies, is not confined to capacitors but is possessed by any pair of conductors between which an electric potential difference may appear. Thus, capacitance may exist between pairs of wires of a transmission line, between turns of a coil of wire (such as the winding of a transformer), between a thundercloud and the earth, or between electrodes of a radio tube. It represents both a useful tool and a nuisance to the electrical engineer, depending on circumstances.

Let us picture a simple capacitor consisting of two separated parallel metal plates immersed in a dielectric. ("*A dielectric is a medium having the property that the energy required to establish an electric field is recoverable, in whole or in part, as electric energy.*"[20] A good practical dielectric possesses also good insulating properties.) Let us imagine, now, that we apply a direct emf between the two plates of the capacitor. Under the influence of the emf, electrons will migrate through the path containing the emf from one plate of the capacitor to the other. Assuming each plate to have had a zero initial net charge, this migration will leave one plate with a deficiency of electrons and will provide the other with a surplus of electrons. The plate having the deficiency will possess a net positive charge, and the plate having the surplus will possess a net negative charge of equal amount. Although the *net* charge on the pair of plates is still zero, the capacitor is said to be "charged." What has changed as a result of the separation of positive and negative charges is not the net charge but the net energy storage of the capacitor.

The discussion of the atomic theory in Chapter 2 showed that an expenditure of energy is required to bring about a separation of electric charges of opposite polarity. This energy is regarded as being stored in the electric field that has been established between the charges in separating them. The energy put into the process of charging a capacitor is stored in the resultant electric field between the plates. For this reason, capacitance may be regarded as that property of an electric circuit which permits the storage of energy in the electrostatic form.

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[20] *Ibid.,* No. 05.30.035.

Capacitance bears to the electric circuit the same relation that mechanical compliance bears to the mechanical rectilineal system. The storage of energy in a capacitor is analogous to the storage of energy in a spring. Both capacitance and compliance handle energy reversibly. Capacitance handles reversibly "potential electric energy," or *electrostatic energy;* compliance handles reversibly potential mechanical energy.

According to Hooke's law, a spring which is subjected to a force that produces deformation provides a counterforce which is proportional to the deformation. Analogously, initially uncharged capacitance to which an emf is applied develops a counter-emf which is proportional to the charge transferred. This counter-emf is, of course, the emf, e, of equation (4-65). From a circuit viewpoint, the counter-emf of pure capacitance must always exactly balance the applied emf. Consequently, the capacitance may be regarded, in circuit analyses, as a sink, although it ultimately may return to the circuit as much energy as it receives from the circuit.

Structurally, capacitance depends upon the geometrical configuration of the system of conductors and dielectrics, and upon the electric permittivities of the dielectrics. Methods of computing capacitance from the geometry of the conductors are developed in standard textbooks on electric fields.[21] Capacitance formulas are available for the precise calculation of capacitance between pairs of conductors of standard forms.[22]

If the permittivity of the dielectric is constant, then, for a given conductor configuration, the capacitance is constant, and we can write

▶ $$C = \frac{Q}{E}, \qquad (4\text{-}66)$$

where:

C = *capacitance for constant permittivity*, in **farads,**

Q = *charge transferred* from one conductor to the other, in **coulombs,**

E = *resulting potential difference* between conductors, in **volts.**

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[21] Attwood, S. S., *Electric and Magnetic Fields.* New York: John Wiley & Sons, Inc., 1941.

[22] Terman, F. E., *Radio Engineers' Handbook,* page 112. New York: McGraw-Hill Book Company, Inc., 1943.

Equation (4-66) may be utilized to define the unit of capacitance, the *farad: If the transfer of one coulomb of electric charge from one to another of a pair of conductors results in a change of potential difference between the conductors of one volt, the capacitance between the conductors is one farad.* In engineering practice, typical capacitance values range between a few hundred microfarads and one micromicrofarad.

EXAMPLE 4.36. What charge resides on each plate of a 20-microfarad capacitor across which there is a potential difference of 100 volts?

$$Q = CE = 20 \times 10^{-6} \times 100 = 2 \times 10^{-3} \text{ coulomb.}$$

4.37. Displacement Current

Referring to Fig. 4.37, we see that during the process of charging a capacitor, an ammeter at either terminal would indicate the presence of electric current. A voltmeter connected between the terminals indicates a potential difference; an energy transformation is taking place between the terminals. In short, we conclude that electric current is present in the capacitor—even between the plates—and yet no charges are moving through the dielectric!

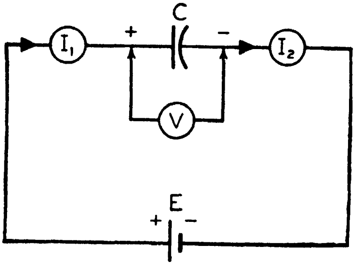


Fig. 4.37. Displacement current is the equivalent of conduction current from a circuit viewpoint.

This phenomenon of electric current flowing through capacitance without any moving electric charges to constitute the current is known as *displacement current.* Displacement current, we observe, is present only when the electric field within the capacitor is in the process of changing in value. It exists only when the capacitor is either charging or discharging.

This concept of displacement current is a most important link between the theory of electric circuits and the theory of electric fields, as we can see when we stop to consider the following facts. Displacement current, like conduction current, establishes about itself a linking magnetic field. As soon as the rate of change of the electric field (the value of the displacement current) changes, the linking magnetic field of self inductance changes, thereby developing an induced emf, which constitutes an electric field component. We have, then, shown that a changing electric field creates a magnetic field and that a changing magnetic field creates an electric field. This mutuality of behavior suggests the possibility that once an electromagnetic disturbance is started, it might be self-sustaining. This phenomenon, of course, underlies radio-wave propagation.

4.38. Complex Impedance for Pure Capacitance

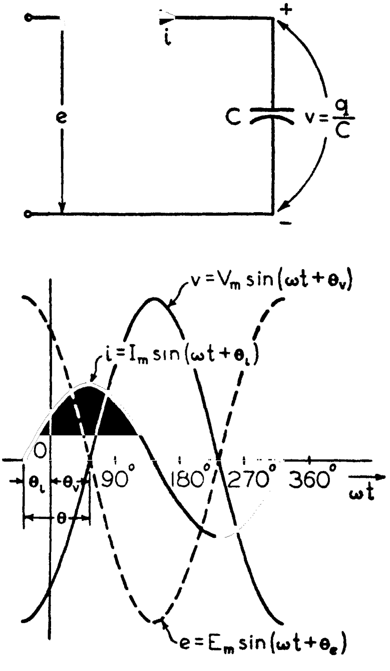

+
i
e
C
$v = \frac{q}{C}$
−
$v = V_m \sin(\omega t + \theta_v)$
$i = I_m \sin(\omega t + \theta_i)$
O
θ_i
θ_v
90°
180°
270°
360°
ωt
θ
$e = E_m \sin(\omega t + \theta_e)$

Fig. 4.38. Current and potential waves for pure capacitance.

We shall now investigate the result of sending a sinusoidal alternating current through a circuit element assumed to possess only the circuit parameter, capacitance (Fig. 4.38). The equation of the current is

$$i = I_m \sin(\omega t + \theta_i). \tag{4-67}$$

Assuming the capacitance to be a true constant, we know that the resulting potential drop must be of the form

$$v = -e = V_m \sin(\omega t + \theta_v). \tag{4-68}$$

Upon substitution of the expression for the emf developed by the capacitance, expressed as a drop in the positive current direction, we have

$$v = V_m \sin(\omega t + \theta_v) = \frac{q}{C}. \tag{4-69}$$

Differentiating with respect to time then gives

$$\frac{dq}{dt} = V_m \omega C \cos(\omega t + \theta_v). \tag{4-70}$$

The time rate of change of charge is, of course, current. Equation (4-70) may then be expressed as a sine function, as follows:

$$i = \frac{V_m}{\left(\frac{1}{\omega C}\right)} \sin\left(\omega t + \theta_v + \frac{\pi}{2}\right). \tag{4-71}$$

Comparing (4-71) with (4-67), we see that

$$I_m = \frac{V_m}{\left(\dfrac{1}{\omega C}\right)} \qquad \text{and} \qquad \theta_i = \left(\theta_v + \frac{\pi}{2}\right). \tag{4-72}$$

This equation tells us that in an a-c circuit energized by a sinusoidal emf, a pure capacitance of constant value plays a dual role:

1. *It determines the amplitude of the current wave relative to the amplitude of the wave of potential drop.*
2. *It causes the wave of current to lead the wave of potential drop by* $\pi/2$ *radians, or* 90°.

Following a procedure similar to that already followed with resistance and inductance, we may now write the following expression for complex impedance for pure capacitance:

$$\mathbf{Z}_C = \frac{V_m}{I_m}\underline{/(\theta_v - \theta_i)} = Z_C\underline{/\theta_C} = \frac{1}{\omega C}\underline{/-\pi/2}. \tag{4-73}$$

The next chapter, "Vectors and Complex Algebra," shows us that we may write, if we so desire,

$$\mathbf{Z}_C = \frac{1}{\omega C}\underline{/-\pi/2} = \frac{-1}{\omega C}\underline{/+\pi/2}. \tag{4-74}$$

Let us now denote $X_C = -1/\omega C$, the *capacitive reactance* of the circuit. Then the complex impedance is

▶ $$\mathbf{Z}_C = Z'_C\underline{/\theta'_C} = X_C\underline{/\pi/2} = \frac{-1}{\omega C}\underline{/90^\circ}, \tag{4-75}$$

where:

$\mathbf{Z}_C$ = *complex impedance for pure capacitance*, in **complex ohms,**

$Z'_C = X_C = \dfrac{-1}{\omega C}$ = *capacitive reactance*, in **ohms,** and

$\theta'_C = -\theta_C = \dfrac{\pi}{2}$ radians = 90° = *angular phase difference for pure capacitance.* (Current through capacitance leads resulting potential drop by 90°.)

Some texts define X_C without the negative sign but with a negative angle in the expression for complex impedance, as in the first part of equation (4-74). Certain advantages will accrue in later work if we define X_C as we already have, with the negative sign, so that values of X_C will always be negative.[23] The use of a positive angle in conjunction with the negative value for X_C preserves the correctness of the expression for complex impedance, as indicated by equation (4-74).

---

[23] In view of the fact that the present author, among others, writes $X_C =$

Example 4.38. The potential drop produced by a sinusoidal current flowing through a 25-microfarad capacitance is $v = 125 \sin (377t + 30°)$. What is the equation of the current wave?

$$\begin{aligned}
X_C &= \frac{-1}{\omega C} = \frac{-1}{377 \times 25 \times 10^{-6}} = -106 \text{ ohms} = Z'_C. \\
I_m &= \frac{V_m}{Z'_C} = \frac{125}{-106} = -1.18 \text{ amperes.} \\
\theta_i &= (\theta_v - \theta'_C) = 30° - 90° = -60°. \\
i_C &= I_m \sin (\omega t + \theta_i) = -1.18 \sin (377t - 60°) \text{ amperes,} \\
\text{or } i_C &= 1.18 \sin (377t - 60° + 180°) = 1.18 \sin (377t + 120°) \text{ amperes.}
\end{aligned}$$

The positive current wave is observed to lead the potential drop by 90°.

4.39. Power in a Purely Capacitive A-C Circuit

The following equation may be written for instantaneous power associated with a pure capacitance through which a sinusoidal current is flowing:

$$\begin{aligned}
p_C = iv_C &= I_m \sin (\omega t + \theta_i) V_m \sin (\omega t + \theta_v) \\
&= V_m I_m \sin (\omega t + \theta_v + 90°) \sin (\omega t + \theta_v) \\
&= V_m I_m \cos (\omega t + \theta_v) \sin (\omega t + \theta_v) \\
&= \frac{V_m I_m}{2} \sin 2(\omega t + \theta_v) \qquad (4\text{-}76)
\end{aligned}$$

or

▶ $$p_C = \frac{V_m I_m}{2} \sin (2\omega t + 2\theta_v), \qquad (4\text{-}77)$$

where: p_C = *instantaneous power at capacitance*, in **watts**, and all other quantities are as previously defined.

The graphical interpretation of equation (4-77) appears in Fig. 4.39. This figure should be compared with Fig. 4.33. As in the case of pure inductance, the instantaneous power associated with a pure capacitance varies sinusoidally with a double frequency. Here, too, the sinusoidal term oscillates about the zero axis, with the result that, for pure capacitance, the average power is zero:

▶ $$P_C = 0, \qquad (4\text{-}78)$$

where: P_C = *average power in pure capacitance*, in **watts.**

$-1/\omega C$ and still other authors write $X_C = 1/\omega C$, the reader is urged to check on this point the work of each author to whom he refers.

A major difference between the power waves of pure self inductance and pure capacitance is their initial phase angles relative to the same point on a current wave. We see that if a given current wave were to be sent through both a pure self inductance and a pure capacitance, the power wave of capacitance would be exactly half of a power cycle out of phase with the power wave of self inductance. This is a most useful fact, as we shall see later.

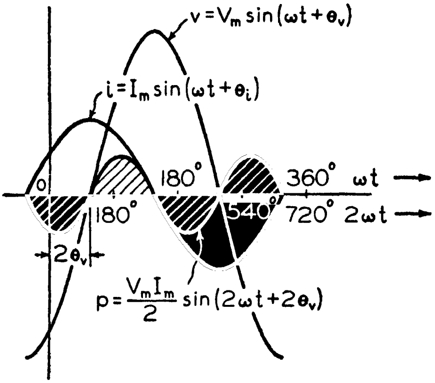


Fig. 4.39. Instantaneous potential, current, and power for pure capacitance.

A physical interpretation of Fig. 4.39 will prove instructive. Here is another case in which, for every negative power loop, there is an equal positive power loop. Thus, on the positive power loops the capacitance is draining energy from the circuit; and on negative power loops the capacitance is returning that energy to the circuit. We note that the positive power loops coincide with the increase of potential drop across the capacitance, whereas the negative power loops coincide with the decrease of potential drop across the capacitance. This distinction is logical, inasmuch as the storage of energy in capacitance is accompanied by the building up of potential across that capacitance by the transfer of charges from one conductor to the other.

The figure likewise enables us to see that the maximum value of current rushing into the capacitor occurs, quite logically, when the opposing potential difference across the capacitor is zero. This condition enables us to understand better the fact that sinusoidal current through constant pure capacitance leads by 90° the potential drop across that capacitance.

4.40. Determination of Energy Storage in Capacitance

At a given instant, the value of electrostatic energy stored in a given capacitance depends solely upon the value of potential difference across the capacitance at that instant. This fact may be shown quite easily. Given the capacitance C, we may write, from equation (4-65),

$$dq = C\,dv, \tag{4-79}$$

where the symbol v indicates the fact that we are dealing with a potential drop in the positive current direction during charging. Dividing through by dt, we obtain

$$i = \frac{dq}{dt} = C\frac{dv}{dt}. \tag{4-80}$$

We may now write

$$p_C = iv = Cv\frac{dv}{dt}. \tag{4-81}$$

This equation represents the rate of energy flow at the capacitance. The positive sign shows it to be energy leaving the circuit and being stored in the electrostatic field. In time dt, the increment, dw_C, of energy thus stored is

$$dw_C = p_C\,dt = Cv\,dv. \tag{4-82}$$

As the potential varies in value from zero to the value v, regardless of the manner of variation, the net energy storage is

▶ $$w_C = C\int_0^v v\,dv = \frac{Cv^2}{2}, \tag{4-83}$$

where:

w_C = *instantaneous net energy storage in capacitance*, in **joules**,
v = *instantaneous value of potential difference*, in **volts**, and
C = *constant capacitance*, in **farads**.

Example 4.40. A potential drop $v = 150 \sin (157t + 10°)$ volts appears across a capacitor having a capacitance of 100 microfarads. What is the instantaneous value of energy stored in the capacitor at $t = 0.2$ second?

$$\omega t = 157 \times 0.2 = 31.4 \text{ radians} = 31.4 \times \frac{180}{\pi} = 1800°.$$

$$v = 150 \sin (1800° + 10°) = 150 \sin 10°$$
$$= 150 \times 0.1737 = 26 \text{ volts}.$$

$$w_C = \frac{Cv^2}{2} = \frac{100 \times 10^{-6} \times (26)^2}{2} = 0.0336 \text{ joule}.$$

PROBLEMS

4-1. What is the rated resistance of a 230-volt, 60-watt incandescent lamp?

4-2. What is the resistance, at 20°C, of an annealed-copper strap having a length of 1 yard and a cross-sectional area of 2 × 0.75 centimeters?

4-3. If an annealed-copper wire 0.0641 inch in diameter is carrying 20 amperes, what will be the potential drop per mile at 20°C?

4-4. A bank of 20 lamps connected in parallel (across the same pair of terminals) is to be supplied by a two-wire supply line. Each lamp takes 0.5 ampere at 120 volts. The lamp bank is 500 feet distant from the power supply.

(a) What must be the diameter of the annealed-copper conductor used in the line wires if the line drop is not to exceed 5 volts?

(b) What gauge of wire is this? (Refer to Table A1 in the Appendix.)

4-5. Express the following areas in circular mils: (a) one square inch; (b) one circular yard; (c) ten square mils; (d) a rectangular area 1 foot long and 3 ten-thousandths of an inch wide; (e) a 1-centimeter-diameter circle.

4-6. What are the d-c conductivities in English units of the following materials at 20°C: (*a*) hard-drawn aluminum; (b) standard annealed copper; (c) tungsten?

4-7. What is the resistance, in circular-mil ohms per foot at 20°C, of 97 per cent conductivity copper?

4-8. A certain solid aluminum conductor 5150 feet long and 0.730 inch in diameter has a resistance of 0.168 ohm at 20°C. What is its per cent conductivity?

4-9. What diameter must a copper wire of 96 per cent conductivity have if it is to be electrically equivalent to the aluminum wire of Prob. 4-8?

4-10. What gauge of copper wire should be used to transmit electric power over a two-wire line a distance of 10 miles if the resistance is not to exceed 25 ohms and the temperature is assumed to be 20°C?

4-11. A stranded cable is to be made having the lay mentioned in Article 4.12. If its resistance per foot is not to exceed that of No. 14 (B. and S.) gauge copper wire at 20°C and if it is to contain 7 strands of equal cross section, what size of wire must be used?

4-12. If a certain cable is made of 19 strands of No. 4 (B. and S.) gauge copper wire, assuming the lay mentioned in Article 4.12,

(a) What is the cable resistance per 1000 feet at 20°C?

(b) What is the circular-mil area of the cable?

4-13. A No. 0 (B. and S.) gauge copper-clad steel wire of 30 per cent conductivity is to be made of 97 per cent conductivity annealed copper surrounding a steel core. If the area of the steel core is to be 79.5 per cent of the entire cross section, what must be the resistivity at 20°C of the steel used in the core?

4-14. A copper-clad steel wire having an outside diameter of 0.410

inch is made from copper having a resistivity at 20°C of 10.4 circular-mil ohms per foot. If the copper occupies 30 per cent of the total cross-sectional area, what is the per cent conductivity of the copper-clad wire?

4-15. A 125-volt d-c motor which draws 5 kilowatts under its condition of maximum loading is to be supplied by wire run through a metal raceway. What is the smallest permissible wire (a) if the wire is rubber-insulated; (b) if the wire is asbestos-varnished cambric-insulated?

4-16. What maximum power may be handled by a No. 0 (B. and S.) gauge two-wire d-c line at 500 volts in an enclosed space?

4-17. Derive the following expression:

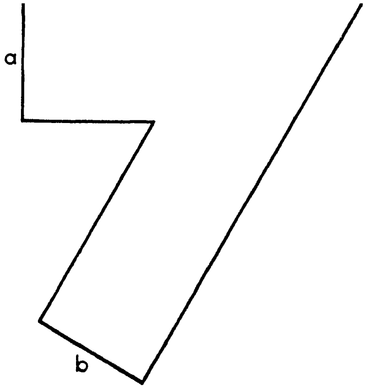
$$\alpha_2 = \frac{\alpha_1}{1 + \alpha_1(t_2 - t_1)}.$$

4-18. What will be the resistance of a copper wire at 5°C if the resistance at 50°C is 3 ohms?

4-19. The "cold" (room-temperature) resistance of a certain machine armature was 1.12 ohms and the "hot" resistance of the armature was 1.30 ohms. If the room temperature remained constant at 70°F, what was the temperature rise of the armature in degrees Fahrenheit?

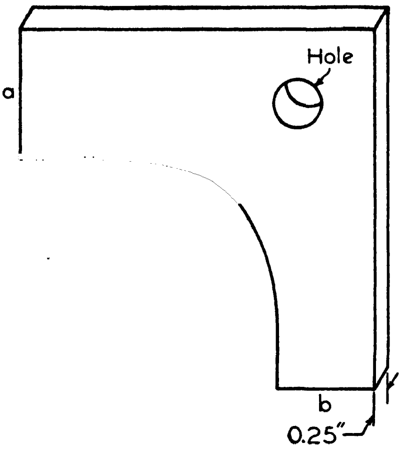
4-20. For certain types of machine insulation, it is specified that the temperature must not rise more than 65°C on full load. The cold resistance of a set of field coils whose insulation is of this type was found to be 128 ohms at 20°C. After a run of 3½ hours at rated load, the resistance was found to be 159 ohms. Did the machine meet the above specification?

4-21. Find the resistance between faces *a* and *b* of the copper conductor shown in the accompanying figure. The copper has a uniform thickness of 125 mils and a resistivity of 10.4 circular-mil ohms per foot.



Prob. 4.21.

4-22. Find the resistance between faces *a* and *b* of the aluminum conductor shown in the accompanying figure.



Prob. 4.22.

4-23. A certain coil has an air core. Laboratory measurements yield the following data:

Test	*Applied Potential*	*Current*	*Power*
D-C	8 *v*	5 A	40 W
A-C	98 *v*	1.1 A	2.18 W

(a) Determine the d-c resistance of the coil.

(b) Determine the a-c resistance of the coil.

(c) Account for any difference between the answers to (a) and (b).

(d) What would be the effect on resistance of inserting an insulated brass core in the coil? Explain.

(e) What would be the effect on resistance of substituting for the brass core an iron core of the same size and shape? Explain.

4-24. The following current flows through a resistive element of 20 ohms: $i = 10 \sin (377t + 45°)$ amperes.

(a) Find the complex impedance.

(b) Write the expression for v as a function of time.

(c) Write the expression for p as a function of time.

(d) Find the average power by integration of the answer to (c).

(e) Sketch the potential, current, and power waves to the same coordinate axes.

(f) What significance do you attach to the fact that the power wave never crosses the axis of time?

4-25. The wattmeter of a certain circuit indicates that an average power of 500 watts is being taken by a bank of incandescent lamps. The potential drop across the lamp bank is

$$v = 162.5 \cos 157t.$$

(a) Write the equation of the current taken by the lamp bank as a function of time.

(b) Determine the over-all resistance of the lamp bank.

(c) Sketch waves of current, potential, and power on the same coordinate axes.

4-26. What emf will be induced in a 5-millihenry coil in which the current is changing at a rate of 2000 amperes per second?

4-27. An emf of 3100 volts is induced at a certain instant in a 2000-turn coil. If the self inductance of the coil is known to be 350 millihenrys, what must be the rate of change of current?

4-28. What was the inductance of the coil of Prob. 3-10, if the initial current value was 100 amperes?

4-29. What value of flux is established by a current of 20 amperes flowing through a 250-turn coil of 60 millihenrys inductance?

4-30. A concentrated coil of 100 turns of fine wire has a self inductance of 0.010 henry. If the number of turns is increased to 300, what is the self inductance? (Watch for the "bear trap" in this one!)

4-31. An oscillographic study of a particular circuit element reveals that a current,

$$i = -5 \sin 377t \text{ amperes,}$$

produces a potential drop through the element equal to

$$v = -200 \cos 377t \text{ volts.}$$

(a) Make a sketch of the current and potential waves.

(b) Write the expression for the complex impedance.

(c) Determine the nature and magnitude of the circuit parameter.

4-32. The following potential drop is measured across a 90-millihenry inductance:

$$v = 170 \sin 314t \text{ volts.}$$

(a) Write the equation for current as a function of time, employing numerical coefficients.

(b) Write the expression for power as a function of time, employing numerical coefficients.

(c) Find the average power by integration of the answer to (b).

(d) Sketch the potential, current, and power waves to the same coordinate axes.

(e) In terms of energy flow, what is the significance of the positive loops of the power wave? Of the negative loops?

4-33. The following current flows through an inductance of 0.1 henry:

$$i = 5 \cos (157t + 30°) \text{ amperes.}$$

(a) Find the complex impedance.

(b) Write the expression for v as a function of time, employing numerical coefficients.

(c) Find the maximum instantaneous energy storage in the magnetic field associated with the inductance.

(d) Find the inductive energy storage at the instant $t = 0.1$ second.

4-34. Find the initial energy storage in the magnetic field of the coil of Prob. 3-10, if the initial current value was 100 amperes.

4-35. A long cylindrical coil of uniformly and compactly wound wire is termed a *solenoid.* The approximate formula for the average flux linkage with a solenoid whose length is much greater than its diameter is, in any rationalized system of units,

$$\lambda = (N\varphi)_{\text{ave}} = \frac{N^2IA}{s}\,\mu,$$

where:

λ = *average flux linkage,*
N = *number of turns,*
φ = *flux,*
I = *current* through the conductor,
μ = *magnetic permeability of medium containing flux,*
A = *cross-sectional area* perpendicular to axis,
s = *length* of solenoid.

If no ferromagnetic materials are present, what is the self inductance of an 1100-turn solenoid having a mean diameter of 3 inches and a length of 52 inches?

4-36. A two-wire transmission line can be considered to constitute a long, thin single-turn coil which is, of course, linked by flux established by its own current. The formula for the inductance of such a line in any rationalized system of units is:

$$L = \frac{s}{4\pi}\left(4\mu \ln \frac{D}{R} + \mu_{\text{wire}}\right),$$

where:

L = *self-inductance of line,*
s = *length of line,*
D = *distance between conductor centers,*
R = *radius of conductor,*
μ = *magnetic permeability of surrounding medium,*
μ_{wire} = *magnetic permeability of conductor material.*

What is the inductance per mile of a two-wire transmission line made up of two No. 0000 (B. and S.) gauge copper wires spaced at a distance of 9 feet? The permeability of copper may be assumed the same as that of air, which may be assumed the same as that of free space.

4-37. A given capacitor has a capacitance of 10 microfarads with transil oil as a dielectric. What would be its capacitance with each of the following as a dielectric: (a) air; (b) kraft paper; (c) distilled water?

4-38. The approximate formula for capacitance between two plane parallel plates which are separated by a distance that is small relative to the dimensions of the plate area is, in any rationalized system of units,

$$C = \frac{\epsilon A}{s},$$

where:

C = *capacitance between plane parallel plates,*
ϵ = *permittivity of dielectric between plates,*
A = *cross-sectional area of one plate,*
s = *distance between plates.*

One plate of a certain capacitor consists of 500 sheets of aluminum connected electrically in parallel to a common terminal. The other plate consists of 499 sheets of aluminum foil interleaved with the sheets of the first plate and connected electrically in parallel to the second terminal. Between each pair of adjacent sheets is a sheet of 3-mil paraffin paper having an effective area between sheets of aluminum foil of 144 square inches. If the dielectric constant of paraffin paper can be assumed the same as that of paraffin, what is the capacitance of this capacitor?

4-39. What will be the capacitance of the capacitor of Prob. 4-38 if mica is substituted for the paraffin paper?

4-40. The following equation gives approximately the potential difference between wires of a two-wire transmission line in any rationalized system of units:

$$E = \frac{\lambda}{\pi\epsilon} \ln \frac{D}{R},$$

where:

E = *potential difference between wires,*
λ = *charge on one wire per unit length of line,*
ϵ = *permittivity of medium between wires,*
D = *distance between wire centers,*
R = *radius of wire.*

What is the capacitance of a 150-mile two-wire transmission line consisting of two No. 0000 (B. and S.) gauge copper conductors with 9-foot spacing? Ignore the effect of the ground and towers.

4-41. The following current flows through a given circuit parameter:

$$i = -8 \sin (377t - 20°) \text{ amperes.}$$

The resulting potential drop is

$$v = 240 \cos (377t + 340°) \text{ volts.}$$

(a) Sketch the current and potential waves.

(b) Write the expression for the complex impedance of the element.

(c) Determine the nature and magnitude of the circuit parameter.

4-42. The following potential drop is measured across a capacitance of 300 microfarads:

$$v = 170 \sin 314t \text{ volts.}$$

(a) Write the equation for current as a function of time, employing numerical coefficients.

(b) Write the expression for power as a function of time, employing numerical coefficients.

(c) Find the average power by integration of the answer to (b).

(d) Sketch the potential, current, and power waves to the same coordinate axes.

(e) In terms of energy flow, what is the significance of the positive loops of the power wave? Of the negative loops?

4-43. The following current flows through a capacitive element of 100 microfarads: $i = 0.05 \sin (7540t - 30°)$ amperes.

(a) Find the complex impedance.

(b) Write the expression for v as a function of time, employing numerical coefficients.

(c) Find the maximum instantaneous energy storage in the electric field associated with the capacitance.

(d) Find the capacitive energy storage at the instant $t = 0.2$ second.

4-44. Find the energy storage in the capacitor of Prob. 4-37 for each of the three dielectrics, if the potential difference is, in all cases, 1000 volts.

4-45. What value of energy would be dissipated as heat by the resulting arc if the capacitor of Prob. 4-39 were to break down because of an overvoltage? Assume that the power source has been removed just previous to the breakdown.

4-46. An isolated three-mile section of the transmission line of Prob. 4-40 becomes charged between conductors to a potential of 500 volts through the effect of atmospheric electricity. What energy, in joules, would be discharged through the body of a lineman who inadvertently allowed himself to become part of a conducting path between the two conductors?

CHAPTER 5

VECTORS AND COMPLEX ALGEBRA

5.01. The Need for Another Mathematical Tool

Preceding chapters have developed relations for current, potential, and power associated with the individual pure circuit parameters—resistance, self inductance, and capacitance—under the influence of sinusoidal applied emf's. They have also shown that the general solution of circuits containing combinations of circuit parameters and emf's is based on the application of Kirchhoff's two laws, Ohm's law, and the power law. We realize that application of Kirchhoff's laws calls for the addition and subtraction of currents and of potentials. Application of Ohm's law requires multiplication and division of currents, potentials, and impedance quantities. Application of the power law involves multiplication and division of currents, potentials, and power quantities.

We have seen that, even for the simple cases of individual pure circuit parameters, the mathematical manipulation of currents and potentials can be cumbersome if these quantities must be expressed as sinusoidal functions of time. The more advanced problem of many parameters and emf's in complicated combinations would appear hopelessly involved if we were forced to rely solely on the mathematical methods which have been presented thus far.

To facilitate the mathematical handling of problems involving alternating currents and potentials, a special adaptation of an existing branch of mathematics is employed in electrical engineering. This chapter develops the basic ideas of that method, known as the *vector method* or the *method of complex quantities.* This method is one of the most important tools that an electrical engineer possesses. Real facility in its use should be developed.

5.02. Vector Representation of Current and Potential

The new mathematical tool which we are acquiring is based on a new scheme of representing sinusoidal current and potential waves.

If we are to obtain from a sinusoidal wave equation the instantaneous value of current or potential, we must be able to specify two types of information: the *amplitude* of the wave and the instantaneous value of *phase angle* of the wave. Thus, to compute the value, at any particular instant, of the current

$$i = I_m \sin(\omega t + \theta_i) = I_m \sin \alpha, \qquad (5\text{-}1)$$

we must be able to specify the value of I_m and the value of α. Any alternative method of representation, in addition to allowing for the sinusoidal variation of i, must also be able to specify the two properties of amplitude and phase angle.

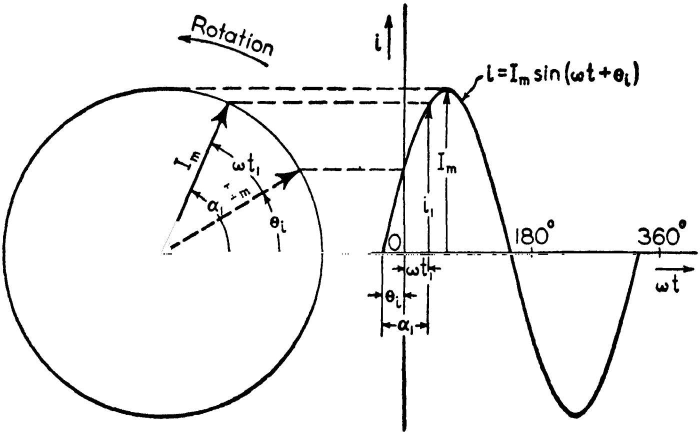


Fig. 5.02A. Mathematical generation of a sine wave by a rotating vector.

The vector method, or method of complex quantities, utilizes a vector[1] having the usual vector properties of length and direction, to represent, respectively, amplitude and phase angle, of a wave for which it will stand. Sinusoidal variation is achieved by requiring this vector to rotate about its origin at a uniform angular velocity in one plane. The angular position of this vector then varies as a linear function of time, just as the phase angle ($\alpha = \omega t$) of equation

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[1] Inasmuch as the time-varying currents, potentials, and fluxes associated with an alternating-current circuit are not vectors in the same sense as are the vectors of electric and magnetic fields, it has been proposed that either the term "phasor" or the term "sinor" be used in place of "vector" to describe the time-varying circuit quantities. This proposal is meeting with increasing favor.

(5-1) is a linear function of time. The vertical projection of such a vector will vary sinusoidally in value.

Figure 5.02A shows, for a particular case, the vector representation of the sine wave of equation (5-1). As the figure reveals, a sine wave may be considered as the development in rectangular coordinates of a uniformly rotating vector. Standard practice defines the counterclockwise direction of rotation as the positive direction, and all simple potential and current vectors are assumed to rotate in that direction with a uniform angular velocity. The sinusoidal wave of this figure is the curve obtained by plotting in rectangular coordinates the vertical projection of the vector as a function of angular displacement, α. One revolution of this vector corresponds to one cycle of the wave for which it stands.

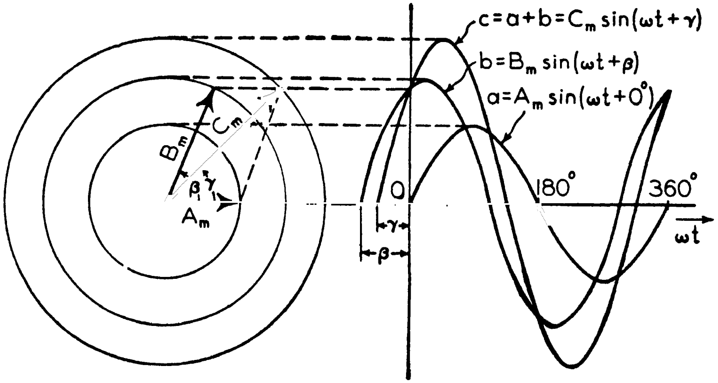

$c = a + b = C_m \sin(\omega t + \gamma)$
$b = B_m \sin(\omega t + \beta)$
$a = A_m \sin(\omega t + 0^\circ)$
C_m
B_m
A_m
β
γ
0
180°
360°
ωt
γ
β

Fig. 5.02B. Addition of two sine waves and of their generating vectors.

If two current (or two potential) waves of the same frequency are to be added by Kirchhoff's current (or potential) law, they may be represented by two vectors rotating at the same speed about the same origin in the same plane, as indicated by Fig. 5.02B. Inasmuch as these two vectors rotate at the same angular velocity, the angular phase difference between them remains constant. These two vectors may then be added vectorially, by finding the diagonal of the parallelogram, to give the resultant vector. This vector has a constant length and rotates at the same angular velocity as the other two, thereby maintaining the same relative phase position. This resultant vector, which is the sum of the original two vectors, generates a sine wave which is the sum of the original two sine waves. It is in this property that the vector method finds its first usefulness.

5.03. Vector Diagrams for Individual Pure Circuit Parameters

Instead of employing diagrams of waves of potential and current associated with a particular circuit or circuit element, we may employ *vector diagrams.* These vector diagrams can provide us with just as much information as can the wave diagrams, and, in addition, are simpler to draw.

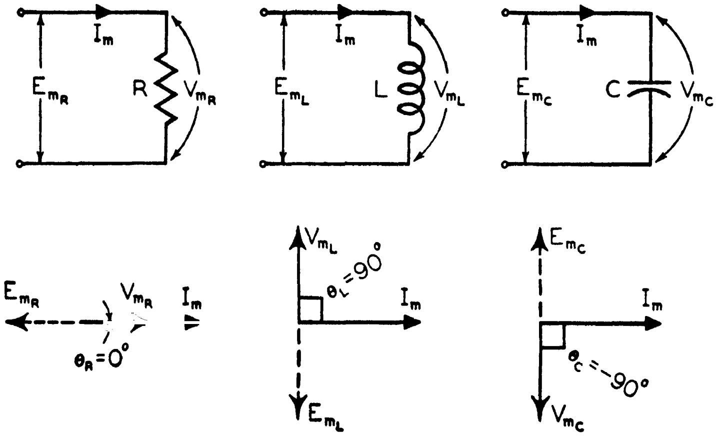


Fig. 5.03. Vector diagrams for individual circuit parameters.

Figure 5.03 shows the vector diagrams applying to the individual "pure" circuit parameters, *R*, *L*, and *C*. In each of these diagrams, current has arbitrarily been drawn horizontally to the right, so that it serves as reference. This position corresponds to a phase angle $\alpha = (\omega t + \theta_i) = 0$. The current could just as well have been drawn at any other angle with respect to the positive horizontal axis. The angular phase positions of the vectors of potential drop must always remain the same relative to the current vectors, whatever the positions of the current vectors, because these angular phase differences between current and potential drop are determined by the complex impedance of the circuit element in question. Thus, we see that current and potential drop are in phase for pure resistance, that current lags potential drop by 90° for pure inductance, and that current leads potential drop by 90° for pure capacitance.

5.04. Vectors as Complex Quantities

One of the simplest analytical methods for adding any two coplanar vectors calls for resolving both vectors into horizontal and vertical components. The algebraic sum of the horizontal components is, then, the horizontal component of the resultant, and the algebraic sum of the vertical components is the vertical component of the resultant. These statements immediately suggest that we may successfully handle our alternating-current circuit vectors by methods borrowed from the branch of mathematics known as *functions of a complex variable.* We shall develop for use here a portion of this branch of mathematics, which we might term the *algebra of complex quantities,* or *complex algebra.*

The word "complex" as used here does *not* mean "complicated"! As a matter of fact, complex algebra can greatly simplify our work. As used here, the word "complex" means "composed of two or more parts." This description fits our rotating vectors of current and potential, which are composed of two parts: magnitude and angle. There are several possible ways of incorporating these two parts into a single complex quantity. We shall learn all these forms of notation for complex quantities, inasmuch as each has a particular field of special usefulness.

Because the rotating vectors of a-c electric circuits are treated mathematically as complex quantities, we often hear all of the complex quantities of electrical engineering erroneously referred to as "vector quantities." Only sinusoidally varying quantities, such as potential and current, may be represented by rotating vectors, and the term "vector" is properly applied only to such quantities. The only exceptions which we shall make to this rule are those established by long usage.

In order to distinguish between complex quantities, including vectors, and algebraic quantities, or scalars, we may adopt any one of the three standard notations given by equations (5-2). Each of these equations makes the statement, "The complex quantity, **A,** equals the sum of the complex quantity, **B,** and the complex quantity, **C,**" or "The vector **A** is the resultant of vector **B** plus vector **C.**"

Such addition is known as *complex addition*, or *vector addition.* The quantities have been added *vectorially.*

▶ $$\bar{A} = \bar{B} + \bar{C}. \tag{5-2a}$$

▶ $$\dot{A} = \dot{B} + \dot{C}. \tag{5-2b}$$

▶ $$\mathbf{A} = \mathbf{B} + \mathbf{C}. \tag{5-2c}$$

Occasionally, the line or the dot appears beneath the letter to which it is applied. Although either of the first two schemes is more readily adaptable to pencil or chalk work, the third scheme—that of employing **boldface type** to denote complex quantities, including vectors—is recommended for print and will be adhered to in this text.

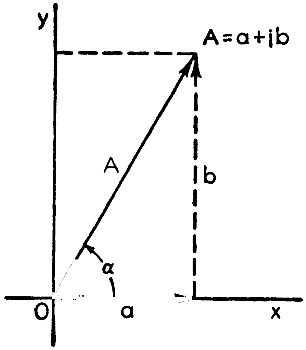


Fig. 5.05A. Resolution of a vector into its horizontal and vertical components.

5.05. Rectangular Complex Notation

Any vector lying within the x-y-plane may be specified in both magnitude and direction in terms of its x-axis projection and its y-axis projection. Thus, the vector **A** in Fig. 5.05A is completely specified in terms of the components a and b. We may easily obtain from these components both the magnitude and the angle of the vector, which are inherent in them. Thus, the magnitude of **A** is

▶ $$A = |\mathbf{A}| = \sqrt{a^2 + b^2}, \tag{5-3a}$$

and the angle of **A** is

▶ $$\alpha = \tan^{-1} \frac{b}{a}. \tag{5-3b}$$

If the vector itself is to be expressed as the sum of two rectangular components, we need some means of distinguishing the vertical component from the horizontal component. This distinction is obtained by placing the symbol j in front of the vertical components. Those components of vectors which lack the j are understood to be horizontal, and those which have the j are understood to be vertical. Thus, the vector, **A**, of Fig. 5.05A may be completely described using *rectangular complex notation* as follows:

▶ $$\mathbf{A} = a + jb. \tag{5-4}$$

The algebraic signs of the rectangular components determine which quadrant a vector or complex quantity occupies, as a study of the following example should make clear.

EXAMPLE 5.05. Write the rectangular complex expressions for vectors **A, B, C,** and **D,** which lie, respectively, in the first, second, third, and fourth quadrants, and have horizontal components of 4 units and vertical components of 3 units. Determine also the magnitude and phase angle of each vector.

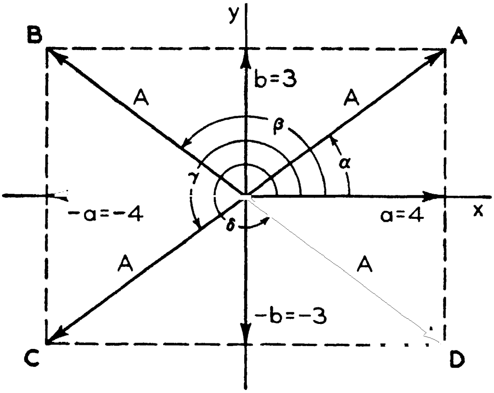


Fig. 5.05B. Rectangular components of vectors in the four quadrants.

The answers are readily obtained by inspection of the vector diagram of Fig. 5.05B, constructed in accordance with the specifications of the problem.

ANSWERS

First quadrant:

$$\mathbf{A} = 4 + j3;\ A = \sqrt{4^2 + 3^2} = 5;\ \alpha = \tan^{-1}\left(\frac{3}{4}\right) = 36.9°.$$

Second quadrant:

$$\mathbf{B} = -4 + j3;\ B = \sqrt{(-4)^2 + 3^2} = 5;\ \beta = \tan^{-1}\left(\frac{3}{-4}\right) = 143.1°.$$

Third quadrant:

$$\mathbf{C} = -4 - j3;\ C = \sqrt{(-4)^2 + (-3)^2} = 5;\ \gamma = \tan^{-1}\left(\frac{-3}{-4}\right) = 216.9°.$$

Fourth quadrant:

$$\mathbf{D} = 4 - j3;\ D = \sqrt{4^2 + (-3)^2} = 5;\ \delta = \tan^{-1}\left(\frac{-3}{4}\right) = 323.1°.$$

5.06. Operational Aspect of j

The preceding article regarded the symbol *j* as merely a "tag" to distinguish vertical vector components from horizontal vector components. Mathematically, *j* may be regarded as having certain unique properties which justify its being referred to as a type of *rotational operator.*

Except for its unique function, *j* may be regarded as differing but little in nature from other common mathematical operators, such as plus and minus signs, multiplication and division signs, differential and integral signs, and exponent and radical signs. Each of these operators is a form of shorthand notation indicating that the quantity to which it is attached is to be subjected to a particular form of mathematical manipulation. The operator *j* indicates that the vector to which it is attached has been rotated through 90° in a counterclockwise direction.

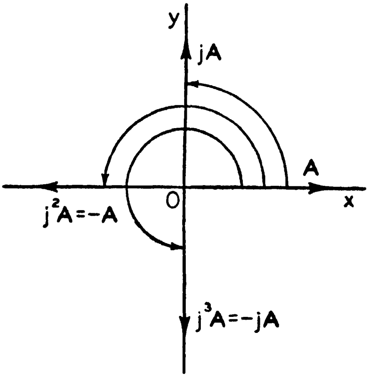


Fig. 5.06. Effect of successively applying j to the vector A.

Figure 5.06 indicates the effects produced by successive applications of the operator *j* upon the vector, **A**, originally directed positively along the horizontal axis.

The operational significance of *j* may be appreciated by first considering that it operates on the horizontal vector, **A**, to rotate it 90° in the counterclockwise direction, thereby causing it to fall along the positive vertical axis. In this position, **A** may be considered a new vector which, in turn, may be operated on by *j* to produce additional 90° rotation in the counterclockwise direction. The result is

$$j \times j\mathbf{A} = j^2\mathbf{A} = -\mathbf{A}. \tag{5-5}$$

From this equation, we conclude that

$$j^2 = -1, \tag{5-6a}$$

and that

▶

$$j = \pm\sqrt{-1}. \tag{5-6b}$$

If we utilize only the positive root, we see that j is identical with the imaginary number, i, of mathematics. In electrical engineering, the symbol j is used instead of i to avoid possible confusion with the symbol for instantaneous current. Because of the mathematical implications of equation (5-6b), the vertical components of vectors are often called the *j-components*, or the *imaginary components*. Correspondingly, the horizontal components are often called the *real components*.

The results of operation by j may be interpreted equally well in a number of ways. Referring to the figure again, we find that the application of j to the vector $j^2\mathbf{A}$ leads to a third vector lying along the negative vertical axis:

$$j^3\mathbf{A} = j^2 \times j\mathbf{A} = -j\mathbf{A}. \tag{5-7}$$

The negative sign may, with equal correctness, be interpreted variously as indicating that this new vector is the negative of the vector $j\mathbf{A}$, that it is the result of 90° counterclockwise (positive) rotation of the vector $-\mathbf{A}$, or that it is the result of 90° clockwise (negative) rotation of the vector **A**. This last interpretation leads to the conclusion that if $+j$ operates to produce 90° positive rotation, $-j$ must operate to produce 90° negative rotation.

EXAMPLE 5.06. Given the vector $\mathbf{A} = 4 + j3$, show that $\mathbf{B} = j\mathbf{A}$ is a vector of the same length but of 90° greater angle.

$$\mathbf{A} = 4 + j3;\ A = \sqrt{4^2 + 3^2} = 5;\ \alpha = \tan^{-1}\left(\frac{3}{4}\right) = 36.9°.$$

$$\mathbf{B} = j\mathbf{A} = j(4 + j3) = j4 + j^2 3 = -3 + j4;$$

$$B = \sqrt{(-3)^2 + (4)^2} = 5;\ \beta = \tan^{-1}\left(\frac{4}{-3}\right) = 126.9°.$$

$$(\beta - \alpha) = (126.9° - 36.9°) = 90°.$$

This computation demonstrates that **B** has a 90° greater angle than **A**.

5.07. Rectangular Form of General Rotational Operator: $(\cos \phi \pm j \sin \phi)$

Reference to Fig. 5.07 reveals that the real and imaginary components of any vector such as **A** may be expressed, respectively, as

$$a = A \cos \alpha, \qquad \text{and} \qquad b = A \sin \alpha. \tag{5-8}$$

The rectangular complex expression for the vector may then be written

$$\blacktriangleright \quad \mathbf{A} = a + jb = A\cos\alpha + jA\sin\alpha = A(\cos\alpha + j\sin\alpha). \qquad (5\text{-}9)$$

From an operational viewpoint, we may interpret the factor $(\cos\alpha + j\sin\alpha)$ as being a rotational operator which has rotated an originally horizontal vector of length A through an angle α.

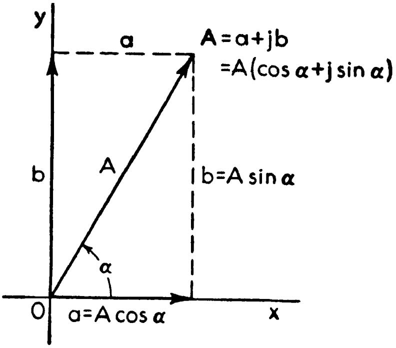


Fig. 5.07. Trigonometric evaluation of rectangular vector components.

It is not difficult to see that if α is positive, counterclockwise (or positive) rotation will result. If α is negative, clockwise (or negative) rotation will result. We note, further, that since $\cos\alpha = \cos(-\alpha)$ and $(-\sin\alpha) = \sin(-\alpha)$, we may write

$$(\cos\alpha - j\sin\alpha) = [\cos(-\alpha) + j\sin(-\alpha)]. \qquad (5\text{-}10)$$

We are now prepared to state that

$$\blacktriangleright \quad (\cos\phi \pm j\sin\phi) = \begin{Bmatrix}\text{rectangular form of} \\ \text{general rotational operator}\end{Bmatrix}, \qquad (5\text{-}11)$$

where: ϕ = *absolute value of the angular rotation produced, the positive sign producing positive rotation and the negative sign producing negative rotation.* Applied to any complex quantity, this operator alters the angle by the amount $\pm\phi$.

We may verify the statement regarding the operational properties of the factor $(\cos\phi \pm j\sin\phi)$ by applying this factor to a general vector (or complex quantity) such as

$$\mathbf{A} = a + jb = A(\cos\alpha + j\sin\alpha). \qquad (5\text{-}12)$$

Applying the general rotational operator in rectangular form, we have

$$\begin{aligned}\mathbf{B} &= \mathbf{A}(\cos\phi \pm j\sin\phi) = A(\cos\alpha + j\sin\alpha)(\cos\phi \pm j\sin\phi)\\ &= A(\cos\alpha\cos\phi \pm j\cos\alpha\sin\phi + j\sin\alpha\cos\phi \pm j^2\sin\alpha\sin\phi)\\ &= A(\cos\alpha\cos\phi \mp \sin\alpha\sin\phi) + j(\sin\alpha\cos\phi \pm \cos\alpha\sin\phi)\\ &= A[\cos(\alpha \pm \phi) + j\sin(\alpha \pm \phi)]. \qquad (5\text{-}13)\end{aligned}$$

This is a vector (or complex quantity) of magnitude A but of angle $(\alpha \pm \phi)$, and it represents the original vector (or complex quantity) changed in angle by an amount $(\pm\phi)$.

EXAMPLE 5.07. Use the general rotational operator in rectangular form to rotate the vector $\mathbf{A} = 6 - j7$ through a negative angle of 30°.

$$\begin{aligned}\mathbf{A}' &= \mathbf{A}(\cos 30^\circ - j \sin 30^\circ) = (6 - j7)(0.866 - j0.50)\\ &= 5.196 - j3.00 - j6.062 + j^2 3.50\\ &= 1.696 - j9.062.\end{aligned}$$

The reader is urged to construct these two vectors on coordinate paper to satisfy himself concerning the angular relation between them.

5.08. Exponential Form of General Rotational Operator: $\epsilon^{\pm j\phi}$

A form of general rotational operator which is more convenient in many cases than the rectangular form is the *exponential form* of general rotational operator:

▶
$$\epsilon^{\pm j\phi} \equiv (\cos\phi \pm j \sin\phi), \tag{5-14}$$

where ϵ is the natural logarithmic base.

Without any attempt at complete mathematical rigor, we may demonstrate the validity of (5-14), which is known as *Euler's equation*, by following the time-honored mathematical procedure of expanding both members of the equation into infinite series which can be shown to be identical. Knowing that the functions of (5-14) are continuous and possess derivatives of all orders, we may employ *Maclaurin's theorem*, which states

$$f(\phi) = f(0) + \frac{f'(0)\phi}{1!} + \frac{f''(0)\phi^2}{2!} + \frac{f'''(0)\phi^3}{3!} + \cdots, \tag{5-15}$$

where:

$f(\phi)$ = *specific function of* ϕ *to be expanded,*
$f(0)$ = *value of function when* ϕ *is zero,*
$f'(0)$ = *value of first derivative when* ϕ *is zero,*
$f''(0)$ = *value of second derivative when* ϕ *is zero,*
and so on.

The terms of equation (5-14) become, after expansion by this theorem:

$$\cos\phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \cdots, \tag{5-16}$$

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \cdots, \tag{5-17}$$

$$\epsilon^{\pm j\phi} = 1 + (\pm j\phi) + \frac{(\pm j\phi)^2}{2!} + \frac{(\pm j\phi)^3}{3!} + \frac{(\pm j\phi)^4}{4!} + \cdots \quad (5\text{-}18)$$

Since, by equation (5-6a), $j^2 = -1$, we may write

$$\epsilon^{\pm j\phi} = \left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \cdots\right) \pm j\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \cdots\right). \quad (5\text{-}19)$$

Comparison of equation (5-19) with (5-16) and (5-17) shows that

▶ $$\epsilon^{\pm j\phi} = (\cos\phi \pm j\sin\phi), \quad (5\text{-}20)$$

which verifies equation (5-14).

A vector (or complex quantity) may be described with this exponential form of notation as follows:

▶ $$\mathbf{A} = a + jb = A(\cos\alpha + j\sin\alpha) = A\epsilon^{j\alpha}. \quad (5\text{-}21)$$

5.09. Polar Form of Notation

Equation (5-21) showed two different ways of describing the same vector or complex quantity. To these we shall add a third, the *polar form* of notation, as exemplified by the right-handmost member of the following equation:

▶ $$\mathbf{A} = a + jb = A(\cos\alpha + j\sin\alpha) = A\epsilon^{j\alpha} = A\underline{/\alpha}. \quad (5\text{-}22)$$

A comparison of the exponential and polar forms will reveal that the polar form of notation is merely a shorthand way of recording the essential facts of *magnitude* and *angle* which are stated plainly—but somewhat more clumsily—by the exponential form.

EXAMPLE 5.09. Write in all three complex forms the complex expression for the vector representing the current $i = 5\sin(\omega t + 30°)$ at the instant $t = 0$.

Polar form:

$$\mathbf{I}_m = I_m\underline{/\theta_i} = 5\underline{/30°} \text{ complex amperes.}$$

Exponential form:

$$\mathbf{I}_m = I_m\,\epsilon^{j\theta_i} = 5\epsilon^{j30°} \text{ complex amperes.}$$

Rectangular form:

$$\begin{aligned}\mathbf{I}_m &= I_m(\cos\theta_i + j\sin\theta_i) = 5(\cos 30° + j\sin 30°)\\ &= 5(0.866 + j0.500) = 4.33 + j2.5 \text{ complex amperes.}\end{aligned}$$

5.10. Vector Position Represented as a Function of Time

Equation (5-22) shows a number of ways of describing a vector of fixed length occupying a *fixed* angular position. The assignment of fixed angles to the rotating vectors that represent sinusoidally varying quantities corresponds to taking a snapshot of that quantity: the vector is "stopped" at a particular instant when it is occupying instantaneously a particular angular position. For most steady-state a-c circuit analysis, this "snapshot-of-circuit-conditions-at-a-particular-instant" technique is satisfactory. Under transient conditions, and in certain special types of steady-state analysis, however, it is not adequate. These types of analysis demand a means of picturing mathematically the position of the rotating vector as a function of angular velocity and time. Such representation is a logical and simple outgrowth of the discussion of the previous article.

In equation (5-22), if we substitute for the fixed angle, α, a time-varying angle, ωt, we have the following representation of a vector whose angular position is a function of time:

▶ $$\mathbf{A} = A\epsilon^{\pm j\omega t} = A\underline{/\pm\omega t} = A(\cos\omega t \pm j\sin\omega t). \qquad (5\text{-}23)$$

If the angular velocity is known, substitution of a particular value of t in equation (5-23) yields the expression for the vector at the position fixed by t. So long as t is not specified, however, the vector continues to rotate at an angular velocity, ω.

Example 5.10. (a) Write as a function of time the expression for the rotating vector which corresponds to the following quantity:

$$i = 10 \sin (157t + 30°) \text{ amperes.}$$

(b) What is the equation for the vector at the instant $t = 0.1$ second?

Solution

(a) The desired vector is:

$$\begin{aligned}\mathbf{I}_m &= 10\epsilon^{j(157t+30°)} = 10\underline{/(157t + 30°)}\\ &= 10[\cos(157t + 30°) + j\sin(157t + 30°)] \text{ complex amperes.}\end{aligned}$$

(b) When $t = 0.1$ second: $\omega t = 157 \times 0.1 = 15.7$ radians $= 900°$.

$$(\omega t + 30°) = 930° = (3 \times 360° - 150°).$$

$$\mathbf{I}_{m_{t=0.1}} = 10\epsilon^{-j150°} = 10\underline{/-150°} = -8.66 - j5.00 \text{ complex amperes.}$$

5.11. Addition of Vectors and Complex Quantities

The two ends of a vector may be termed the *tail* (or *initial point*) and the *head* (or *terminal point*). The sum of any two vectors **A** and **B** may be obtained by moving **B** parallel to itself until its tail is coincident with the head of **A**, as shown in Fig. 5.11. A third vector, **C**, drawn from the tail of **A** to the head of **B**, is the *vector sum* of **A** and **B**.

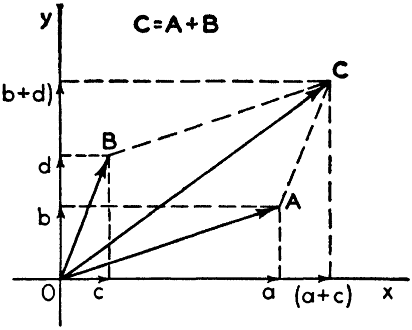


Fig. 5.11. Addition of two vectors.

Since the opposite sides of a parallelogram are equal and parallel, it is clear from the figure that the sum is independent of the order of addition and is equal to the vector diagonal:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} = \mathbf{C}. \qquad (5\text{-}24)$$

It is also apparent from the figure that the horizontal component of **C** is equal to the algebraic sum of the horizontal components of **A** and **B**. Likewise, the vertical component of **C** is equal to the algebraic sum of the vertical components of **A** and **B**.

If these vectors are expressed in the rectangular complex form, the resultant may be obtained in that form by simple addition of complex quantities. For example, let

$$\mathbf{A} = a + jb \quad \text{and} \quad \mathbf{B} = c + jd. \qquad (5\text{-}25)$$

Then

▶ $\mathbf{C} = \mathbf{A} + \mathbf{B} = (a + c) + j(b + d).$ (5-26)

This result may be expressed in polar form as follows:

▶ $\mathbf{C} = \mathbf{A} + \mathbf{B} = \sqrt{(a + c)^2 + (b + d)^2}\,\underline{/\tan^{-1}\left(\frac{b + d}{a + c}\right)}.$ (5-27)

Example 5.11. Find the sum of the following vectors and express the result in polar form: $\mathbf{A} = 10 + j6$; $\mathbf{B} = -7 + j2$; $\mathbf{C} = 6 - j4$.

$$\begin{aligned}\mathbf{A} + \mathbf{B} + \mathbf{C} &= (10 - 7 + 6) + j(6 + 2 - 4) = 9 + j4 \\ &= \sqrt{9^2 + 4^2}\,\underline{/\tan^{-1}\left(\tfrac{4}{9}\right)} = 9.84\underline{/24^\circ}.\end{aligned}$$

It is recommended that the reader verify this result by drawing the vectors to scale and adding them graphically.

5.12. Negative of a Vector or Complex Quantity

A study of Fig. 5.12 will verify the following mathematical statements:

▶ $$-\mathbf{A} = -(a + jb) = -a - jb = (-a) + j(-b), \qquad (5\text{-}28)$$

and

▶ $$-\mathbf{A} = -(A\underline{/\alpha}) = (-A)\underline{/\alpha} = A\underline{/(\alpha \pm 180^\circ)}. \qquad (5\text{-}29)$$

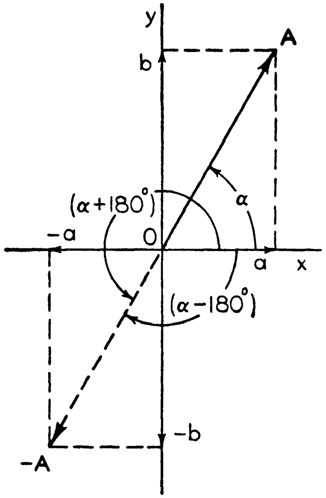


Fig. 5.12. Negative of a vector quantity.

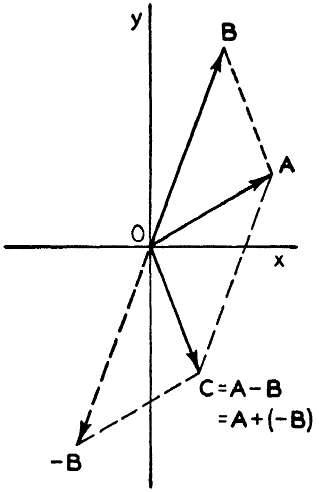


Fig. 5.13. Subtraction of two vectors.

5.13. Subtraction of Vectors and Complex Quantities

The process of subtraction in complex algebra may be accomplished by changing the sign of the quantity to be subtracted and adding. This operation is equivalent to 180° rotation $(-1 = j^2)$ and addition. Symbolically:

▶ $$\begin{aligned}\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) &= (a + jb) + (-c - jd) \\ &= (a - c) + j(b - d). \qquad (5\text{-}30)\end{aligned}$$

This equation is illustrated by Fig. 5.13.

Because of the necessity for adding the horizontal components and the vertical components separately, vectors must be expressed in the *rectangular* form before being *added* or *subtracted* by the methods of complex algebra.

Example 5.13. Given $\mathbf{A} = 6 - j3$; $\mathbf{B} = -2 - j4$; $\mathbf{C} = 7 + j2$, find $\mathbf{D} = \mathbf{A} - \mathbf{B} + \mathbf{C}$.

$$\mathbf{D} = \mathbf{A} + (-\mathbf{B}) + \mathbf{C} = (6 + 2 + 7) + j(-3 + 4 + 2) = 15 + j3.$$

It is recommended that the reader verify this result graphically.

5.14. Multiplication of Complex Quantities

The product of two complex quantities is a third complex quantity which may be obtained by following the rules of simple algebra for finding the product of two binomials. Multiplication of complex numbers may be carried out in any one of the three forms of complex notation, as is indicated by what follows.

Rectangular form:

▶
$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} = (a + jb)(c + jd) &= (ac + jad + jbc + j^2bd)\\ &= (ac - bd) + j(ad + bc).\end{aligned}\tag{5-31}$$

Exponential form:

▶
$$\mathbf{A}\cdot\mathbf{B} = A\epsilon^{j\alpha}\cdot B\epsilon^{j\beta} = A\cdot B\epsilon^{j(\alpha+\beta)}.\tag{5-32}$$

Polar form:

▶
$$\mathbf{A}\cdot\mathbf{B} = (A\underline{/\alpha})(B\underline{/\beta}) = AB\underline{/(\alpha + \beta)}.\tag{5-33}$$

Obviously, the justification for multiplying the magnitudes and adding the angles in the polar form lies in the algebra of the exponential form.

Although multiplication may be carried out in any of the complex forms, it usually is most easily carried out in the exponential or polar forms.

Example 5.14. Given $\mathbf{A} = 2 + j0$; $\mathbf{B} = 3 - j4$; $\mathbf{C} = -2 - j3$, find the product $\mathbf{D} = \mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}$, using both the rectangular and polar forms.

Rectangular form:

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C} &= (2 + j0)(3 - j4)(-2 - j3) = (6 - j8)(-2 - j3)\\ &= (-12 - j18 + j16 + j^2 24) = (-36 - j2).\end{aligned}$$

Polar form:

$$\mathbf{A} = 2\underline{/0^\circ}; \qquad \mathbf{B} = 5\underline{/-53.1^\circ}; \qquad \mathbf{C} = 3.6\underline{/-123.7^\circ}.$$

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C} &= (2\underline{/0^\circ})(5\underline{/-53.1^\circ})(3.6\underline{/-123.7^\circ})\\ &= 36/-176.8^\circ = (-36 - j2).\end{aligned}$$

5.15. Reciprocal of a Vector or Complex Quantity

The reciprocal of a vector or complex quantity may be found by the rules of algebra applying to binomials. If the quantity is in the rectangular form, it is necessary to eliminate the symbol j from the expression for its reciprocal. This process, called *rationalizing*, may be accomplished by multiplying and dividing the reciprocal by the *conjugate* of the original quantity. The conjugate of a complex quantity in the rectangular form is like the original quantity with the sign of the imaginary term reversed. Thus, if we have the complex quantity $\mathbf{A} = a + jb$ the conjugate of $\mathbf{A}$ is $\hat{\mathbf{A}} = a - jb$. In the polar form, the conjugate of a vector has the same magnitude as the original vector, but an angle of equal size but opposite sign. Thus, the conjugate of $\mathbf{A} = A\underline{/\alpha}$ is $\hat{\mathbf{A}} = A\underline{/-\alpha}$.

The following equations show how the reciprocal of the complex quantity $\mathbf{A} = a + jb = A\epsilon^{j\alpha} = A\underline{/\alpha}$ is found in each of the three forms.

Rectangular form:

▶
$$\frac{1}{\mathbf{A}} = \frac{1}{(a + jb)} = \frac{1}{(a + jb)} \cdot \frac{(a - jb)}{(a - jb)} = \frac{a - jb}{a^2 + b^2} = \frac{a - jb}{A^2} = \frac{\hat{\mathbf{A}}}{A^2}. \tag{5-34}$$

Exponential form:

▶
$$\frac{1}{\mathbf{A}} = \frac{1}{A\epsilon^{j\alpha}} = \left(\frac{1}{A}\right)\epsilon^{-j\alpha}. \tag{5-35}$$

Polar form:

▶
$$\frac{1}{\mathbf{A}} = \frac{1}{A\underline{/\alpha}} = \left(\frac{1}{A}\right)\underline{/-\alpha}. \tag{5-36}$$

It will be noted that the rectangular form shows the reciprocal of a complex quantity to be expressible as the conjugate of the quantity divided by the square of the magnitude. The exponential form shows the reciprocal of a complex quantity to be expressible as a new quantity having a magnitude equal to the reciprocal of the original magnitude and an angle equal to the negative of the original angle. The justification for inverting the magnitude and reversing the sign of the angle when finding the reciprocal in the polar form lies in the algebra of the exponential form.

5.16. Division of Vectors and Complex Quantities

The quotient of one complex quantity divided by another may be found by taking the reciprocal of the divisor and multiplying this by the dividend. The following equations show, in the three forms, how the complex quantity $A = a + jb = A\epsilon^{j\alpha} = A\underline{/\alpha}$ may be divided by the complex quantity $B = c + jd = B\epsilon^{j\beta} = B\underline{/\beta}$.

Rectangular form:

▶
$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{a + jb}{c + jd} = \frac{a + jb}{c + jd} \cdot \frac{c - jd}{c - jd} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}$$
$$= \left(\frac{ac + bd}{B^2}\right) + j\left(\frac{bc - ad}{B^2}\right). \qquad (5\text{-}37)$$

Exponential form:

▶
$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{A\epsilon^{j\alpha}}{B\epsilon^{j\beta}} = \frac{A}{B}\epsilon^{j(\alpha-\beta)}. \qquad (5\text{-}38)$$

Polar form:

▶
$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{A\underline{/\alpha}}{B\underline{/\beta}} = \frac{A}{B}\underline{/(\alpha - \beta)}. \qquad (5\text{-}39)$$

Although division may be carried out in any one of the complex forms, it usually is most easily carried out in the exponential or polar forms.

Example 5.16. Given the complex quantities of Example 5.14, find the quotient **B** ÷ **C**, using both the rectangular and polar forms.

Rectangular form:

$$\frac{\mathbf{B}}{\mathbf{C}} = \frac{3 - j4}{-2 - j3} = \frac{(3 - j4)}{(-2 - j3)} \cdot \frac{(-2 + j3)}{(-2 + j3)} = \frac{-6 + j9 + j8 - j^212}{4 + 9}$$
$$= \frac{6 + j17}{13} = 0.461 + j1.310.$$

Polar form:

$$\frac{\mathbf{B}}{\mathbf{C}} = \frac{5\underline{/-53.1^\circ}}{3.6\underline{/-123.7^\circ}} = 1.39\underline{/70.6^\circ} = 0.461 + j1.310.$$

5.17. Powers of a Complex Quantity

A complex quantity may be raised to a given power by the appropriate number of successive multiplications by itself. A simple rule for writing the equation for the nth power of a complex quantity may easily be developed. From what has been presented in the previous articles, we may immediately write this rule in the *exponential form:*

$$\blacktriangleright \qquad \mathbf{A}^n = (A\epsilon^{j\alpha})^n = A^n\epsilon^{jn\alpha}. \tag{5-40}$$

In the *polar form,* this same rule is

$$\blacktriangleright \qquad \mathbf{A}^n = (A\underline{/\alpha})^n = A^n\underline{/n\alpha}. \tag{5-41}$$

By substitution of the rectangular complex operator in (5-40), we obtain this rule in the *rectangular form:*

$$\blacktriangleright \qquad \mathbf{A}^n = A^n\,(\cos n\alpha + j \sin n\alpha). \tag{5-42}$$

5.18. Roots of a Complex Quantity

Extracting the roots of a complex quantity is the inverse of the process of raising that quantity to a given power. Since a root may be considered a fractional power, roots may be extracted by the same formulas employed to find powers. We must realize that there are n nth roots. Only one of these roots would be found by inserting a fractional power in place of the n of equations (5-40) through (5-42). The $(n - 1)$ remaining roots must also be found. This process is not difficult.

We realize that adding any integral multiple of 2π radians (or 360°) to an angle does not affect the values of the trigonometric functions of that angle. Thus,

$$\sin\alpha = \sin(\alpha + 360^\circ) = \sin(\alpha + 2 \times 360^\circ),$$

and so on. If we let n = the *order of the root,* and q = *any integer,* such as $0, 1, 2, 3, \cdots (n - 1)$, we may rewrite equation (5-42) in such form as to constitute a formula for finding the nth root of a complex number in the *rectangular form:*

$$\blacktriangleright \quad \sqrt[n]{\mathbf{A}} = \mathbf{A}^{\frac{1}{n}} = A^{\frac{1}{n}}\left[\cos\left(\frac{\alpha + 2\pi q}{n}\right) + j \sin\left(\frac{\alpha + 2\pi q}{n}\right)\right], \tag{5-43}$$

where: $q = 0, 1, 2, \cdots (n - 1)$.
In the *exponential form* this becomes

$$\blacktriangleright \qquad \sqrt[n]{\mathbf{A}} = \mathbf{A}^{\frac{1}{n}} = A^{\frac{1}{n}}\epsilon^{j\left(\frac{\alpha+2\pi q}{n}\right)}, \tag{5-44}$$

where: $q = 0, 1, 2, \cdots (n - 1)$.

In the *polar form* it is

▶ $$\sqrt[n]{\mathbf{A}} = \mathbf{A}^{\frac{1}{n}} = A^{\frac{1}{n}}\underline{\Big/\left(\frac{\alpha + 2\pi q}{n}\right)}, \tag{5-45}$$

where: $q = 0, 1, 2, \cdots (n - 1)$.

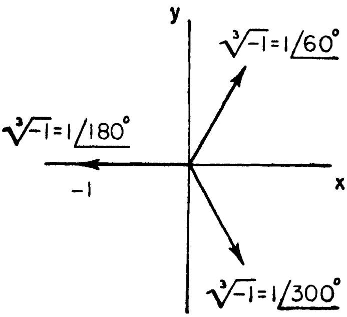

y
$\sqrt[3]{-1} = 1\underline{/60^\circ}$
$\sqrt[3]{-1} = 1\underline{/180^\circ}$
-1
x
$\sqrt[3]{-1} = 1\underline{/300^\circ}$

Fig. 5.18. The three cube roots of (−1).

The test for any one of the n nth roots of a complex quantity is to raise that root to the nth power and then observe whether the result equals the original quantity. As a study of actual cases will show, the n roots of any complex quantity are complex quantities which have equal magnitudes and which are uniformly distributed in angular position about the 360 degrees of a circle. The first root has an angle one-nth that of the original quantity. All the roots have magnitudes equal to the nth root of the magnitude of the original quantity.

EXAMPLE 5.18. Find the three cube roots of (−1).

$$\sqrt[3]{-1} = \sqrt[3]{1\underline{/180^\circ}} = (1\underline{/180^\circ})^{\frac{1}{3}} = 1\underline{\Big/\frac{180^\circ + 360^\circ q}{3}}.$$

First root: $q = 0$.

$$\sqrt[3]{-1} = 1\underline{\Big/\frac{180^\circ}{3}} = 1\underline{/60^\circ}.$$

Second root: $q = 1$.

$$\sqrt[3]{-1} = 1\underline{\Big/\frac{180^\circ + 360^\circ}{3}} = 1\underline{\Big/\frac{540^\circ}{3}} = 1\underline{/180^\circ}.$$

Third root: $q = 2$.

$$\sqrt[3]{-1} = 1\underline{\Big/\frac{180^\circ + 720^\circ}{3}} = 1\underline{\Big/\frac{900^\circ}{3}} = 1\underline{/300^\circ}$$

A value of $q = 3$ yields no new root but produces the first root again; a value of $q = 4$ produces the second root again; and so on. There are only three cube roots of any number. A graphical interpretation of these results is given in Fig. 5.18. It will be observed that the three cube roots are of the same magnitude and are uniformly distributed about 360°, with the first root at an angle equal to one-third that of the original complex quantity.

5.19. Logarithm of a Complex Quantity

In long-line recurrent-network theory it is occasionally necessary to utilize the logarithms of complex quantities. Just as with ordinary numbers, the logarithm of a complex quantity is the mathematical inverse of the exponential of that quantity. By this definition, then,

$$\ln (A\epsilon^{j\alpha}) = \log_\epsilon (A\epsilon^{j\alpha}) = \log_\epsilon A + \log_\epsilon \epsilon^{j\alpha} = \log_\epsilon A + j\alpha \log_\epsilon \epsilon, \tag{5-46}$$

or

▶ $$\ln (A\epsilon^{j\alpha}) = \ln A + j\alpha. \tag{5-47}$$

where:

A = the *magnitude* of the complex quantity, and
α = the *angle* of the complex quantity, in **radians.**

EXAMPLE 5.19. Find the logarithm of the current $\mathbf{I} = 2 + j3$ amperes.

$$\mathbf{I} = 2 + j3 = \sqrt{2^2 + 3^2}\underline{/\tan^{-1}(\tfrac{3}{2})} = 3.606\underline{/56.3^\circ} = 3.606\underline{/0.982 \text{ rad.}}$$
$$\ln \mathbf{I} = \ln 3.606\epsilon^{j0.982} = \ln 3.606 + j0.982 = 1.282 + j0.982.$$

PROBLEMS

5-1. Write the complex rectangular expressions for the vectors which represent the following waves at the instant $t = 0$:
(a) $e_a = 100 \sin (\omega t + 30^\circ)$ volts.
(b) $e_b = 50 \cos (\omega t - 20^\circ)$ volts.
(c) $i_a = -10 \sin (\omega t - 30^\circ)$ amperes.
(d) $i_b = 2 \sin (\omega t - 120^\circ)$ amperes.

5-2. Write as functions of ω and t the equations of the waves which the following vectors represent:
(a) $\mathbf{E}_{m1} = 50 - j70$ complex volts.
(b) $\mathbf{E}_{m2} = 100 + j20$ complex volts.
(c) $\mathbf{I}_{m1} = 2 + j4$ complex amperes.
(d) $\mathbf{I}_{m2} = -7 - j3$ complex amperes.

5-3. Write, in the rectangular, exponential, and polar forms, the expressions for the vectors representing the following waves at the instant $t = 0$:
(a) $a = 14.14 \sin \omega t$.
(b) $b = 8.76 \cos (\omega t + 60^\circ)$.
(c) $c = -2.41 \sin (\omega t - 15^\circ)$.
(d) $d = -9 \cos (\omega t + 20^\circ)$.

5-4. Express the answers to Prob. 5-1 in both the exponential and the polar forms.

5-5. (a) Express the vectors of Prob. 5-2 in the exponential form.
(b) Express the vectors of Prob. 5-2 in the polar form.

5-6. Given: $e_1 = 100 \sin (377t - 30°)$ volts, and $e_2 = 70 \sin (377t + 60°)$ volts. Showing all your work, determine whether the following statement is true:

$$\mathbf{E}_{m1} + \mathbf{E}_{m2} = 121.6 + j10.62 \text{ complex volts.}$$

5-7. Given: $i_1 = 10 \sin (157t + 240°)$ amperes, and $i_2 = 5 \sin (314t - 120°)$ amperes. Showing all your work, determine whether the following statement is true:

$$\mathbf{I}_{m1} - \mathbf{I}_{m2} = -2.5 - j4.33 \text{ complex amperes.}$$

5-8. Referring to Prob. 5-1, write as functions of ω and t the equations for the resultant waves:

$$\text{(a) } e_c = e_a + e_b. \qquad \text{(b) } i_d = i_a - i_b.$$

5-9. Show that $A \sin \alpha = A \left(\dfrac{\epsilon^{j\alpha} - \epsilon^{-j\alpha}}{2j} \right)$.

5-10. Show that $\cos x = \dfrac{\epsilon^{jx} + \epsilon^{-jx}}{2}$.

5-11. (a) Write as a trigonometric function of ω and t the following sum:

$$A\epsilon^{j\omega t} + A\epsilon^{-j\omega t}.$$

(b) Provide a graphical interpretation of the result of part (a).

5-12. Rotate the following vectors through the angles indicated, and express the answers in the original form:
(a) $\mathbf{A} = 10 + j0$ through $+30°$.
(b) $\mathbf{B} = -7 - j4$ through $-20°$.
(c) $\mathbf{C} = 10\epsilon^{j45°}$ through 15°.
(d) $\mathbf{D} = 2\underline{/120°}$ through 90°.

5-13. Find the following products:
(a) $\mathbf{A} = (6 - j7)(3 + j4)$.
(b) $\mathbf{B} = (10\epsilon^{j30°})(2.6\epsilon^{-j15°})$.
(c) $\mathbf{C} = (3 + j4)(4 + j3)$.
(d) $\mathbf{D} = (25\underline{/90°})(2\underline{/-120°})$.

5-14. Use the rectangular form to evaluate:

(a) $\dfrac{1}{\cos \alpha + j \sin \alpha}$. (b) $\dfrac{1}{1 + j}$. (c) $\dfrac{1}{j}$.

5-15. Repeat Prob. 5-14, using the exponential form, and check by converting the results into the rectangular form and comparing with the results obtained through use of that form.

5-16. Using the rectangular form, find the following quotients, and convert the answers to the polar form:

(a) $\dfrac{3+j4}{8-j6}$. (b) $\dfrac{2-j8}{1+j4}$. (c) $\dfrac{-6+j3}{-3-j4}$. (d) $\dfrac{10+j0}{0-j5}$.

5-17. Repeat Prob. 5-16, using the exponential form.

5-18. The conjugate of a complex quantity expressed in the rectangular form is equal to the original quantity with the sign of the imaginary term reversed. To what is the conjugate of a complex quantity expressed in the exponential form equal?

5-19. The following rotational vector operator is useful in later applications of complex algebra to electric circuit analyses:

$$a \equiv \epsilon^{j120^\circ} = (\cos 120^\circ + j \sin 120^\circ) = (-0.5 + j0.866).$$

(a) Write the expressions for a^2, a^3, a^4, and a^5 in each of the above forms.

(b) Evaluate analytically the following functions of a, stating the answers in both rectangular and polar forms:

(1) $a + a^2 + 1$.
(2) $a + a^2$.
(3) $a - 1$.
(4) $a^2 - 1$.
(5) $1 + a$.
(6) $1 + a^2$.
(7) $(1 + a)^2$
(8) $(1 + a^2)^2$.
(9) $(1 + a)(1 + a^2)$.
(10) $(1 - a)(1 - a^2)$.
(11) $\dfrac{1+a}{1+a^2}$.
(12) $\dfrac{1-a}{1-a^2}$.

(c) Draw a single polar diagram of vectors representing a and all the answers to parts (a) and (b) of this problem.

5-20. Determine the 6 sixth roots of unity, and draw a vector diagram to show them graphically.

5-21. Determine the 5 fifth roots of (-1), and draw a vector diagram to show them graphically.

5-22. Evaluate the following:
(a) $\ln (3 - j4)$. (b) $\log_{10} (7 + j3)$.

5-23. Evaluate the following:
(a) $\epsilon^{j(5+j7)}$. (b) $\epsilon^{j(10\epsilon^{j30^\circ})}$.

5-24. Reduce the following expression to a single complex quantity expressed in the rectangular form:

$$\frac{(10 - j6)^3 - 17\epsilon^{j30^\circ}}{4\underline{/20^\circ} - j7}.$$

5-25. Reduce the following expression to a single complex quantity expressed in the exponential form:

$$\frac{\sqrt{-7 + j20} + \log_\epsilon (6\underline{/-10^\circ})}{j^5}.$$

PART II

SIMPLE CIRCUITS

CHAPTER 6

PARAMETERS IN SERIES

6.01. Introduction

Chapter 4 developed the basic potential, current, and power relations for each of the circuit parameters—resistance, inductance, and capacitance. Only hypothetically can "pure" R, L, or C be obtained. In practice, for example, it is impossible to construct an inductance coil with no resistance; no capacitor is entirely free of heat-dissipating characteristics (effective resistance); and no resistor can be entirely free of inductive and capacitive properties. Careful design and construction can, in a given case, reduce the undesired properties to a minimum, but under no conditions can an absolutely "pure" circuit parameter be obtained. This chapter will be devoted to a discussion of the potential, current, and power relations for various *series*[1] combinations of the three basic parameters. These combinations occur so frequently in practice that they are often treated as basic circuit elements for which complex impedance values must be specified.

6.02. Instantaneous Current and Potential for R and L in Series

Let us assume that a sinusoidal current is passing through a series combination of constant R and L, such as that shown by Fig. 6.02. The instantaneous current may be expressed as

$$i = I_m \sin (\omega t + \theta_i). \qquad (6\text{-}1)$$

From Kirchhoff's potential law, we know that the total instantaneous potential drop across the series combination is

$$v_{RL} = Ri + L\frac{di}{dt} = V_{mRL} \sin (\omega t + \theta_v). \qquad (6\text{-}2)$$

---

[1] A *series* combination of circuit elements is one in which the current flowing through one element flows through the other elements of the combination in a continuous path.

Substituting equation (6-1) in (6-2), we obtain

$$v_{RL} = RI_m \sin(\omega t + \theta_i) + L\frac{d}{dt}[I_m \sin(\omega t + \theta_i)], \qquad (6\text{-}3)$$

or

$$v_{RL} = RI_m \sin(\omega t + \theta_i) + \omega L I_m \cos(\omega t + \theta_i), \qquad (6\text{-}4)$$

which may be written

$$\blacktriangleright \quad \begin{aligned} v_{RL} &= V_{mRL} \sin(\omega t + \theta_v) \\ &= RI_m \sin(\omega t + \theta_i) + X_L I_m \sin(\omega t + \theta_i + 90°), \end{aligned} \qquad (6\text{-}5)$$

where: $X_L = \omega L =$ *inductive reactance*, in **ohms**.

These waves are shown in Fig. 6.02, for a particular case.

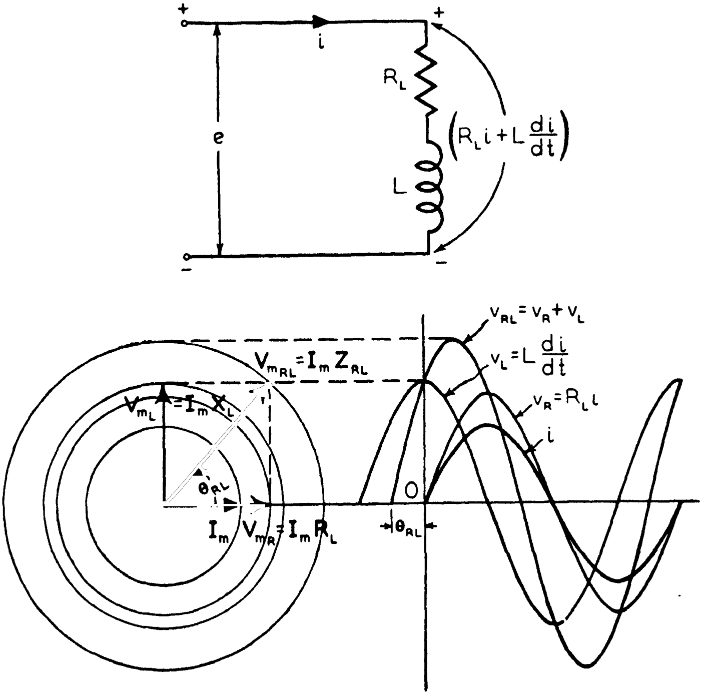


Fig. 6.02. Vectors and waves of potential and current for resistance and inductance in series.

6.03. Current and Potential Vectors for R and L in Series

A study of equation (6-5) and of Fig. 6.02 reveals that the wave of potential drop is made up of two components: the resistive drop, which is in phase with the current, and the inductive drop, which leads the current by 90°. The vector diagram of Fig. 6.02 shows the vectors corresponding to the current and the two component potential drops, as well as the resultant potential vector corresponding to the total potential drop.

In terms of complex quantities, the equations of these vectors may be written as follows, for the case where $(\omega t + \theta_i) = 0$, or current is used as reference.

Current:

▶ $$\mathbf{I}_m = I_m + j0 = I_m\underline{/0^\circ}. \tag{6-6}$$

Resistive drop:

$$\mathbf{V}_{mR} = RI_m + j0 = RI_m\underline{/0^\circ}. \tag{6-7}$$

Inductive drop:

$$\mathbf{V}_{mL} = 0 + jX_LI_m = X_LI_m\underline{/90^\circ}, \tag{6-8}$$

where: $X_L = \omega L$.

Total drop:

▶ $$\mathbf{V}_{mRL} = RI_m + jX_LI_m \tag{6-9a}$$

$$= \sqrt{(RI_m)^2 + (X_LI_m)^2}\,\underline{/\tan^{-1}\left(\frac{X_LI_m}{RI_m}\right)}. \tag{6-9b}$$

The reader should compare the vector statement of equations (6-9) with the instantaneous-value statement of equation (6-5). Term for term, they will be seen to correspond, just as the vectors of Fig. 6.02 correspond to the waves of that figure.

6.04. A-C Version of Ohm's Law

Article 4.25 defined complex impedance as:

$$\mathbf{Z} = \frac{V_m}{I_m}\underline{/(\theta_v - \theta_i)} = Z\underline{/\theta}. \tag{6-10}$$

From our knowledge of complex algebra, we may rewrite this as

▶ $$\mathbf{Z} = Z\underline{/\theta} = \frac{V_m}{I_m}\underline{/(\theta_v - \theta_i)} = \frac{V_m\underline{/\theta_v}}{I_m\underline{/\theta_i}} = \frac{\mathbf{V}_m}{\mathbf{I}_m}. \tag{6-11}$$

where:

$\mathbf{Z}$ = ***complex impedance*****, in complex ohms,**

$\mathbf{V}_m$ = ***vector potential drop*****, in vector volts, and**

$\mathbf{I}_m$ = ***vector current*****, in vector amperes.**

This relation is often called the *a-c version of Ohm's law* or *Ohm's law of the a-c circuit,* because of its similarity to the d-c circuit relation first established by G. S. Ohm and expressed by equation (4-7).

If we refer to equations (4-36), (4-54), and (4-73), we can now see that the "complex impedance" quantities expressed by those equations are, in reality, the a-c version of Ohm's law applied to the "pure" circuit parameters.

6.05. Complex Impedance for R and L in Series

By equation (6-11), we may utilize equations (6-9) and (6-6) to write the equation for complex impedance of R and L in series combination:

$$\mathbf{Z}_{RL} = \frac{\mathbf{V}_{mRL}}{\mathbf{I}_m} = \frac{\sqrt{(RI_m)^2 + (X_L I_m)^2}\,\underline{/\tan^{-1}\left(\frac{X_L I_m}{RI_m}\right)}}{I_m\underline{/0^\circ}}, \tag{6-12}$$

or

▶ $$\mathbf{Z}_{RL} = Z_{RL}\underline{/\theta_{RL}} = \sqrt{R^2 + X_L^2}\,\underline{/\tan^{-1}\left(\frac{X_L}{R}\right)}$$
$$= \sqrt{R^2 + (\omega L)^2}\,\underline{/\tan^{-1}\left(\frac{\omega L}{R}\right)}, \tag{6-13}$$

where:

$\mathbf{Z}_{RL}$ = *complex impedance for R-L series combination,* in **complex ohms,** and

θ_{RL} = *phase angle of R-L series combination,* in **radians** or **degrees,** and all other quantities are as previously defined.

Figure 6.05 shows the graphical interpretation of equation (6-13) for a particular case. This figure may be called the *impedance triangle* for a series combination of R and L. It might be the impedance triangle for a practical inductor. The horizontal leg of this triangle is the resistance of the series combination, and the vertical leg is the inductive reactance of the combination. It must be noted that in drawing impedance triangles, it is customary to draw the resistance horizontally, with the reactance drawn vertically at the right-hand end of resistance. No arrowheads are shown on impedance triangles, because complex impedance, although expressible as a complex quantity, is not a rotating vector like the vectors of potential difference and current. Complex impedance does not, for example, rotate at uniform angular velocity and generate mathematically a "wave of impedance." To borrow a mathematical term, impedance is an "operator" which determines relative magnitudes and phase positions of vectors of potential and current in a-c circuits.

We may obtain the rectangular form of expression for the complex impedance of equation (6-13), either by direct conversion of this equation or by dividing equation (6-9a) by I_m. We may then write the following expressions for complex impedance of a series combination of R and L:

▶ $$\mathbf{Z}_{RL} = R + jX_L = \sqrt{R^2 + X_L^2}\,\underline{/\tan^{-1}\left(\frac{X_L}{R_L}\right)}, \qquad (6\text{-}14)$$

where: $X_L = \omega L = 2\pi f L$.

Equation (6-13) shows clearly that the complex impedance of a series combination of R and L, though not a function of time, does depend on frequency. This statement is true of both the magnitude and the angle. The higher the value of frequency, the greater the relative importance of the ωL terms, and the greater the magnitude and the greater the phase angle of the impedance. If the amplitude of the applied potential wave is constant, an increase of frequency results in a decrease of current amplitude and an increase of current lag. This aspect of impedance will be discussed more completely in Article 9.09, "Locus Diagrams."

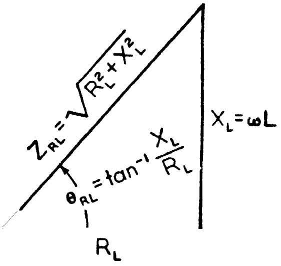

$Z_{RL} = \sqrt{R_L^2 + X_L^2}$
$X_L = \omega L$
$\theta_{RL} = \tan^{-1}\frac{X_L}{R_L}$
R_L

Fig. 6.05. Impedance triangle for series combination of resistance and inductance.

In the *d-c case*, where frequency is zero, $\omega L = 2\pi f L = 0$, and $\theta_{RL} = 0$. Thus, inductance, which operates only when current is changing, exerts no influence on a d-c circuit in which steady-state conditions prevail. In this d-c case, Ohm's law is

▶ $$\mathbf{Z}_{RL_{\text{d-c}}} = R_{\text{d-c}} = \frac{V_{\text{d-c}}}{I_{\text{d-c}}}. \qquad (6\text{-}15)$$

This is the same as equation (4-7), for Ohm's law.

EXAMPLE 6.05. The current $i = 2 \sin (377t + 30°)$ produces a potential drop $v = 26 \sin (377t + 45°)$ in a certain circuit element. Find the values of the circuit parameters comprising this element, assuming a series combination.

The vectors of current and potential drop are

$$\mathbf{I}_m = 2\underline{/30^\circ} \qquad \text{and} \qquad \mathbf{V}_m = 26\underline{/45^\circ}.$$

The complex impedance is

$$\mathbf{Z} = R + jX = \frac{V_m}{I_m}\underline{/(\theta_v - \theta_i)} = \frac{26}{2}\underline{/45^\circ - 30^\circ} = 13\underline{/15^\circ} = 12.55 + j3.36.$$

From this, we conclude that

$$R = 12.55 \text{ ohms}, \qquad \text{and} \qquad X = 3.36 \text{ ohms}.$$

The positive value for X shows that the reactance is inductive. Then

$$X_L = \omega L,$$

or $$L = \frac{X_L}{\omega} = \frac{3.36}{377} = 0.00893 \text{ henry} = 8.93 \text{ millihenrys}.$$

6.06. Instantaneous Current and Potential for R and C in Series

Let us send the sinusoidal current

▶ $$i = I_m \sin(\omega t + \theta_i) \tag{6-16}$$

through a series combination of constant R and C, as shown by Fig. 6.06. Kirchhoff's potential law gives us the following expression for total potential drop:

$$v_{RC} = Ri + \frac{q}{C} = Ri + \frac{1}{C}\int i\,dt = V_{m_{RC}} \sin(\omega t + \theta_v). \tag{6-17}$$

Substitution of (6-16) gives

$$\begin{aligned} v_{RC} &= RI_m \sin(\omega t + \theta_i) + \frac{1}{C}\int I_m \sin(\omega t + \theta_i)\,dt \\ &= RI_m \sin(\omega t + \theta_i) - \frac{I_m}{\omega C}\cos(\omega t + \theta_i) + K_2. \end{aligned} \tag{6-18}$$

The constant of integration, K_2, will prove important in our study of Transients, in Chapter 17. In the present steady-state analysis, $K_2 = 0$. Utilizing this fact, and noting that

$$\begin{aligned} -\frac{I_m}{\omega C}\cos(\omega t + \theta_i) &= -\frac{I_m}{\omega C}\sin(\omega t + \theta_i + 90^\circ) \\ &= X_C I_m \sin(\omega t + \theta_i + 90^\circ), \end{aligned}$$

we may write

$$\blacktriangleright \quad v_{RC} = V_{m_{RC}} \sin(\omega t + \theta_v)$$
$$= RI_m \sin(\omega t + \theta_i) + X_C I_m \sin(\omega t + \theta_i + 90^\circ), \quad (6\text{-}19)$$

where: $X_C = \dfrac{-1}{\omega C}$ = *capacitive reactance*, in **ohms.**

These waves are shown in Fig. 6.06 for a particular case.

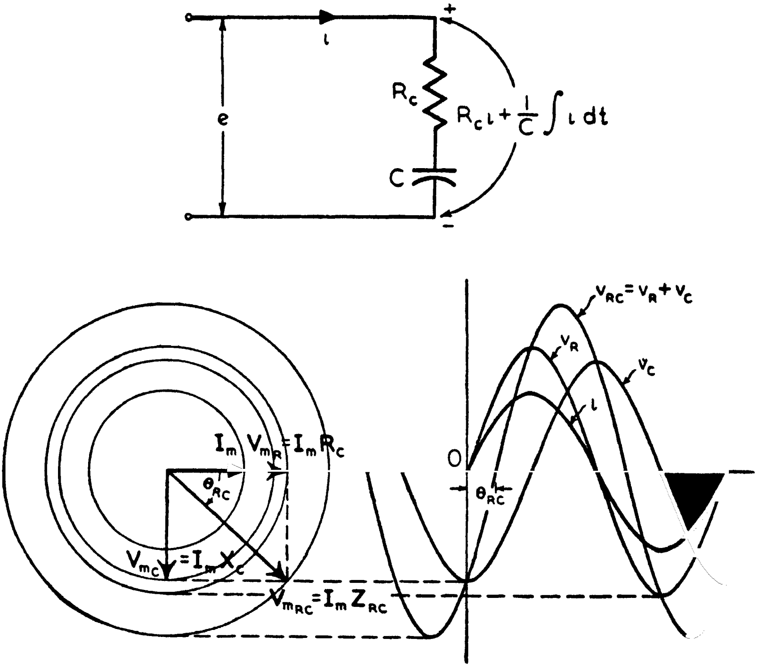


Fig. 6.06. Vectors and waves of potential and current for resistance and capacitance in series.

6.07 Current and Potential Vectors for R and C in Series

By a procedure similar to that of Article 6.03 dealing with the *R-L* case, we may write the following complex expressions for the vectors associated with a series combination of *R* and *C*:

Current:

$$\blacktriangleright \quad \mathbf{I}_m = I_m + j0 = I_m\underline{/0^\circ}. \qquad (6\text{-}20)$$

Resistive drop:

$$\mathbf{V}_{m_R} = RI_m + j0 = RI_m\underline{/0^\circ}. \tag{6-21}$$

Capacitive drop:

$$\mathbf{V}_{m_C} = 0 + jX_CI_m = X_CI_m\underline{/90^\circ}, \tag{6-22}$$

where: $X_C = \dfrac{-1}{\omega C}$.

Total drop:

$$\blacktriangleright \quad \mathbf{V}_{m_{RC}} = RI_m + jX_CI_m \tag{6-23a}$$

$$= \sqrt{(RI_m)^2 + (X_CI_m)^2}\underline{/\tan^{-1}\left(\frac{X_CI_m}{RI_m}\right)}. \tag{6-23b}$$

Equations (6-19) and (6-23) should be compared, in the light of Fig. 6.06.

6.08. Complex Impedance for R and C in Series

From equations (6-11), (6-20), and (6-23), we may write the following expression for complex impedance of R and C in series combination:

$$\blacktriangleright \quad \mathbf{Z}_{RC} = R + jX_C = \sqrt{R^2 + X_C^2}\underline{/\tan^{-1}\left(\frac{X_C}{R}\right)}, \tag{6-24}$$

where: $X_C = \dfrac{-1}{\omega C} = \dfrac{-1}{2\pi fC}$.

Figure 6.08 shows the impedance triangle for a particular R-C series combination.

Equation (6-24) shows that the complex impedance of a combination of R and C in series is not a function of time but is a function of frequency. This statement is true of both its magnitude and its angle. An increase of frequency in a series combination of R and C produces effects which are the inverse of those produced in a series combination of R and L. In the R-C series combination, the higher the frequency, the less the relative importance of the $(-1/\omega C)$ term, and the less the magnitude and the less the angle of the complex impedance. If the amplitude of the wave of potential drop is held constant, an increase of frequency results in a current wave of increased amplitude but decreased angle of lead.

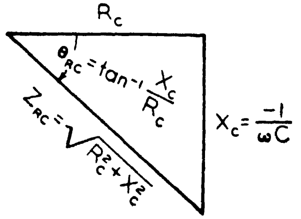

R_C
$\theta_{RC} = \tan^{-1} \frac{X_C}{R_C}$
$Z_{RC} = \sqrt{R_C^2 + X_C^2}$
$X_C = \frac{-1}{\omega C}$

Fig. 6.08. Impedance triangle for series combination of resistance and capacitance.

In the *d-c case,* where frequency is zero,

$$\blacktriangleright\quad X_{C_{d\text{-}c}} = -\frac{1}{\omega C} = \frac{-1}{2\pi f C} = \frac{-1}{0} = -\infty, \qquad \text{and} \qquad \theta_{d\text{-}c} = 0.$$

Thus, capacitance offers an infinite opposition to the flow of direct current under steady-state conditions. Under steady-state conditions in a d-c circuit, capacitance is the equivalent of an open switch; direct current cannot flow in a series capacitive circuit.

Example 6.08. The current $i = 2 \sin (377t - 120°)$ produces a potential drop $v = 110 \sin (377t + 160°)$ in a certain circuit element. Find the values of the circuit parameters comprising this element, assuming a series combination.

The vectors of current and potential drop are

$$\mathbf{I}_m = 2\underline{/-120°} \qquad \text{and} \qquad \mathbf{V}_m = 110\underline{/160°}.$$

The complex impedance is

$$\mathbf{Z} = R + jX = \frac{V_m}{I_m}\underline{/(\theta_v - \theta_i)} = \frac{110}{2}\underline{/160° + 120°}$$
$$= 55\underline{/280°} = 9.55 - j54.1.$$

From this, we conclude that

$$R = 9.55 \text{ ohms} \qquad \text{and} \qquad X = -54.1 \text{ ohms}.$$

The negative value for X shows that the reactance is capacitive. Then

$$X_C = \frac{-1}{\omega C}$$

and $C = \dfrac{-1}{\omega X_C} = \dfrac{-1}{377 \times (-54.1)} = 49.1 \times 10^{-6}$ farad $= 49.1$ microfarads.

6.09. Instantaneous Current and Potential for R, L, and C in Series

Let us now send the current

$$\blacktriangleright\qquad i = I_m \sin (\omega t + \theta_i) \qquad (6\text{-}25)$$

through a series combination of all three parameters, R, L, and C, as indicated by Fig. 6.09. Kirchhoff's potential law tells us that the total potential drop is

$$v_{RLC} = v_R + v_L + v_C = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt. \qquad (6\text{-}26)$$

From our previous work, we can see that substitution of (6-25) in this equation will yield

$$v_{RLC} = RI_m \sin(\omega t + \theta_i) + \omega L I_m \cos(\omega t + \theta_i) + \frac{-I_m}{\omega C}\cos(\omega t + \theta_i). \tag{6-27}$$

Combining the last two terms, we have

$$\begin{aligned} v_{RLC} &= V_m \sin(\omega t + \theta_v) \\ &= RI_m \sin(\omega t + \theta_i) + \left(\omega L + \frac{-1}{\omega C}\right) I_m \cos(\omega t + \theta_i), \end{aligned} \tag{6-28}$$

or

$$\blacktriangleright\ \begin{aligned} v_{RLC} &= V_m \sin(\omega t + \theta_v) \\ &= RI_m \sin(\omega t + \theta_i) + X_{LC} I_m \sin(\omega t + \theta_i + 90°), \end{aligned} \tag{6-29}$$

where: $X_{LC} = (X_L + X_C) = \left(\omega L + \dfrac{-1}{\omega C}\right)$ = *total reactance,* in **ohms.**

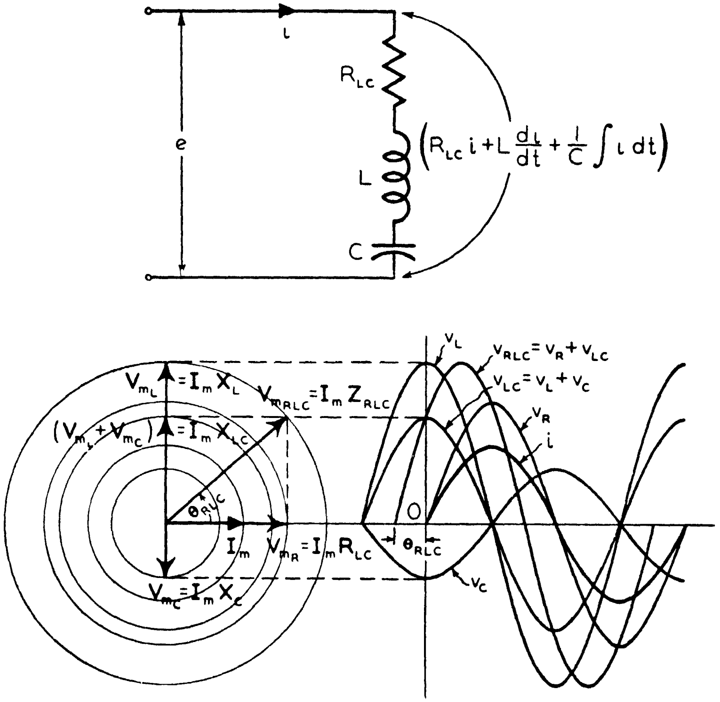


Fig. 6.09. Vectors and waves of potential and current for resistance, inductance, and capacitance in series.

Figure 6.09 shows the graphical interpretation of equation (6-29) for a particular case.

6.10. Current and Potential Vectors for R, L, and C in Series

Figure 6.09 shows not only the waves of equation (6-29) but also the vectors representing those waves at the instant $(\omega t + \theta_i) = 0$. The complex expressions for these vectors may be obtained directly from equation (6-29). They are

Current:

▶ $$\mathbf{I}_m = I_m + j0 = I_m\underline{/0^\circ}. \tag{6-30}$$

Resistive drop:

$$\mathbf{V}_{m_R} = RI_m + j0 = RI_m\underline{/0^\circ}. \tag{6-31}$$

Inductive drop:

$$\mathbf{V}_{m_L} = 0 + jX_LI_m = X_LI_m\underline{/90^\circ}, \tag{6-32}$$

where: $X_L = \omega L$.

Capacitive drop:

$$\mathbf{V}_{m_C} = 0 + jX_CI_m = X_CI_m\underline{/90^\circ}, \tag{6-33}$$

where: $X_C = \dfrac{-1}{\omega C}$.

Total reactive drop:

$$\mathbf{V}_{m_{LC}} = 0 + jX_{LC}I_m = X_{LC}I_m\underline{/90^\circ}, \tag{6-34}$$

where: $X_{LC} = X_L + X_C$

Total potential drop:

▶ $$\mathbf{V}_{m_{RLC}} = RI_m + jX_{LC}I_m \tag{6-34a}$$

$$= \sqrt{(RI_m)^2 + (X_{LC}I_m)^2}\,\underline{/\tan^{-1}\left(\frac{X_{LC}I_m}{RI_m}\right)}, \tag{6-34b}$$

where: $X_{LC} = (X_L + X_C) = \left(\omega L + \dfrac{-1}{\omega C}\right)$ = *net reactance*, in **ohms.**

6.11. Complex Impedance for R, L, and C in Series

Utilizing equations (6-30) and (6-34) in the a-c version of Ohm's law, we obtain the following expression for complex impedance for a series combination of R, L, and C:

$$\blacktriangleright \quad \mathbf{Z}_{RLC} = \frac{\mathbf{V}_{mRLC}}{\mathbf{I}_m} = R + jX_{LC} = \sqrt{R^2 + X_{LC}^2}\,\underline{/\tan^{-1}\left(\frac{X_{LC}}{R}\right)}, \tag{6-35}$$

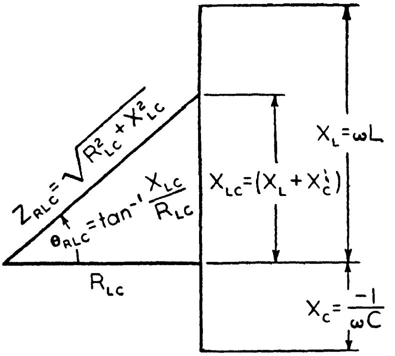
where:

$\mathbf{Z}_{RLC}$ = *complex impedance of R-L-C series combination*, in **ohms,**

$X_{LC} = (X_L + X_C) = \left(\omega L + \frac{-1}{\omega C}\right)$ = *net reactance*, in **ohms.**

Figure 6.11 shows the impedance triangle for a particular series combination of R, L, and C.

Equation (6-35) tells us that $\mathbf{Z}_{RLC}$, although independent of time, is a function of frequency. Any variation of frequency causes the net reactance, $X_{LC} = \left(\omega L + \frac{-1}{\omega C}\right)$, to vary in value in such a manner that the net reactance may appear to be capacitive, equal to zero, or inductive, depending on the relative values of $|\omega L|$ and $|-1/\omega C|$. This equation assumes considerable importance when we realize that it may be considered the general expression for the impedance of series combinations of parameters. If either R or L is absent from the circuit, then R or X_L will equal zero in (6-35). If no series capacitance is present in the closed circuit, the capacitive reactance, X_C, equals zero. This condition is equivalent to the presence of a series capacitance of value $C = \infty$.



$Z_{RLC} = \sqrt{R_{LC}^2 + X_{LC}^2}$
$\theta_{RLC} = \tan^{-1} \frac{X_{LC}}{R_{LC}}$
R_{LC}
$X_L = \omega L$
$X_{LC} = (X_L + X_C)$
$X_C = \frac{-1}{\omega C}$

Fig. 6.11. Impedance triangle for series combination of resistance, inductance, and capacitance.

In the steady-state *d-c case*, where $\omega = 0$, the impedance of a series combination including capacitance is infinite. This case is equivalent to an open circuit. If the series combination contains no capacitance but consists only of R and L, the inductance exerts no influence, and the net impedance equals the resistance.

Example 6.10. If a current $i = 2 \sin (377t)$ is sent through a certain series combination of circuit parameters, the resulting net potential drop is $v = 100 \sin (377t + 80°)$. If a current $i = 2 \sin (157t)$ is sent through the same series combination of parameters, the resulting net potential drop is $v = 20 \sin(157t - 29.6°)$. Find the values of the circuit parameters.

Test one ($\omega = 377$):

$$\mathbf{I}_{m_1} = 2\underline{/0}; \qquad \mathbf{V}_{m_1} = 100\underline{/80°}; \qquad \mathbf{Z}_1 = \frac{100\underline{/80°}}{2\underline{/0°}} = 50\underline{/80°} \doteq 8.69 + j49.2.$$

$$R_1 = 8.69 \text{ ohms}; \qquad X_1 = +49.2 \text{ ohms}.$$

Test two ($\omega = 157$):

$$\mathbf{I}_{m_2} = 2\underline{/0°}; \qquad \mathbf{V}_{m_2} = 20\underline{/-29.6°};$$

$$\mathbf{Z}_2 = \frac{20\underline{/-29.6°}}{2\underline{/0°}} = 10\underline{/-29.6°} = 8.69 - j4.94.$$

$$R_2 = 8.69 \text{ ohms}; \qquad X_2 = -4.94 \text{ ohms}.$$

Resistance is observed to remain constant with the change of frequency, but reactance is observed to change from an inductive to a capacitive nature as frequency decreases. This behavior suggests that both inductance and capacitance are present in series. We may now write the following two simultaneous equations:

$$X_1 = \left(\omega_1 L + \frac{-1}{\omega_1 C}\right); \qquad 49.2 = \left(377L - \frac{1}{377C}\right)$$

and

$$X_2 = \left(\omega_2 L + \frac{-1}{\omega_2 C}\right); \qquad -4.94 = \left(157L - \frac{1}{157C}\right).$$

Simultaneous solution of these equations yields

$$L = 0.165 \text{ henry} = 165 \text{ millihenrys},$$

and

$$C = 206.6 \times 10^{-6} \text{ farad} = 206.6 \text{ microfarads}.$$

6.12. General Equation for Instantaneous Power

Let us consider the general case of sinusoidal current passing through an energy sink in an electric circuit. This current may be expressed as

▶ $$i = I_m \sin(\omega t + \theta_i). \qquad (6\text{-}36)$$

Assuming the sink to possess a linear potential-current characteristic, the resulting potential drop is

▶ $$v = V_m \sin(\omega t + \theta_v). \qquad (6\text{-}37)$$

The general expression for the instantaneous power associated with this pair of waves at a sink is

▶ $$p = vi = V_m \sin(\omega t + \theta_v) I_m \sin(\omega t + \theta_i). \tag{6-38}$$

Recognizing that

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

we may rewrite (6-38) as:

$$p = V_m I_m(\sin\omega t\cos\theta_v + \cos\omega t\sin\theta_v)(\sin\omega t\cos\theta_i + \cos\omega t\sin\theta_i). \tag{6-39}$$

Multiplying,

$$\begin{aligned} p = V_m I_m(&\sin^2\omega t\cos\theta_v\cos\theta_i + \sin\omega t\cos\omega t\cos\theta_v\sin\theta_i \\ &+ \sin\omega t\cos\omega t\sin\theta_v\cos\theta_i + \cos^2\omega t\sin\theta_v\sin\theta_i), \end{aligned} \tag{6-40}$$

or

$$\begin{aligned} p = V_m I_m[&\sin^2\omega t\,(\cos\theta_v\cos\theta_i) \\ &+ \sin\omega t\cos\omega t\,(\cos\theta_v\sin\theta_i + \sin\theta_v\cos\theta_i) \\ &+ \cos^2\omega t(\sin\theta_v\sin\theta_i)]. \end{aligned} \tag{6-41}$$

We shall now use the identities

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}; \qquad \sin\alpha\cos\alpha = \frac{\sin 2\alpha}{2}; \qquad \cos^2\alpha = \frac{1+\cos 2\alpha}{2}.$$

Equation (6-41) then becomes

$$\begin{aligned} p = V_m I_m \Bigg[&\frac{1-\cos 2\omega t}{2}(\cos\theta_v\cos\theta_i) \\ &+ \frac{\sin 2\omega t}{2}(\cos\theta_v\sin\theta_i + \sin\theta_v\cos\theta_i) \\ &+ \frac{1+\cos 2\omega t}{2}(\sin\theta_v\sin\theta_i)\Bigg]. \end{aligned} \tag{6-42}$$

Rearranging terms, we obtain

$$\begin{aligned} p = V_m I_m \Bigg[&\frac{\cos\theta_v\cos\theta_i + \sin\theta_v\sin\theta_i}{2} \\ &- \frac{\cos\theta_v\cos\theta_i - \sin\theta_v\sin\theta_i}{2}\cos 2\omega t \\ &+ \frac{\cos\theta_v\sin\theta_i + \sin\theta_v\cos\theta_i}{2}\sin 2\omega t\Bigg]. \end{aligned} \tag{6-43}$$

By substituting the trigonometric identities,

$$\begin{aligned} \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta, \\ \cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta, \\ \sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta, \end{aligned}$$

we have

$$p = \frac{V_m I_m}{2}[\cos(\theta_v - \theta_i) - \cos(\theta_v + \theta_i)\cos 2\omega t + \sin(\theta_v + \theta_i)\sin 2\omega t], \quad (6\text{-}44)$$

or

$$\blacktriangleright\ p = \left[\frac{V_m I_m}{2}\cos(\theta_v - \theta_i)\right] - \left[\frac{V_m I_m}{2}\cos(\theta_v + \theta_i)\right]\cos 2\omega t + \left[\frac{V_m I_m}{2}\sin(\theta_v + \theta_i)\right]\sin 2\omega t. \quad (6\text{-}45)$$

This is the *general equation for a wave of instantaneous power associated with sinusoidal potential and current waves.* In deriving this equation, no particular combination of circuit parameters was specified at the receiver, and no particular values of θ_v and θ_i were specified.

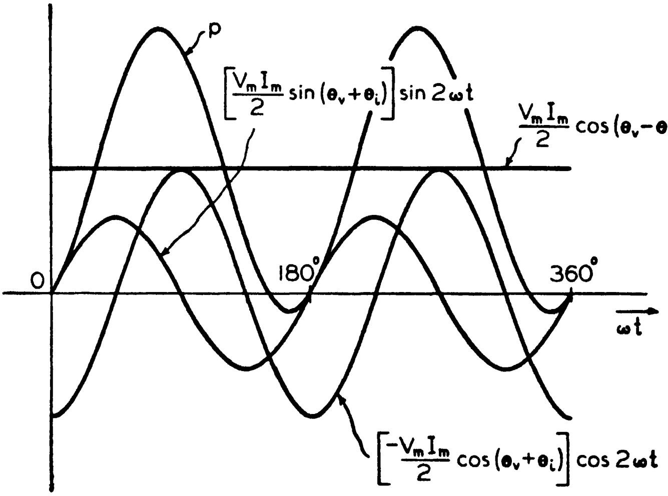


Fig. 6.12. Graphical representation of equation (6-45) for a particular case in which $\theta_i = 0$ degrees.

A study of equation (6-45) will reveal the wave of instantaneous power to consist of a d-c component of uniform value, and two double-frequency sinusoidal components. The value of the d-c component is determined by the amplitudes and the angular phase difference of the potential and current waves. The amplitudes of the sinusoidal components depend on the amplitudes and the sum of the initial phase angles of the potential and current waves. It will be observed that one of these sinusoidal components is always a cosine wave and that the other is always a sine wave. This puts a 90° phase angle between them. Figure 6.12 shows the instantaneous power wave for a particular case.

The *average power* is found by integrating equation (6-45) over a complete cycle of potential or current. Because of their sinusoidal nature, the last two terms will average zero, whereas the unvarying first term will be, in fact, the average power. Mathematically,

▶ $$P_{\text{ave}} = \frac{V_m I_m}{2} \cos\,(\theta_v - \theta_i), \tag{6-46a}$$

or

▶ $$P_{\text{ave}} = \frac{V_m I_m}{2} \cos\,\theta. \tag{6-46b}$$

The positive sign indicates that the average flow of energy is away from the circuit, as we should expect at a sink.

Although the *average* flow is away from the circuit, we can easily see from Fig. 6.12 that, in general, the energy flow is not entirely unidirectional. The predominant positive pulses alternate with smaller negative pulses. This reaction indicates, of course, that the net reactance of the receiver is not zero. It functions alternately to receive energy and to return energy to the circuit. In the special case where $(\theta_v - \theta_i) = \theta = 0$, the net reactance is zero, and the instantaneous power wave is entirely positive. In this case the receiver appears to be purely resistive, and the energy supplied to the receiver by the circuit is supplied irreversibly in a series of unidirectional pulses.

For the *d-c case,* where $\omega = 0$, $\theta = (\theta_v - \theta_i)$ is indeterminate, and equations (6-46) cannot be used. Average power may be determined in this d-c case by use of equation (3-4), which was obtained directly from the fundamental power equation. This *equation for d-c power* is

▶ $$P_{\text{d-c}} = I_{\text{d-c}} V_{\text{d-c}}. \tag{6-47}$$

6.13. Instantaneous Power Quantities for Parameters in Series

Because circuit parameters connected in series all carry the same current, it is convenient for this case to utilize the quantity, current, as "reference" in writing equations. We do this by letting $\theta_i = 0$ and then choosing the instant when $t = 0$ to write the complex expressions for the vectors. The initial phase angle of the potential drop then becomes equal to the angular phase difference between the potential and current waves, since $\theta = (\theta_v - \theta_i) = \theta_v$. We then have the following expressions for current and potential drop:

$$i = I_m \sin \omega t, \tag{6-48}$$
$$v = V_m \sin (\omega t + \theta). \tag{6-49}$$

Equation (6-45) for instantaneous power assumes the following form when current is used as "reference":

$$p = \left[\frac{V_m I_m}{2} \cos \theta\right] - \left[\frac{V_m I_m}{2} \cos \theta\right] \cos 2\omega t + \left[\frac{V_m I_m}{2} \sin \theta\right] \sin 2\omega t.$$

We shall rewrite this equation in such a way as to group the first two terms:

$$\blacktriangleright \quad p = \left[\frac{V_m I_m}{2} \cos \theta\right] (1 - \cos 2\omega t) + \left[\frac{V_m I_m}{2} \sin \theta\right] \sin 2\omega t = p_a + p_r. \tag{6-51}$$

where:

p = *instantaneous power*, in **watts**, and, by definition,

$p_a = \left[\frac{V_m I_m}{2} \cos \theta\right] (1 - \cos 2\omega t)$ = *instantaneous active power*, in **watts**, and

$p_r = \left[\frac{V_m I_m}{2} \sin \theta\right] \sin 2\omega t$ = *instantaneous reactive power* in **vars.**[2]

Figure 6.13 shows the graphical interpretation of this equation.

The reason for designating "active" and "reactive" components of instantaneous power is made clear by reference to Fig. 6.13. We see that the *active* component is entirely *unidirectional*, and, for the case shown, represents irreversible energy flow away from the circuit in the form of heat dissipated by the receiver resistance. Contrastingly, the *reactive* component *alternates sinusoidally*. This alternation represents surging of energy alternately into and back out of storage in the net reactance. It constitutes an oscillation of energy between the source and the receiver, superimposed on the pulsating unidirectional flow of energy from the source to the receiver. Active power is "active" in the sense that it represents energy conversion to a form which leaves the circuit and therefore is inherently useful, whereas reactive power is not inherently useful in doing work outside of the circuit.

---

[2] Because reactive power consists of the product of volts and amperes and yet must be distinguished from real power, the term "var" was coined from the first letters of the words "volt-ampere, reactive." It was adopted by the International Electrotechnical Commission in 1930.

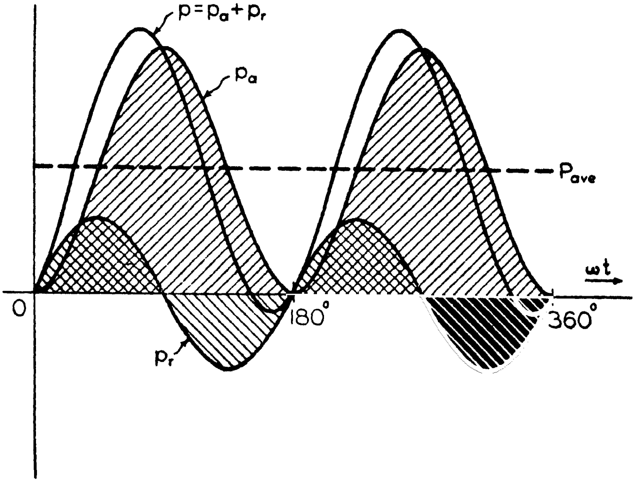


Fig. 6.13. Active and reactive components of instantaneous power for a particular case in which $\theta_i = 0$ degrees.

6.14. Active and Reactive Power Associated with Series Parameters

By definition, *active power* is the constant coefficient of the instantaneous active power equation when either i or v is used as reference. From equations (6-51) and (6-46b), we see that the *active power* associated with parameters in series (i used as reference) is equal to the average power:

$$\blacktriangleright \qquad P = \frac{V_m I_m}{2} \cos \theta = P_{ave}, \qquad (6\text{-}52)$$

where: P = *active power*, in watts.

Also, by definition, *reactive power* is the constant coefficient of the instantaneous reactive power equation when either i or v is used as reference. From equation (6-51),

▶ $$Q = \frac{V_m I_m}{2} \sin \theta, \tag{6-53}$$

where: Q = *reactive power*, in **vars.**

6.15. Vector Power Associated with Series Parameters

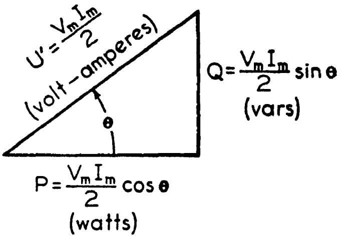
A comparison of equations (6-52) and (6-53) reveals them both to be functions of the power-factor angle, θ, such that

▶ $$U' = \sqrt{P^2+Q^2} = \sqrt{\left(\frac{V_m I_m}{2} \cos \theta\right)^2 + \left(\frac{V_m I_m}{2} \sin \theta\right)^2} = \frac{V_m I_m}{2}, \tag{6-54}$$

where: $U' = \dfrac{V_m I_m}{2}$ = *vector power*, in **volt-amperes.**

Graphically, the relationship between active, reactive, and vector power is shown by Fig. 6.15A.

We observe that vector power, although not a rotating vector like those of potential or current (since it is not a simple sine function of time), may be expressed as a complex quantity, just as impedance is expressed as a complex quantity:



$U' = \frac{V_m I_m}{2}$
(volt-amperes)
$Q = \frac{V_m I_m}{2} \sin \theta$
(vars)
θ
$P = \frac{V_m I_m}{2} \cos \theta$
(watts)

Fig. 6.15A. Triangle of power quantities at a sink, potential drops considered positive, and inductive reactive vars considered positive (current is reference).

▶ $$\mathbf{U}' = P + jQ = \sqrt{P^2+Q^2}\,\underline{/\tan^{-1}\left(\frac{Q}{P}\right)} = U'\,\underline{/\tan^{-1}\left(\frac{Q}{P}\right)}, \tag{6-55}$$

where: $\mathbf{U}'$ = *vector power*, in **complex volt-amperes.**
For the case of a potential drop in the positive current direction (a receiver), the equation for vector power is

▶ $$\mathbf{U}' = \left(\frac{V_m I_m}{2} \cos \theta\right) + j\left(\frac{V_m I_m}{2} \sin \theta\right) = \frac{V_m I_m}{2}\underline{/\theta}. \tag{6-56}$$

The vector power of Fig. 6.15A and equations (6-54) through (6-56) lies in either the first or fourth quadrants (depending on θ), because while working with an energy sink we chose to call potential drops positive. This convention makes active power leaving the circuit positive.

If we were to deal with a source instead of a sink and were we to continue to call drops positive, the rise at the source would be considered a negative drop. This convention would place the vector power in either the third or second quadrants (depending on θ) and would make active power entering the circuit negative. In this case, the vector power is removed by 180° from the position it would occupy if it were the power of a sink, rather than the power of a source. Figure 6.15B shows this situation.

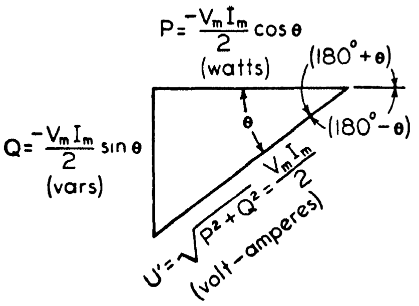


Fig. 6.15B. Triangle of power quantities at a source, potential drops considered positive and inductive reactive vars considered positive (current is reference).

It is instructive to note that if, when dealing with sources, we were to call potential *rises* positive, we would substitute E for $(-V)$ in the power equations and put the resulting vector power in either the first or fourth quadrant. To work only with the first or fourth quadrants, electrical engineers often utilize positive rises when drawing power triangles at sources; positive drops when drawing power triangles at sinks.

EXAMPLE 6.15. Given the current $i = 2 \sin 377t$ and the potential drop $v = 100 \sin (377t + 30°)$. (a) Write the expression for the instantaneous power, and point out the active and reactive components. (b) Write the complex expression for the vector power.

Instantaneous power:

$$p = \left[\frac{2 \times 100}{2} \cos 30°\right] (1 - \cos 2 \times 377t) + \left[\frac{2 \times 100}{2} \sin 30°\right] \sin 2 \times 377t$$

$$= 86.6(1 - \cos 754t) + 50 \sin 754t \text{ watts;}$$

$$p_a = 86.6(1 - \cos 754t) \text{ watts,}$$

$$p_r = 50 \sin 754t \text{ vars.}$$

Vector power:

$$U' = 86.6 + j50 = 100\underline{/30^\circ} \text{ complex volt-amperes;}$$
$$P = 86.6 \text{ watts,}$$
$$Q = 50 \text{ vars.}$$

PROBLEMS

6-1. A 25-cycle alternating emf wave having an amplitude of 153 volts is applied to a series circuit containing resistance totaling 8 ohms and inductance totaling 0.050 henry. Using numerical coefficients,

(a) Write the expression for complex impedance.

(b) Write the expression for the applied emf as a function of ω and t, making de/dt negative and $e = 0$ at $t = 0$.

(c) Write the expression for the total potential drop as a function of ω and t.

(d) Write the expression for current as a function of ω and t.

(e) Write the expression for power as a function of ω and t.

(f) Compute the average power.

(g) Write the complex expressions for the vectors associated with all the foregoing current and potential quantities, and draw the corresponding vector diagrams.

(h) Sketch all of the above waves to the same set of coordinate axes.

6-2. A current, $i = 2 \sin (377t - 20^\circ)$ amperes, causes a potential drop, $v = 17.8 \cos (377t - 75^\circ)$ volts, in a certain circuit element.

(a) Write the numerical expression for the complex impedance of the element.

(b) Assuming the element to consist of no more than two parameters in series, find the values of the parameters.

(c) Employing numerical coefficients, write as functions of ω and t the equations for the potential drops across the two circuit parameters.

(d) Employing numerical coefficients, write the expression for power as a function of ω and t.

(e) Compute the average power.

(f) Sketch all the above waves to the same set of rectangular coordinate axes.

(g) Write the complex expressions for the vectors associated with all the foregoing currents and potentials, and draw the corresponding vector diagrams.

(h) Show that the total vector potential is the vector sum of the individual vector potentials.

6-3. An alternating emf wave,

$$e = 14.14 \cos (6283t - 90^\circ) \text{ volts,}$$

when applied to a particular circuit element, results in a current wave,

$$i = 0.1 \sin (6283t + 10^\circ) \text{ amperes.}$$

Carry out the requirements of Prob. 6-2 for these two waves.

6-4. A current, $i = 5 \sin (\omega t - 30°)$ amperes, flows through a certain circuit element. The resulting power wave is

$$p = 171 + 321 \cos 2\omega t - 383 \sin 2\omega t \text{ watts.}$$

(a) What is the equation of the wave of potential drop?

(b) Assuming no more than two circuit parameters to be present, what are the values of the two parameters?

6-5. (a) What is the instantaneous active power of Prob. 6-3?

(b) What is the instantaneous reactive power of Prob. 6-3?

(c) What is the active power of Prob. 6-3?

(d) What is the reactive power of Prob. 6-3?

6-6. (a) Write the equation of the power wave for the resistive parameter of Prob. 6-2.

(b) Write the equation of the power wave for the reactive parameter of Prob. 6-2.

(c) Show that the sum of the solutions to (a) and (b) is equivalent to the total instantaneous power wave, as obtained by substitution in equation (6-45).

6-7. A current, $i = 2.12 \sin 157t$ amperes, flows through a series R-L-C combination in which $R = 5$ ohms, $L = 100$ millihenrys, and $C = 100$ microfarads. Employing numerical coefficients,

(a) Write as a function of ω and t the expression for the potential drop, Ri, across the resistance.

(b) Repeat (a) for the potential drop, $L\,di/dt$.

(c) Repeat (a) for the potential drop, $1/C\int i\,dt$.

(d) Add (a), (b), and (c) to obtain the potential drop across the entire series combination, and simplify to a single sine wave.

(e) Write the numerical expression for the complex impedance of the entire combination.

(f) Write the expression for the total instantaneous power as a function of ω and t.

(g) What is the average power delivered?

(h) What is the reactive power of the inductance?

(i) What is the reactive power of the capacitance?

(j) What is the total reactive power?

(k) Write, as a function of ω and t, the equation for the instantaneous power delivered to the resistor.

6-8. (a) Write the complex expression for vector power associated with the following vector quantities:

$$\mathbf{V}_m = 100 + j0 \text{ complex volts,}$$
$$\mathbf{I}_m = 3 + j4 \text{ complex amperes.}$$

(b) Draw the corresponding power triangle.

6-9. (a) Draw to scale the power triangles for the circuits of Probs. 6-1 and 6-2. (Watch for the trap!)

(b) Compute the vector powers associated with the triangles of (a).

6-10. Repeat the requirements of Prob. 6-8 for the circuits of Probs. 6-3 and 6-4. (Look out for the trap!)

6-11. (a) Write the complex expression for vector power associated with the following waves:

$$p = 250 - 250 \cos 2\omega t + 125 \sin 2\omega t \text{ watts.}$$

(b) Draw the corresponding power triangle.

6-12. (a) Write the complex expression for the vector power of Prob. 6-7.

(b) Draw the corresponding power triangle.

CHAPTER 7

PARAMETERS IN PARALLEL

7.01. Introduction

Many of the circuit elements encountered in practice may be represented electrically as *parallel*[1] combinations of the basic circuit parameters, R, L, and C. Although we may find the complex impedance of such a combination, it is often convenient to find the reciprocal of the complex impedance—a quantity called *complex admittance.*

7.02. Instantaneous Potential and Current for R and C in Parallel

Let us assume the sinusoidal potential drop

$$\blacktriangleright \qquad v = V_m \sin(\omega t + \theta_v) \qquad (7\text{-}1)$$

to appear across a parallel combination of constant R and C, as shown by Fig. 7.02. Kirchhoff's current law gives us the following expression for total instantaneous current approaching the combination:

$$i_{RC} = i_R + i_C. \qquad (7\text{-}2)$$

From equation (4-80),

$$i_C = C\frac{dv}{dt}. \qquad (7\text{-}3)$$

Equation (7-2) may then be written

$$i_{RC} = \frac{v}{R} + C\frac{dv}{dt}. \qquad (7\text{-}4)$$

Substituting (7-1) yields

$$\begin{aligned} i_{RC} &= \frac{1}{R} V_m \sin(\omega t + \theta_v) + C\frac{d}{dt}[V_m \sin(\omega t + \theta_v)] \\ &= \frac{1}{R} V_m \sin(\omega t + \theta_v) + \omega C V_m \cos(\omega t + \theta_v), \qquad (7\text{-}5) \end{aligned}$$

---

[1] A *parallel* combination of circuit parameters is one such that the parameters are all connected across the same potential difference and share among themselves the total current which approaches the combination.

or

▶ $$i_{RC} = I_{m_{RC}} \sin(\omega t + \theta_i)$$
$$= \left(\frac{1}{R}\right) V_m \sin(\omega t + \theta_v) + \left(\frac{-1}{X_C}\right) V_m \sin(\omega t + \theta_v + 90^\circ), \tag{7-6}$$

where: $X_C = \dfrac{-1}{\omega C}$ = *capacitive reactance*, in ohms.

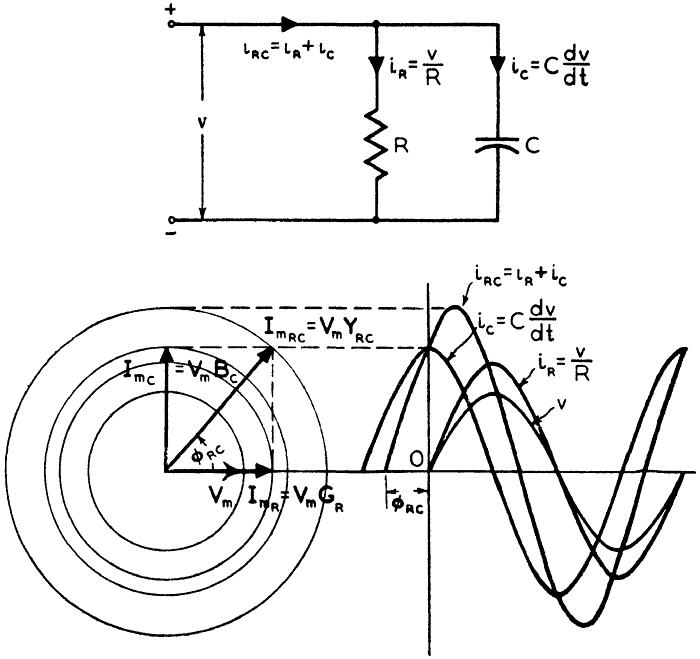


Fig. 7.02. Vectors and waves of potential difference and current for resistance and capacitance in parallel.

Let us now denote $G_R = 1/R$ and $B_C = -1/X_C$. Then equation (7-6) becomes:

▶ $$i_{RC} = G_R V_m \sin(\omega t + \theta_v) + B_C V_m \sin(\omega t + \theta_v + 90^\circ), \tag{7-7}$$

where:

$G_R = \dfrac{1}{R}$ = *conductance for parallel parameters*, in mhos, and

$B_C = \frac{-1}{X_C} = \frac{-1}{\frac{-1}{\omega C}} = \omega C =$ *capacitive susceptance for parallel parameters,* in **mhos.**

These waves are shown in Fig. 7.02 for a particular case.

7.03. Potential and Current Vectors for R and C in Parallel

The vectors corresponding to the waves of the preceding article may be written as follows, assuming $(\omega t + \theta_v) = 0$.

Potential drop:

$$\mathbf{V}_m = V_m + j0 = V_m\underline{/0^\circ}. \tag{7-8}$$

Resistive current :

$$\mathbf{I}_{m_R} = G_R V_m + j0 = G_R V_m\underline{/0^\circ}, \tag{7-9}$$

where: $G_R = \frac{1}{R}$.

Capacitive current:

$$\mathbf{I}_{mC} = 0 + jB_C V_m = B_C V_m\underline{/90^\circ}, \tag{7-10}$$

where: $B_C = \frac{-1}{X_C}$.

Total current:

▶ $$\mathbf{I}_{mRC} = G_R V_m + jB_C V_m \tag{7-11a}$$

$$= \sqrt{(G_R V_m)^2 + (B_C V_m)^2}\underline{/\tan^{-1}\left(\frac{B_C V_m}{G_R V_m}\right)}. \tag{7-11b}$$

Figure 7.02 shows the graphical interpretation of these equations for a particular case. Recalling that $B_C = \omega C$ is positive, we may see from these equations that the current through the capacitance leads the potential drop by 90°. This result is in accord with our previously developed concepts.

7.04. Complex Admittance

From the a-c version of Ohm's law we may write

$$\frac{1}{\mathbf{Z}} = \frac{\mathbf{I}_m}{\mathbf{V}_m}. \tag{7-12}$$

We shall now define *complex admittance* as follows:

▶ $$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}_m}{\mathbf{V}_m}, \tag{7-13}$$

where: $\mathbf{Y}$ = *complex admittance,* in **mhos,** and all other quantities are as previously defined.

7.05. Complex Admittance for R and C in Parallel

From equations (7-8) and (7-11), we may obtain the equation for the complex admittance of a parallel combination of R and C:

$$\blacktriangleright\ \mathbf{Y}_{RC} = \frac{\mathbf{I}_{mRC}}{\mathbf{V}_m} = Y_{RC}\underline{/\phi_{RC}} = \sqrt{G_R^2 + B_C^2}\,\underline{\Big/\tan^{-1}\left(\frac{B_C}{G_R}\right)} = G_R + jB_C, \tag{7-14}$$

where:

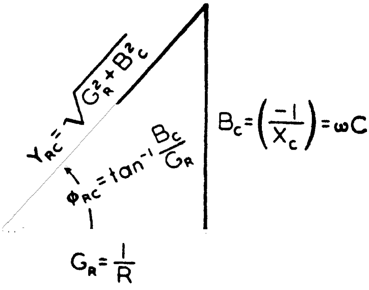
$\mathbf{Y}_{RC}$ = *complex admittance for R-C parallel combination,* in **complex mhos,** and

ϕ_{RC} = *admittance angle for R-C parallel combination,* in **radians** or **degrees;**

$G_R = \dfrac{1}{R}$ = *conductance for parallel parameters,* in **mhos,** and

$B_C = \dfrac{-1}{X_C} = \dfrac{-1}{\dfrac{-1}{\omega C}} = \omega C$ = *capacitive susceptance for parallel parameters,* in **mhos,**

Figure 7.05 shows the admittance triangle for a particular R-C parallel combination.



$Y_{RC} = \sqrt{G_R^2 + B_C^2}$
$\phi_{RC} = \tan^{-1} B_C/G_R$
$B_C = \left(\frac{-1}{X_C}\right) = \omega C$
$G_R = \frac{1}{R}$

Fig. 7.05. Admittance triangle for parallel combination of resistance and capacitance.

We note that complex admittance for R and C in parallel is not a function of time but is a function of frequency. An increase in frequency produces an increase in both the magnitude of admittance and the admittance angle. If the amplitude of the applied potential wave is constant, an increase of frequency results in an increase in the amplitude of total current and an increase in the angle by which current leads the potential drop.

In the *d-c case,* where $\omega = 0$, then $B_C = 0$, and the capacitive path is equivalent to an open switch in parallel with resistance.

Example 7.05. The current $i = 2 \sin (377t - 120°)$ produces a potential drop $v = 110 \sin (377t + 160°)$ across a certain circuit element. Assuming a simple parallel combination, find the values of the parameters comprising this element.

The vectors of current and potential drop are

$$\mathbf{I}_m = 2\underline{/-120^\circ} \qquad \text{and} \qquad \mathbf{V}_m = 110\underline{/160^\circ}.$$

The complex admittance is

$$\mathbf{Y} = G_R + jB = \frac{\mathbf{I}_m}{\mathbf{V}_m} = \frac{2\underline{/-120^\circ}}{110\underline{/160^\circ}} = 0.0182\underline{/80^\circ} = 0.00316 + j0.0179.$$

Then $G_R = 0.00316$ mho, and $B = 0.0179$ mho. The positive value for B shows it to be capacitive. Then

$$B_C = \omega C$$

or $$C = \frac{B_C}{\omega} = \frac{0.0179}{377} = 0.0000475 \text{ farad} = 47.5 \text{ microfarads.}$$

Also $$R = \frac{1}{G_R} = \frac{1}{0.00316} = 316 \text{ ohms.}$$

7.06. Instantaneous Potential and Current for R and L in Parallel

Let us assume that a sinusoidal potential drop appears across a parallel combination of constant R and L, such as shown by Fig. 7.06. The instantaneous potential drop may be expressed as

▶ $$v = V_m \sin (\omega t + \theta_v). \tag{7-15}$$

From Kirchhoff's current law, we know that the total instantaneous current approaching the combination is the sum of the two individual currents:

$$i_{RL} = i_R + i_L. \tag{7-16}$$

From Ohm's law,

$$i_R = \frac{v}{R}, \tag{7-17}$$

and from equation (4-42), recalling that $v = -e$,

$$i_L = \frac{1}{L}\int v\, dt. \tag{7-18}$$

Equation (7-16) may then be written

$$i_{RL} = \left(\frac{1}{R}\right) v + \frac{1}{L}\int v\, dt = I_{m_{RL}} \sin (\omega t + \theta_i). \tag{7-19}$$

Substituting (7-15) in this equation yields

$$i_{RL} = \left(\frac{1}{R}\right) V_m \sin (\omega t + \theta_v) + \frac{1}{L}\int V_m \sin (\omega t + \theta_v)\, dt, \tag{7-20}$$

or

$$
\begin{aligned}
i_{RL} &= I_{mRL} \sin (\omega t + \theta_i) \\
&= \left(\frac{1}{R}\right) V_m \sin (\omega t + \theta_v) - \frac{V_m}{\omega L} \cos (\omega t + \theta_v) + K_3. \qquad (7\text{-}21)
\end{aligned}
$$

Under steady-state conditions, $K_3 = 0$. We may then write

$$
\begin{aligned}
i_{RL} &= I_{mRL} \sin (\omega t + \theta_i) \\
&= \left(\frac{1}{R}\right) V_m \sin (\omega t + \theta_v) - \frac{V_m}{\omega L} \sin (\omega t + \theta_v + 90^\circ), \qquad (7\text{-}22)
\end{aligned}
$$

or

▶
$$
\begin{aligned}
i_{RL} &= I_{mRL} \sin (\omega t + \theta_i) \\
&= \frac{1}{R} V_m \sin (\omega t + \theta_v) + \left(\frac{-1}{X_L}\right) V_m \sin (\omega t + \theta_v + 90^\circ), \qquad (7\text{-}23)
\end{aligned}
$$

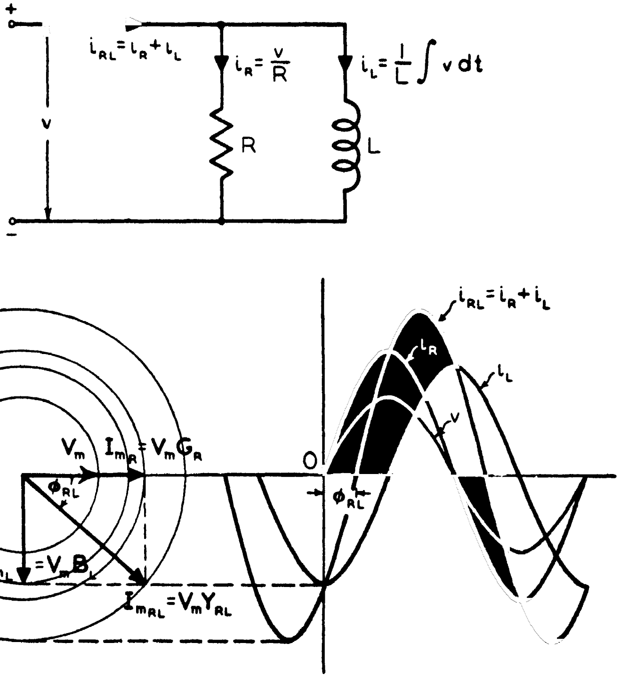


Fig. 7.06. Vectors and waves of potential difference and current for resistance and inductance in parallel.

where: $X_L = \omega L$ = *inductive reactance,* in **ohms.**

Let us now denote

$$G_R = 1/R \qquad \text{and} \qquad B_L = (-1/X_L).$$

Then (7-23) becomes

▶ $$i_{RL} = G_R V_m \sin(\omega t + \theta_v) + B_L V_m \sin(\omega t + \theta_v + 90^\circ), \quad \text{(7-24)}$$

where:

$G_R = \dfrac{1}{R}$ = *conductance for parallel parameters,* in **mhos,** and

$B_L = \left(\dfrac{-1}{X_L}\right) = \left(\dfrac{-1}{\omega L}\right)$ = *inductive susceptance for parallel parameters,* in **mhos.**

These waves are shown in Fig. 7.06 for a particular case.

7.07. Potential and Current Vectors for R and L in Parallel

From our knowledge of the relationship between waves and rotating vectors, we may write the following complex expressions for the vectors associated with a parallel combination of R and L, assuming $(\omega t + \theta_v) = 0$:

Potential drop:

▶ $$\mathbf{V}_m = V_m + j0 = V_m\angle 0^\circ. \qquad \text{(7-25)}$$

Resistive current:

$$\mathbf{I}_{mR} = G_R V_m + j0 = G V_m\angle 0^\circ, \qquad \text{(7-26)}$$

where: $G_R = \dfrac{1}{R}$.

Inductive current:

$$\mathbf{I}_{mL} = 0 + jB_L V_m = B_L V_m\angle 90^\circ, \qquad \text{(7-27)}$$

where: $B_L = \dfrac{-1}{X_L}$.

Total current:

▶ $$\mathbf{I}_{mRL} = G_R V_m + jB_L V_m \qquad \text{(7-28a)}$$

$$= \sqrt{(G_R V_m)^2 + (B_L V_m)^2}\angle \tan^{-1}\left(\frac{B_L V_m}{G_R V_m}\right). \qquad \text{(7-28b)}$$

Figure 7.06 shows the graphical interpretation of these equations. A study of equations (7-25), (7-26), and (7-27) reveals that resistive current is in phase with potential drop, and (since B_L is negative) inductive current lags the potential drop by 90°. This result is in accord with our previously developed concepts.

7.08. Complex Admittance for R and L in Parallel

We may substitute equations (7-11) and (7-14) in (7-16) to obtain the complex admittance for R and L in parallel combination:

$$\mathbf{Y}_{RL} = \frac{\mathbf{I}_{mRL}}{\mathbf{V}_m} = \frac{\sqrt{(G_R V_m)^2 + (B_L V_m)^2}\,\underline{/\tan^{-1}\left(\frac{B_L V_m}{G_R V_m}\right)}}{V_m\underline{/0^\circ}}, \qquad (7\text{-}29)$$

or

$$▶\quad \mathbf{Y}_{RL} = Y_{RL}\underline{/\phi_{RL}} = \sqrt{G_R^2 + B_L^2}\,\underline{/\tan^{-1}\frac{B_L}{G_R}} = G_R + jB_L, \qquad (7\text{-}30)$$

where:

$\mathbf{Y}_{RL}$ = *complex admittance for R-L parallel combination,* in **complex mhos**, and

ϕ_{RL} = *admittance angle for R-L parallel combination,* in **radians** or **degrees**;

$G_R = \frac{1}{R}$ = *conductance for parallel parameters,* in **mhos**, and

$B_L = \frac{-1}{X_L} = \frac{-1}{\omega L}$ = *inductive susceptance for parallel parameters,* in **mhos**.

Figure 7.08 shows the admittance triangle for an R-L parallel combination obtained by graphing equation (7-30).

Equation (7-30) shows that complex admittance for R and L in parallel, though not a function of time, is a function of frequency. An increase of frequency produces a decrease in both the magnitude of admittance and the admittance angle. If the amplitude of the applied potential wave is constant, an increase of frequency results in a decrease in the amplitude of total current and a decrease of total current lag behind potential drop.

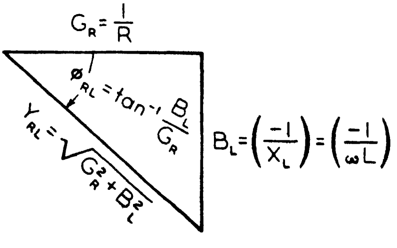

$G_R = \frac{1}{R}$
$\phi_{RL} = \tan^{-1}\frac{B_L}{G_R}$
$Y_{RL} = \sqrt{G_R^2 + B_L^2}$
$B_L = \left(\frac{-1}{X_L}\right) = \left(\frac{-1}{\omega L}\right)$

Fig. 7.08. Admittance triangle for parallel combination of resistance and inductance.

In the *d-c case,* where $\omega = 0$, then $X_L = \omega L = 0$, and $B_L = (-1/X_L) = -\infty$. The inductive path, far from offering opposition to the flow of direct current, constitutes an infinitely conducting short circuit in parallel with the resistance.

EXAMPLE 7.08. The current $i = 2 \sin (377t + 30°)$ produces a potential drop $v = 26 \sin (377t + 45°)$ across a certain circuit element. Find the values of the circuit parameters comprising this element, assuming a simple parallel combination of two pure parameters.

The vectors of current and potential drop are:

$$\mathbf{I}_m = 2\underline{/30°} \qquad \text{and} \qquad \mathbf{V}_m = 26\underline{/45°}.$$

The complex admittance is:

$$\mathbf{Y} = G_R + jB = \frac{\mathbf{I}_m}{\mathbf{V}_m} = \frac{2\underline{/30°}}{26\underline{/45°}} = 0.0769\underline{/-15°} = 0.0742 - j0.0199.$$

Then $G_R = 0.0742$ mho, and $B = -0.0199$ mho. The negative value for B shows it to be inductive. Then

$$B_L = \frac{-1}{X_L} = \frac{-1}{\omega L}$$

or $$L = \frac{-1}{\omega B_L} = \frac{-1}{377 \times (-0.0199)} = 0.1333 \text{ henry} = 133.3 \text{ millihenrys.}$$

Also $$R = \frac{1}{G_R} = \frac{1}{0.0742} = 13.5 \text{ ohms.}$$

7.09. Instantaneous Potential and Current for R, L, and C in Parallel

From our previous work, we can see that the current approaching a parallel combination of R, L, and C will contain three components, one for each parameter. We can also see that the application of a potential difference

▶ $$v = V_m \sin (\omega t + \theta_v) \tag{7-31}$$

will result in a total current

$$i_{RLC} = i_R + i_L + i_C = \frac{1}{R} v + \frac{1}{L} \int v\, dt + C \frac{dv}{dt}, \tag{7-32}$$

which, upon substitution of (7-31), becomes:

$$i_{LRC} = G_R V_m \sin (\omega t + \theta_v) + B_L V_m \cos (\omega t + \theta_v) + B_C V_m \cos (\omega t + \theta_v). \tag{7-33}$$

This may be rewritten:

▶
$$\begin{aligned} i_{RLC} &= I_{m_{RLC}} \sin(\omega t + \theta_i) \\ &= G_R V_m \sin(\omega t + \theta_v) + (B_L + B_C) V_m \cos(\omega t + \theta_v) \\ &= G_R V_m \sin(\omega t + \theta_v) + B_{LC} V_m \sin(\omega t + \theta_v + 90^\circ), \end{aligned} \tag{7-34}$$

where: $B_{LC} = (B_L + B_C) = \left(\dfrac{-1}{\omega L} + \omega C\right) =$ *total susceptance*, in **mhos.**

Figure 7.09 shows the graphical interpretation of this equation for a particular case.

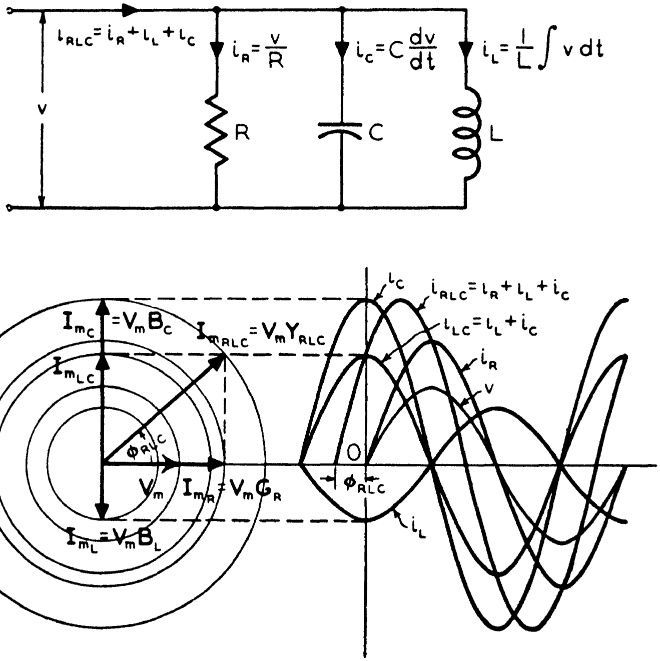


Fig. 7.09. Vectors and waves of potential difference and current for resistance, inductance, and capacitance in parallel.

7.10. Potential and Current Vectors for R, L, and C in Parallel

Figure 7.09 includes the vectors of equation (7-34) at the instant $(\omega t + \theta_v) = 0$. The complex expressions for these vectors are:

Potential drop:

▶
$$\mathbf{V}_m = V_m + j0 = V_m/0^\circ. \tag{7-35}$$

Resistive current:

$$\mathbf{I}_{m_R} = G_R V_m + j0 = G_R V_m\underline{/0^\circ}. \tag{7-36}$$

Inductive current:

$$\mathbf{I}_{m_L} = 0 + jB_L V_m = B_L V_m\underline{/90^\circ}, \tag{7-37}$$

where: $B_L = \dfrac{-1}{X_L} = \dfrac{-1}{\omega L}.$

Capacitive current:

$$\mathbf{I}_{m_C} = 0 + jB_C V_m = B_C V_m\underline{/90^\circ}, \tag{7-38}$$

where: $B_C = \dfrac{-1}{X_C} = \omega C.$

Total current:

$$\blacktriangleright \quad \mathbf{I}_{m_{RLC}} = G_R V_m + jB_{LC} V_m \tag{7-39a}$$

$$= \sqrt{(G_R V_m)^2 + (B_{LC} V_m)^2}\underline{\Big/ \tan^{-1}\left(\frac{B_{LC} V_m}{G_R V_m}\right)}, \tag{7-39b}$$

where: $B_{LC} = (B_L + B_C) = \left(\dfrac{-1}{\omega_L} + \omega C\right) =$ *net susceptance,* in **mhos.**

7.11. Complex Admittance for R, L, and C in Parallel

The complex admittance for a parallel combination of R, L, and C is·

$$\blacktriangleright \quad \mathbf{Y}_{RLC} = \frac{\mathbf{I}_{m_{RLC}}}{\mathbf{V}_m} = Y_{RLC}\underline{/\phi_{RLC}} = \sqrt{G_R^2 + B_l^2}\underline{\Big/ \tan^{-1}\left(\frac{B_{LC}}{G_R}\right)}$$

$$= G_R + jB_l, \tag{7-40}$$

where:

Y_{RLC} = *complex admittance for R-L-C parallel combination,* in **complex mhos,**

ϕ_{RLC} = *admittance angle for R-L-C parallel combination,* in **radians** or **degrees.**

$G_R = \dfrac{1}{R}$ = *conductance for parallel parameters,* in **mhos,** and

$B_{LC} = (B_L + B_C) = \left(\dfrac{-1}{\omega L} + \omega C\right)$ = *net susceptance for parallel parameters,* in **mhos.**

Figure 7.11 shows the admittance triangle for a particular *R-L-C* combination.

Complex admittance for R, L, and C in parallel is a function of frequency. Any variation of frequency causes the net susceptance, $B_{LC} = [(-1/\omega L) + \omega C]$, to vary in such manner that it may appear to be inductive, equal to zero, or capacitive, depending on the relative values of $|-1/\omega L|$ and $|\omega C|$. Equation (7-40) may be considered the general expression for the admittance of parallel combinations of parameters. If C is absent from the circuit, then B_C will equal zero in (6-40). If no parallel inductance is present in the combination, the inductive susceptance, B_L, equals zero. This condition is equivalent to the presence of a parallel inductance of value $L = \infty$. Similarly, the absence of a conductance is equivalent to the presence of a resistive path of value $R = \infty$. In this case, $G_R = 1/R = 0$.

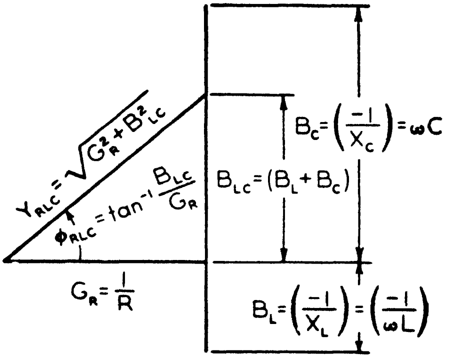

$Y_{RLC} = \sqrt{G_R^2 + B_{LC}^2}$
$\phi_{RLC} = \tan^{-1} \frac{B_{LC}}{G_R}$
$G_R = \frac{1}{R}$
$B_C = \left(\frac{-1}{X_C}\right) = \omega C$
$B_{LC} = (B_L + B_C)$
$B_L = \left(\frac{-1}{X_L}\right) = \left(\frac{-1}{\omega L}\right)$

Fig. 7.11. Admittance triangle for parallel combination of resistance, inductance, and capacitance.

In the steady-state *d-c case*, where $\omega = 0$, the admittance of a parallel combination of pure parameters which includes inductance is infinite. Under this condition the inductance is equivalent to a short circuit. If the parallel combination contains no inductance but consists only of R and C, the capacitance exerts no influence (it resembles an open switch), and the net admittance equals the conductance.

EXAMPLE 7.11. If a potential drop $v = 100 \sin (377t)$ appears across a simple parallel combination of circuit parameters, the corresponding total current is $i = 2 \sin (377t + 10^\circ)$. If a potential drop $v = 100 \sin (157t)$ appears across the combination, the corresponding total current is $i = 20 \sin (157t - 84.35^\circ)$. Find the values of the circuit parameters.

Test one ($\omega = 377$): $\mathbf{V}_{m_1} = 100\underline{/0^\circ}$; $\mathbf{I}_{m_1} = 2\underline{/10^\circ}$;

$$\mathbf{Y}_1 = \frac{2\underline{/10^\circ}}{100\underline{/0^\circ}} = 0.02\underline{/10^\circ} = 0.01968 + j0.00347 = G_1 + jB_1.$$

Test two ($\omega = 157$): $\mathbf{V}_{m_2} = 100\underline{/0^\circ}$; $\mathbf{I}_{m_2} = 20\underline{/-84.35^\circ}$;

$$\mathbf{Y}_2 = \frac{20\underline{/-84.35^\circ}}{100\underline{/0^\circ}} = 0.2\underline{/-84.35^\circ} = 0.01968 - j1.995 = G_2 + jB_2.$$

The value of conductance for this simple parallel combination of parameters is observed to remain constant, and the susceptance is observed to change from a capacitive to an inductive nature as frequency decreases. This behavior suggests that both inductance and capacitance are present in the parallel combination. We may now write the following two simultaneous equations:

$$B_1 = \left(\frac{-1}{\omega_1 L} + \omega_1 C\right); \qquad 0.00347 = \left(\frac{-1}{377L} + 377C\right),$$

and

$$B_2 = \left(\frac{-1}{\omega_2 L} + \omega_2 C\right); \qquad -1.995 = \left(\frac{-1}{157L} + 157C\right).$$

Simultaneous solution of these equations yields:

$$L = 0.00260 \text{ henry} = 2.60 \text{ millihenrys},$$

and

$$C = 2.70 \times 10^{-3} \text{ farad} = 2700 \text{ microfarads}.$$

Also,

$$R = \frac{1}{G_R} = \frac{1}{0.01968} = 50.9 \text{ ohms}.$$

7.12. Instantaneous Power Quantities for Parameters in Parallel

Because circuit parameters connected in parallel all experience the same potential difference, it is convenient to utilize the quantity, potential drop, as "reference" in writing the power equation for a combination of parameters in parallel. Referring to equation (6-45), we may write the power equation thus by setting $\theta_v = 0$ and then choosing the instant when $t = 0$ to write the complex expressions for potential and current vectors. The negative of the initial phase angle of the resultant current then becomes equal to the angular phase difference between the potential and current waves, since $\theta = (\theta_v - \theta_i) = -\theta_i$. We then have the following expressions for potential drop and current:

▶ $$v = V_m \sin \omega t, \tag{7-41}$$

and

▶ $$i = I_m \sin (\omega t - \theta) \tag{7-42}$$

Equation (6-45) for instantaneous power assumes the following form when potential drop is used as reference:

$$p = \left[\frac{V_mI_m}{2}\cos(\theta)\right] - \left[\frac{V_mI_m}{2}\cos(-\theta)\right]\cos 2\omega t + \left[\frac{V_mI_m}{2}\sin(-\theta)\right]\sin 2\omega t, \quad (7\text{-}43)$$

or

$$p = \left[\frac{V_mI_m}{2}\cos\theta\right] - \left[\frac{V_mI_m}{2}\cos\theta\right]\cos 2\omega t - \left[\frac{V_mI_m}{2}\sin\theta\right]\sin 2\omega t. \quad (7\text{-}44)$$

This equation is observed to be identical with (6-50) except for the sign of the last term. Let us rewrite this equation as:

▶ $$p = \left[\frac{V_mI_m}{2}\cos\theta\right](1 - \cos 2\omega t) + \left[\frac{-V_mI_m}{2}\sin\theta\right]\sin 2\omega t = p_a + p_r, \quad (7\text{-}45)$$

where:

p = *instantaneous power*, in **watts**, and

$p_a = \left[\frac{V_mI_m}{2}\cos\theta\right](1 - \cos 2\omega t)$ = *instantaneous active power*, in **watts**,

$p_r = \left[\frac{-V_mI_m}{2}\sin\theta\right]\sin 2\omega t$ = *instantaneous reactive power*, in **vars**.

Figure 7.12 shows the graphical interpretation of this equation. It

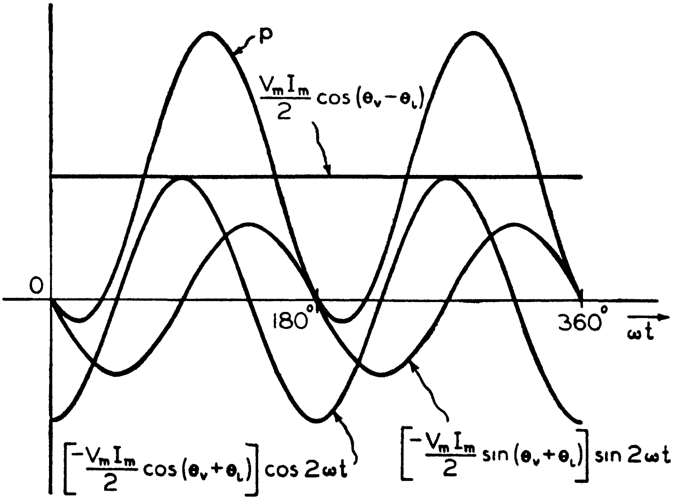


Fig. 7.12. Graphical representation of equation (6-45) for a particular case in which $\theta_v = 0$ degrees.

will be observed that only the sign of reactive power changes as we shift from the equation which employs current as references to that which employs potential drop as reference.

7:13. Active and Reactive Power Associated with Parallel Parameters

From previous definitions (Article 6.14) we may write the following equations for *active power* associated with parameters in simple parallel combination:

▶ $$P = \frac{V_m I_m}{2} \cos \theta = P_{\text{ave}}, \tag{7-46}$$

where: P = *active power*, in **watts.**

Also, we may write the following equation for *reactive power* associated with parameters in simple parallel combination:

▶ $$Q = \frac{-V_m I_m}{2} \sin \theta \tag{7-47}$$

where: Q = *reactive power*, in **vars.**

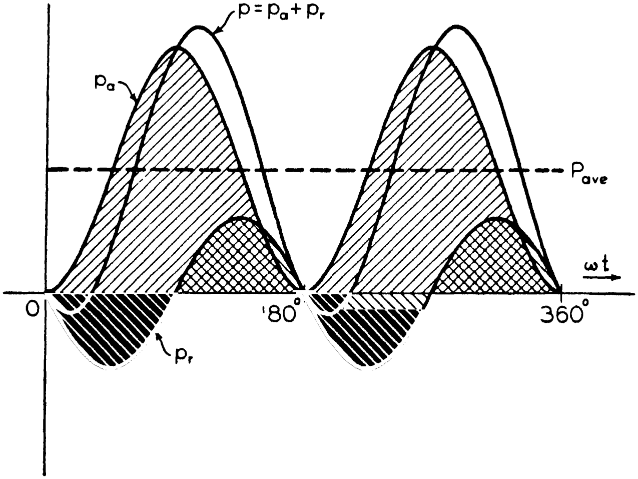


Fig. 7.13. Active and reactive components of instantaneous power for a particular case in which $\theta_v = 0$ degrees.

This is like equation (6-53) for the series case, except for algebraic sign.

For thirteen years the American Institute of Electrical Engineers adhered to a definition, adopted in 1935 by the International Electrotechnical Commission, which arbitrarily designated all inductive reactive power as negative and all capacitive reactive power as positive. In January 1948 the Standards Committee of the Institute officially recommended that this arbitrary designation be reversed to conform with what appears to be predominant practice in the United States. As we can see from the derivation of equations (6-51) and (7-45), it really makes no difference in solving a given problem, as long as inductive and capacitive reactive power are given opposite algebraic signs. The notation which we have employed leads to negative vars for the inductive case if current is used as reference, and to positive vars for the inductive case if potential drop is used as reference.

7.14. Vector Power Associated with Parallel Parameters

The definition of vector power given by equation (6-55) may be utilized for the case of parallel parameters:

$$\blacktriangleright\ \mathbf{U}' = P + jQ = \sqrt{P^2 + Q^2}\,\underline{/\tan^{-1}\left(\frac{Q}{P}\right)} = U'\,\underline{/\tan^{-1}\left(\frac{Q}{P}\right)}$$

$$= \left(\frac{V_m I_m}{2}\cos\theta\right) + j\left(\frac{-V_m I_m}{2}\sin\theta\right) = \frac{V_m I_m}{2}\underline{/-\theta}, \quad (7\text{-}48)$$

where: $\theta = (\theta_v - \theta_i)$ = *phase angle between potential drop and current.*

Figure 7.14 shows the power triangle for a particular set of values of R, L, and C in parallel.

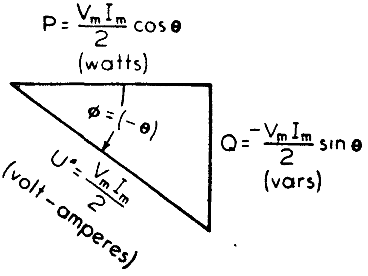


Fig. 7.14. Triangle of power quantities at a sink, potential drops considered positive and inductive reactive vars considered negative (potential is reference).

Example 7.14. Given the potential drop $v = 100 \sin (377t)$ and the associated current $i = 2 \sin (377t - 30°)$.

(a) Write the expression for the instantaneous power, and point out the active and reactive components.
(b) Write the complex expression for the vector power.

(*a*) *Instantaneous power:*

$$p = \left[\frac{2 \times 100}{2}\cos 30°\right](1 - \cos 2 \times 377t) + \left[-\frac{2 \times 100}{2}\sin 30°\right]\sin 2 \times 377t$$

$$= 86.6(1 - \cos 754t) - 50 \sin 754t \text{ watts};$$

$$p_a = 86.6(1 - \cos 754t) \text{ watts}; \qquad p_r = -50 \sin 754t \text{ vars}.$$

(*b*) *Vector power:*

$$\mathbf{U}' = 86.6 - j50 = 100\underline{/-30°} \text{ complex volt-amperes;}$$

$$P = 86.6 \text{ watts}; \qquad Q = -50 \text{ vars.}$$

PROBLEMS

7-1. An alternating emf wave,

$$e = 14.14 \cos (6283t - 90°) \text{ volts,}$$

when applied to a particular circuit element, results in a current wave,

$$i = 0.1 \sin (6283t + 10°) \text{ amperes.}$$

(a) Write the numerical expression for the complex admittance of the element.

(b) Assuming the element to consist of no more than two pure parameters in parallel, find the values of the parameters.

(c) Employing numerical coefficients, write as functions of ω and t the equations for the currents through the individual parallel branches.

(d) Employing numerical coefficients, write the expression for power as a function of ω and t.

(e) Compute the average power.

(f) Sketch all the above waves to the same set of rectangular coordinate axes.

(g) Write the complex expressions for the vectors associated with all the foregoing currents and potentials, and draw the corresponding vector diagrams.

(h) Show that the total vector current is the vector sum of the individual vector currents.

7-2. An emf, $e = 50 \sin \omega t$ volts, is applied to a circuit element consisting of two pure parameters in parallel. The equation for the resulting power wave is:

$$p = 500 - 500 \cos 2\omega t - 700 \sin 2\omega t \text{ watts.}$$

(a) What is the equation of the wave of total current?

(b) What are the values of the circuit parameters?

7-3. A 25-cycle alternating emf wave having an amplitude of 163 volts is applied to a circuit consisting of 8 ohms of pure resistance in parallel with 0.050 henry of pure inductance. Using numerical coefficients,

(a) Write the expression for complex admittance in both rectangular and polar forms.

(b) Write the expression for the applied emf as a function of ω and t, making de/dt negative and $e = 0$ at $t = 0$.

(c) Write the expression for the total current as a function of ω and t.

(d) Write the expressions for the two branch currents as functions of ω and t.

(e) Write the expression for power as a function of ω and t.

(f) Compute the average power.

(g) Write the complex expressions for the vectors associated with all the foregoing current and potential quantities, and draw the corresponding vector diagrams.

(h) Sketch all the above waves to the same set of coordinate axes.

7-4. A current, $i = 2 \sin (377t - 35°)$ amperes, causes a potential drop, $v = 17.8 \cos (377t - 90°)$ volts, in a certain element. Carry out the requirements for Prob. 7-1 for these two waves.

7-5. (*a*) What is the instantaneous active power of Prob. 7-1?

(b) What is the instantaneous reactive power of Prob. 7-1?

(c) What is the active power of Prob. 7-1?

(d) What is the reactive power of Prob. 7-1?

7-6. (a) Write the equation of the power wave for the resistive parameter of Prob. 7-4.

(b) Write the equation for the power wave for the reactive parameter of Prob. 7-4.

(c) Show that the sum of the solutions to (a) and (b) is equivalent to the instantaneous power wave, as obtained by substitution in equation (7-44).

7-7. A current, $i = 2.12 \sin 157t$ amperes, flows through a circuit element consisting of pure R, pure L, and pure C in parallel combination, in which $R = 5$ ohms, $L = 100$ millihenrys, and $C = 100$ microfarads. Employing numerical coefficients,

(a) Write as a function of ω and t the expression for the resistive current, $(1/R)v = G_R v$.

(b) Repeat (a) for the current, $C \dfrac{dv}{dt}$.

(c) Repeat (b) for the current, $1/L \int v \cdot dt$.

(d) Add (a), (b), and (c) to obtain the total current approaching the parallel combination, and simplify to a single sine wave.

(e) Write the numerical expression for the complex admittance of the entire combination.

(f) Write the expression for the total instantaneous power as a function of ω and t.

(g) What is the average power delivered?

(h) What is the reactive power of the capacitance?

(i) What is the reactive power of the inductance?

(j) What is the total reactive power?

(k) Write, as a function of ω and t, the equation of the instantaneous power delivered to the resistor.

7-8. (a) Write the complex expression for vector power associated with the following vector quantities.

$$\mathbf{V}_m = 100 + j0 \text{ complex volts,}$$
$$\mathbf{I}_m = 3 + j4 \text{ complex amperes.}$$

(b) Draw the corresponding power triangle.

(c) Find the values of the two pure parameters in parallel combination which the above current would approach upon application of the above potential difference.

7-9. (a) Draw to scale the power triangle for the circuit of Prob. 7-7.

(b) Write the complex expression for vector power in both rectangular and polar forms.

CHAPTER 8

DUALS; EFFECTIVE CURRENT AND POTENTIAL; COMPLEX POWER COMPUTATIONS

8.01. Principle of Duality

The parallel-parameter relations of Chapter 7 were purposely developed in a pattern resembling as closely as possible the development of series-parameter relations in Chapter 6 to facilitate grasping the principle of *duality* as it applies to electric circuits. This principle may be stated broadly by saying that *two circuits are "duals" (or are "dual to each other") if their differential equations have the same form but the coefficients and variables are interchanged as follows: series connections are replaced by parallel connections, potentials by currents, charges by flux linkages, potential sources by current sources, open circuits by short circuits, loops by nodes, inductance by capacitance, resistance by reciprocal resistance, and reciprocal capacitance by reciprocal inductance; or vice versa.*

A careful comparison of the equations developed in Chapter 7 for parameters in parallel, with the corresponding equations developed in Chapter 6 for parameters in series, will point out this duality strikingly. We note, for example, that the following pairs of equations are reciprocal in nature:

$$v = Ri \qquad \text{and} \qquad i = G_R v, \tag{8-1}$$

and

$$v = \mathrm{L}\frac{di}{dt} \qquad \text{and} \qquad i = C\frac{dv}{dt}. \tag{8-2}$$

An extension of this comparison reveals that Kirchhoff's current and potential laws constitute a pair of dual relationships. In terms of instantaneous values, these are

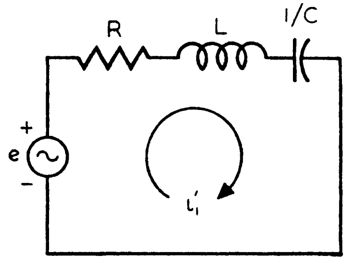
$$\Sigma i = 0 \qquad \text{and} \qquad \Sigma v = 0. \tag{8-3}$$

The first states that the algebraic sum of all currents approaching a given point in a circuit must be zero; and the second states that the algebraic sum of potential differences around a closed path traversed through an electric circuit is zero. We can make effective use of this particular duality in our study of networks in a later portion of this book. Figure 8.01 illustrates in terms of circuit diagrams the duality existing between the following two equations from Chapters 6 and 7:

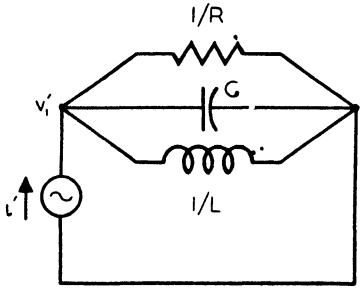
$$v = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \tag{8-4a}$$

and

$$i = G_R v + C\frac{dv}{dt} + \frac{1}{L}\int v\,dt. \tag{8-4b}$$



(a) Series circuit of equation (8-4a).



(b) Parallel circuit of equation (8-4b).

Fig. 8.01. A single-mesh circuit and its dual, the single-node circuit.

The reader can probably appreciate already at least one way in which the principle of duality may be useful. This lies in the fact that if the analysis of one member of the dual pair is known, it can be applied to the other directly by merely interchanging symbols. Thus, all the relations of Chapter 7 could have been obtained immediately by substitution of dual quantities in the relations of Chapter 6. In Chapter 17, we shall shorten our analyses of transients considerably by developing only those relations applying to various combinations of parameters in series. The corresponding relations for parallel combination easily follow. The reader would do well, from time to time, to determine for himself the duals of the relations which he is currently studying in this book, even though the text may make no special mention of them.

Another way in which the principle of duality may be useful lies in the fact that a knowledge of duality may suggest alternative methods of accomplishing a given result with a saving of time or effort (see Prob. 8-2). It may even suggest new theorems or relations, particularly in the more advanced analyses of complicated networks and of machinery.

8.02. Effective Value of Alternating Current

Thus far, we have worked exclusively with sinusoidal waves of potential or current, or with rotating vectors whose lengths were equal to the amplitudes of the waves for which they stood. We may well wonder what an *a-c ammeter*, or electric current-measuring instrument, indicates when it is utilized to measure the value of alternating current in a circuit. It cannot indicate the average value of the current, since the average value of a sinusoid is always zero, and most alternating currents consist of one or more sinusoidal components. A glance at Fig. 3.27A should convince us that the maximum value does not mean much unless the wave is sinusoidal. The maximum value of a nonsinusoidal wave does not tell anything about the manner in which the current varies in attaining that maximum value.

When an alternating-current ammeter tells us that a given conductor is carrying, for example, one ampere, it means that the resistance of that conductor is converting electric energy to heat at the same rate as though one ampere of direct current were flowing through the conductor instead. *The ampere value of an alternating current is the same as that of a direct current which could produce the same average heating effect in a given resistance.* Mathematically,

▶ $$RI^2 = RI_{\text{d-c}}^2 \tag{8-5}$$

where:

I = *effective value of alternating current,* in **amperes,** and

$I_{\text{d-c}}$ = *value of equivalent direct current,* in **amperes.**

We should like to discover some means of relating the effective value of an alternating current to its equation as a function of time. This result can be accomplished by equating (8-5) to the expression for average rate of heat production over one cycle of the alternating current. Mathematically,

$$RI^2 \quad \frac{1}{T}\int_0^T Ri^2\,dt, \tag{8-6}$$

where:

T = *the period, or duration of one cycle,* in **seconds,** and
i = *instantaneous current,* in **amperes.**

This equation may be solved for I, to give the following *root-mean-square* (*rms*) formula for the effective value of an alternating current:

$$I = \sqrt{\frac{1}{T}\int_0^T i^2\,dt}, \tag{8-7}$$

where:

I = *effective,* or *root-mean-square* (*rms*) *value* of the alternating current, in **amperes,**
i = *instantaneous value* of the current, in **amperes,**
t = *time,* in **seconds,** and
T = *period* of one cycle of the current wave, in **seconds.**

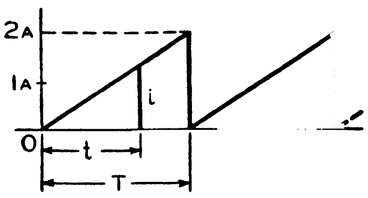


Fig. 8.02. Current of Example 8.02.

This equation is *perfectly general* in that it applies to periodic currents of any wave form. If the equation of the wave, i, may be written as a function of time, the effective value of that wave may be determined analytically through the use of equation (8-7). If the equation of i cannot be written but the wave form can be graphed, the procedure implied by (8-7) can be carried out graphically. Both procedures are employed in the next article for the sine wave of Fig. 8.03.

EXAMPLE 8.02. The current of a given circuit starts at zero and increases linearly to 2 amperes, after which it drops back to zero in negligible time. Find the effective value of this current cycle.

Referring to Fig. 8.02, we can see that

$$\frac{i}{t} = \frac{2}{T}, \qquad \text{or} \qquad i = \frac{2}{T}t.$$

Then $I = \sqrt{\frac{1}{T}\int_0^T \left(\frac{2}{T}t\right)^2 dt} = \sqrt{\frac{4}{T^3}\left[\frac{t^3}{3}\right]_0^T} = \frac{2}{\sqrt{3}}$ amperes.

8.03. Effective Value of Sinusoidal Current

Because sine waves of current and potential are so extensively used in alternating-current circuit computations, it is convenient to derive an expression for the effective value of a sinusoid in terms of its amplitude. Let us consider the sinusoidal current

$$i = I_m \sin(\omega t + \theta_i). \tag{8-8}$$

The effective value is

$$I = \sqrt{\frac{1}{T}\int_0^T I_m^2 \sin^2(\omega t + \theta_i)\,dt} = \sqrt{\frac{I_m^2}{2T}\int_0^T [1 - \cos 2(\omega t + \theta_i)]\,dt}$$

$$= \sqrt{\frac{I_m^2}{2T}\left[t - \frac{1}{2\omega}\sin 2\left(\frac{2\pi}{T}t + \theta_i\right)\right]_0^T} = \sqrt{\frac{I_m^2}{2}} \tag{8-9}$$

and

▶ $$I = \frac{I_m}{\sqrt{2}}, \tag{8-10}$$

where:

I = *effective value of a sinusoidal current,* in **amperes,** and
I_m = *amplitude of sinusoidal current,* in **amperes.**

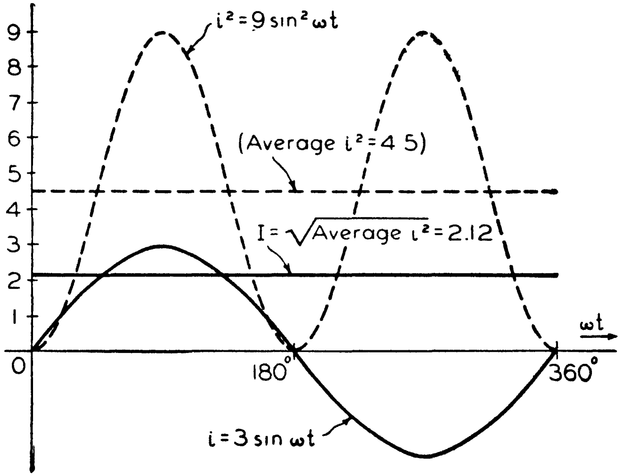


Fig. 8.03. Effective value of a particular sine wave.

Figure 8.03 shows a graphical interpretation of equation (8-7) for the case of a sinusoidal current. The effective value is observed to be independent of time.

8.04. Average Value of Alternating Current

By definition (Article 2.06), the average value of a true alternating current is zero over an integral number of cycles. It is sometimes useful to determine the average value of one loop of an alternating-current wave. This *half-cycle average value* is what is usually meant by the term "average value" of an alternating current. It equals the average value of a full-wave rectified alternating current. The *average value* of one loop of an alternating current wave is expressed mathematically as

$$\blacktriangleright \qquad I_{\text{ave}} = \frac{2}{T}\int_0^{T/2} i\,dt, \qquad (8\text{-}11)$$

where time measurement is understood to begin at the point where the current wave crosses the zero axis.

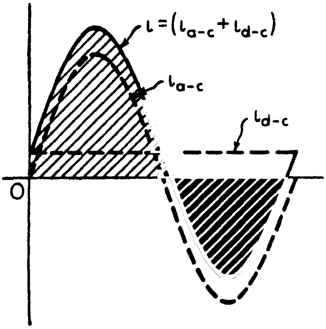


Fig. 8.04. Equivalence of a displaced a-c wave to a symmetrical sine wave plus a d-c component.

If the current is not truly an alternating current by the definition of Article 2.06 but has positive and negative loops which differ in average value, then the actual full-cycle average value represents the value of a d-c component present in the wave. Figure 8.04 indicates, for example, that a displaced a-c wave is equivalent to a true a-c wave plus a d-c component.

Example 8.04. Find the average value of the current wave of Example 8.02.

We found that $i = \frac{2}{T}t$. We also note that, in this case, we are seeking the average value over a full cycle, inasmuch as this wave is unidirectional as well as periodic (see Fig. 8.02). We may then write

$$I_{\text{ave}} = \frac{1}{T}\int_0^T i\,dt = \frac{1}{T}\int_0^T \frac{2}{T}t\,dt$$
$$= \frac{2}{T^2}\left[\frac{t^2}{2}\right]_0^T = 1 \text{ ampere.}$$

8.05. Average Value of Sinusoidal Current

The average value of a sinusoidal current is defined as the average value of one complete loop (one half cycle), inasmuch as the average value of a sinusoid taken over a complete cycle is zero. The average value of a sinusoidal current is

$$I_{\text{ave}} = \frac{2}{T}\int_0^{T/2} I_m \sin \omega t\, dt = \frac{2I_m}{\omega T}\left[-\cos \omega t\right]_0^{T/2} \qquad (8\text{-}12)$$

or, since $\omega = 2\pi f = 2\pi/T$,

▶ $$I_{\text{ave}} = \frac{2}{\pi} I_m, \qquad (8\text{-}13)$$

where: I_{ave} = *average value* (*over one loop*) of *sinusoidal current,* in **amperes.**

8.06. Effective and Average Values of Alternating Potential

The expressions for effective and average values of alternating potential may be obtained by substituting the symbols E and e (or V and v) for the symbols I and i in the expressions for effective and average current.

8.07. Form Factor of a Symmetrical Alternating Quantity

"*The form factor of a symmetrical alternating quantity is the ratio of the effective value of the quantity to its half-period average value.*"[1] Mathematically, form factor for a potential wave is:

▶ $$\text{Form factor} = \frac{E}{E_{\text{ave}}} = \frac{\sqrt{\dfrac{1}{T}\displaystyle\int_0^T e^2\, dt}}{\dfrac{2}{T}\displaystyle\int_0^{T/2} e\, dt}. \qquad (8\text{-}14)$$

Form factor has but little physical significance, since it gives no certain indication of wave form. A peaked wave usually has a higher value than a flat-topped wave, although the complexity of the wave form may render this statement invalid in certain cases. Form factor is used principally as at least a rough indication of the relative hysteresis loss that will exist when a potential difference is impressed on a coil wound on an iron core, and for determining effective induced emf's in such a coil.

8.08. Crest Factor of a Periodic Quantity

"*The crest factor of a periodic quantity is the ratio of the crest value to the effective value of the quantity.*"[2] The *crest value* is the maximum value which the quantity attains. For a sinusoid, it is the amplitude. The crest factor for a potential wave is

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[1] *American Standard Definitions of Electrical Terms,* No. 05.05.260. New York: American Institute of Electrical Engineers, 1942.

[2] *Ibid.,* No. 05.05.266.

$$\blacktriangleright \qquad \text{Crest factor} = \frac{E_m}{E}. \tag{8-15}$$

This ratio is sometimes called also the *peak factor* or the *amplitude factor*. Since the dielectric stress to which insulation is subjected depends upon the maximum value of the impressed potential difference, this factor is important in insulation testing.

8.09. Vector Notation Employing Effective Values

Great emphasis has been placed thus far on the fact that the rotating current and potential vectors employed in electrical engineering must, to "generate" waves of the proper amplitude, possess lengths equal to the amplitudes. Consequently, all complex representation of currents and potentials thus far has utilized maximum values of the quantity being represented.

In practice, computations are frequently based on values of current, potential, and power indicated by instruments placed in the circuit under study. Normally, the instruments employed in measurements of alternating potentials and currents indicate effective values. As a consequence, assuming sinusoidal quantities, theoretically we must multiply such a-c ammeter and voltmeter indications by $\sqrt{2}$ to convert[3] effective values to maximum values before we may write either the equations of the time-varying waves or the complex expressions for the associated vectors. After performing computations, we usually desire, in practice, to obtain the results as effective (or "meter-reading") values. This step requires dividing the computational results by $\sqrt{2}$, if they are alternating potentials or currents. We realize, however, that in terms of obtaining correct results, we might have accomplished the same thing by bothering neither to multiply nor to divide by the $\sqrt{2}$, but by employing effective, rather than maximum, values in the complex representation of current and potential vectors. The following equations exemplify this principle for the case of vector addition. If we are given

$$\mathbf{I}_{m_a} + \mathbf{I}_{m_b} = \mathbf{I}_{m_c}, \tag{8-16}$$

---

[3] In the *d-c case*, effective, maximum, and average values of a current or of a potential are one and the same, and no conversion factor is needed. This identity is shown by making i and e constant in the defining equations.

we may divide through by $\sqrt{2}$ without affecting the equality:

$$\frac{\mathbf{I}_{m_a}}{\sqrt{2}} + \frac{\mathbf{I}_{m_b}}{\sqrt{2}} = \frac{\mathbf{I}_{m_c}}{\sqrt{2}}. \qquad (8\text{-}17)$$

This equation may, however, be written in terms of effective values, thus demonstrating the principle:

$$\mathbf{I}_a + \mathbf{I}_b = \mathbf{I}_c. \qquad (8\text{-}18)$$

In line with this principle, it has become customary to minimize labor by expressing the lengths of potential and current vectors in terms of effective, rather than maximum, values. Those cases in which maximum values are employed are readily recognizable by the presence of the subscript *m* on all symbols for current or potential. In the absence of this subscript, these quantities are understood to be expressed in terms of effective values.

Despite this labor-saving custom, the reader must never forget that there are no such things as steady-state time-varying "waves" of effective value generated by rotating vectors expressed in terms of effective values. Under steady-state conditions, the effective value of a wave appears graphically as a line parallel to the time axis—it is the "d-c equivalent" of the time-varying quantity. For this reason, it is never correct to employ effective values instead of maximum values in writing equations of alternating potentials or currents as functions of time.

Let us now look back through the relations which we have developed for circuits containing sinusoidal currents and potentials. Many of these contain the maximum value of potential or current. By substitution of $\sqrt{2}\, I = I_m$ and $\sqrt{2}\, E = E_m$, we may obtain these relations in terms of effective, or "meter-reading," values. As examples, three of these relations follow, expressed both in original form and in terms of effective values.

The "a-c version" of Ohm's law:

▶ $$\mathbf{Z} = \frac{\mathbf{V}_m}{\mathbf{I}_m} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{\mathbf{Y}}. \qquad (8\text{-}19)$$

Complex expression for a vector quantity:

▶ $$\left.\begin{cases} \mathbf{I}_m = I_m\underline{/\theta_i} = (I_m \cos\theta_i + jI_m \sin\theta_i) \\ \mathbf{I} = I/\theta_i = (I\cos\theta_i + jI\sin\theta_i) \end{cases}\right\}. \qquad (8\text{-}20)$$

Vector power:

▶
$$\begin{aligned}\mathbf{U}' &= \left(\frac{V_mI_m}{2}\cos\theta\right) + j\left(\frac{\pm V_mI_m}{2}\sin\theta\right) = \frac{V_mI_m}{2}\underline{/\pm\theta}\\ &= (VI\cos\theta) + j(\pm VI\sin\theta) = VI\underline{/\pm\theta}\\ &= P + jQ = U'\underline{/\pm\theta}. \end{aligned} \tag{8-21}$$

Example 8.09. The currents $i_1 = 14.14 \sin(\omega t + 30^\circ)$ amperes and $i_2 = 7.07 \sin(\omega t - 15^\circ)$ amperes approach a common junction point in an electric circuit. (a) Determine the indication of an ammeter carrying the resultant current leaving that junction point. (b) Write the equation of the resultant current as a function of time.

(a) In terms of effective values:

$$\mathbf{I}_1 = \frac{14.14}{\sqrt{2}}\underline{/30^\circ} = 10\underline{/30^\circ} = 10(0.866 + j0.500) = 8.66 + j5.00,$$

$$\mathbf{I}_2 = \frac{7.07}{\sqrt{2}}\underline{/-15^\circ} = 5\underline{/-15^\circ} = 5(0.966 - j0.259) = 4.83 - j1.29,$$

$$\mathbf{I}_{12} = \mathbf{I}_1 + \mathbf{I}_2 = 13.49 + j3.71 = 13.99\underline{/15.4^\circ} = I_{12}\underline{/\theta_{12}} \text{ complex amperes.}$$

The ammeter indication $I_{12} = 14.85$ amperes.

(b) The equation of the resultant current leaving the junction is

$$\begin{aligned} i_{12} = I_{m_{12}} \sin(\omega t + \theta_{12}) &= 14.85\sqrt{2}\sin(\omega t + 25^\circ)\\ &= 21 \sin(\omega t + 25^\circ) \text{ amperes.}\end{aligned}$$

8.10. Power Computations Employing Complex Quantities

Figure 8.10 shows the real and imaginary components of a current vector

▶
$$\mathbf{I} = I\underline{/\theta_i} = I_x + jI_y, \tag{8-22}$$

and a potential drop vector

▶
$$\mathbf{V} = V\underline{/\theta_v} = V_x + jV_y. \tag{8-23}$$

From equation (8-21), we know that the active power is given by

$$P = VI\cos\theta. \tag{8-24}$$

Referring to Figure 8.10, we may rewrite this as follows:

$$\begin{aligned} P &= VI\cos(\theta_v - \theta_i)\\ &= VI(\cos\theta_v\cos\theta_i + \sin\theta_v\sin\theta_i)\\ &= VI\left(\frac{V_x}{V}\cdot\frac{I_x}{I} + \frac{V_y}{V}\cdot\frac{I_y}{I}\right). \end{aligned} \tag{8-25}$$

This equation reduces to:

▶
$$P = (V_xI_x + V_yI_y), \tag{8-26}$$

where:

P = *active power*, in **watts**,
V_x, V_y = *real, imaginary components of vector potential drop*, in **volts**,
I_x, I_y = *real, imaginary components of vector current*, in **amperes.**

Due regard must be paid to the algebraic signs of the vector components in using this equation. A positive value of power indicates power *leaving* the circuit, inasmuch as V_x and V_y are components of the vector of potential *drop*.

In similar manner, we may derive the equation for reactive power associated with the two vectors of Fig. 8.10:

$$\begin{aligned} Q &= \pm VI \sin \theta \\ &= \pm VI \sin (\theta_v - \theta_i) \\ &= \pm VI(\sin \theta_v \cos \theta_i - \cos \theta_v \sin \theta_i) \\ &= \pm VI\left(\frac{V_y}{V}\cdot\frac{I_x}{I} - \frac{V_x}{V}\cdot\frac{I_y}{I}\right). \end{aligned} \tag{8-27}$$

This simplifies to

▶
$$Q = \pm(V_yI_x - V_xI_y), \tag{8-28}$$

where:

Q = *reactive power*, in **vars**,
V_x, V_y = *real, imaginary components of vector potential drop*, in **volts**,
I_x, I_y = *real, imaginary components of vector current*, in **amperes.**

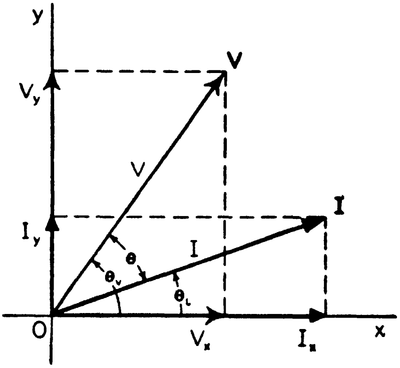


Fig. 8.10. Components of potential and current vectors.

As was pointed out in Article 7.13, the choice of algebraic sign depends on the choice of "reference." If current is considered to be the reference, the positive sign is used in equation (8-28), and inductive reactive power is positive. If potential drop is considered to be the reference, the negative sign is used in equation (8-28), and inductive reactive power is negative.

8.11. Complex Vector Power by the Method of Conjugates

We have seen that, if potential and current are expressed as complex quantities in terms of effective values, the *magnitude* of vector power is equal to the *product of the magnitudes* of potential and current:

$$U' = VI. \tag{8-29}$$

It is natural to wonder whether, similarly, the vector product of the complex potential multiplied by the complex current equals the complex vector power. The answer is that this product has no physical significance, and that

$$\mathbf{U}' \neq \mathbf{VI}. \tag{8-30}$$

Despite the fact that the vector product of complex potential and complex current is meaningless, it turns out that if we take the vector product of one of the two quantities multiplied by the *conjugate* of the other, the result is the expression for complex vector power. If we take the *conjugate of current,* inductive vars will be *positive.* If we take the *conjugate of potential,* inductive vars will be *negative.* Otherwise, it makes no difference which conjugate is employed. These statements are easily verified as follows.

Let us assume the complex expressions of current and potential drop expressed in terms of effective values to be

$$\mathbf{I} = I_x + jI_y = I\underline{/\theta_i}, \tag{8-31}$$

and

$$\mathbf{V} = V_x + jV_y. \tag{8-32}$$

The *conjugates* of these two quantities are, respectively,

$$\hat{\mathbf{I}} = I_x - jI_y = I\underline{/-\theta_i}, \tag{8-33}$$

and

$$\hat{\mathbf{V}} = V_x - jV_y = V\underline{/-\theta_v}. \tag{8-34}$$

Using first the *conjugate of I* and employing the polar form, we obtain

$$\blacktriangleright\ \mathbf{U}' = \mathbf{V}\hat{\mathbf{I}} = (V\underline{/\theta_v})(I\underline{/-\theta_i}) = VI\underline{/(\theta_v - \theta_i)}$$
$$= VI\underline{/\theta} = (VI\cos\theta + jVI\sin\theta) = P + jQ. \tag{8-35a}$$

Using instead the *conjugate of V* and employing the polar form, we obtain

▶ $$\begin{aligned} \mathbf{U}' = \hat{\mathbf{V}}\mathbf{I} &= (V\angle{-\theta_v})(I\angle{\theta_i}) = VI\angle{-(\theta_v - \theta_i)} \\ &= VI\angle{-\theta} = (VI\cos\theta - jVI\sin\theta) = P + jQ. \end{aligned} \quad \text{(8-35b)}$$

These two results are seen to differ only in the sign of reactive power.

Equivalent results can be obtained through the use of the rectangular form, in which case the final relations are

▶ $$\begin{aligned} \mathbf{U}' = \mathbf{V}\hat{\mathbf{I}} &= (V_xI_x + V_yI_y) + j(V_yI_x - V_xI_y) \\ &= P + jQ, \end{aligned} \quad \text{(8-36a)}$$

and

▶ $$\begin{aligned} \mathbf{U}' = \hat{\mathbf{V}}\mathbf{I} &= (V_xI_x + V_yI_y) - j(V_yI_x - V_xI_y) \\ &= P + jQ. \end{aligned} \quad \text{(8-36b)}$$

The form of equations (8-36) is interesting in that it gives us handy rules for computing values of active and reactive power from the real and imaginary components of potential and current:

If the rectangular forms of potential and current are given, active power is the product of the real components plus the product of the imaginary components.

If the rectangular forms of potential and current are given, reactive power is the product of one real component and one imaginary component minus the product of the other real component and the other imaginary component.

EXAMPLE 8.11. Given the current $\mathbf{I} = 3 - j4 = 5\angle{-53.1°}$ amperes and the potential drop $\mathbf{V} = 86.6 + j50 = 100\angle{30°}$ volts. (a) Using the polar form, find the complex vector power. (b) Repeat with rectangular form.

$$\begin{aligned} &\text{(a)} & \mathbf{U}' = \mathbf{V}\hat{\mathbf{I}} &= (100\angle{30°})(5\angle{+53.1°}) = 500\angle{83.1°} \\ && &= 60 + j496.4 \text{ complex volt-amperes.} \\ && P &= 60 \text{ watts;} \qquad Q = 496.4 \text{ vars.} \\ &\text{(b)} & \mathbf{U}' = \mathbf{V}\hat{\mathbf{I}} &= (86.6 + j50)(3 + j4) \\ && &= 59.8 + j496.4 \text{ complex volt-amperes.} \\ && P &= 59.8 \text{ watts;} \qquad Q = 496.4 \text{ vars.} \end{aligned}$$

The two sets of results are seen to check well.

8.12. In-phase and Quadrature Components of Potential and Current

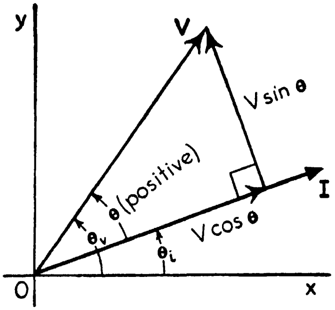
The following equations have been developed for computing active and reactive power:

$$P = VI\cos\theta = (V_xI_x + V_yI_y), \tag{8-37}$$

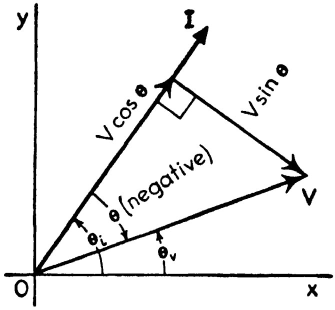
and

$$Q = \pm VI\sin\theta = \pm(V_yI_x - V_xI_y), \tag{8-38}$$

where $\theta = (\theta_v - \theta_i)$ is the angle by which **V** leads **I** (or by which **I** lags **V**). We have stated that the choice of algebraic sign for Q depends on whether **V** or **I** is used as reference. It is the purpose of this article to discuss that statement more thoroughly.

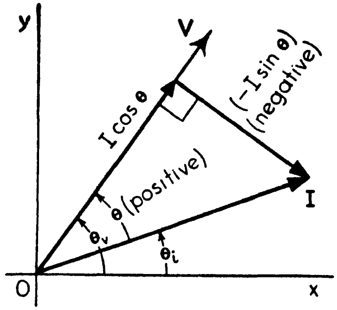


(a) Inductive case

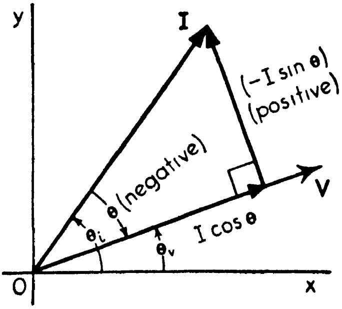


(b) Capacitive case.

Fig. 8.12A. In-phase and quadrature components of potential.



(a) Inductive case.



(b) Capacitive case.

Fig. 8.12B. In-phase and quadrature components of current.

Figure 8.10 showed vectors of potential and current for a particular physical situation. Figure 8.12A shows these same vectors resolved into different components.

From this figure, it is clear that we may rewrite equations (8-37) and (8-38) as follows:

▶ $$P = I(V\cos\theta), \tag{8-39}$$

and

▶ $$Q = I(+V \sin \theta). \qquad (8\text{-}40)$$

In this figure and in these two equations, we have employed the vector **I** as reference in finding two components of potential. The first of these, ($V \cos \theta$), is the *in-phase component* of V; and the second, ($V \sin \theta$), is the *quadrature component* of V. Because θ, for the inductive case, is positive, and the quadrature component of V is a "leading" vector, the positive sign for this component is justified. *This condition makes Q positive for the inductive case when I is used as reference.*

Figure 8.12B shows an alternative interpretation of Fig. 8.10. In this case, the vector **V** is used as reference in finding two components of current. The corresponding interpretations of equations (8-37) and (8-38) are

▶ $$P = V(I \cos \theta), \qquad (8\text{-}41)$$

and

▶ $$Q = V(-I \sin \theta), \qquad (8\text{-}42)$$

where ($I \cos \theta$) is the *in-phase component* of **I**, and ($-I \sin \theta$) is the *quadrature component* of **I**. Because θ for the inductive case is positive and yet the quadrature component of I is a "lagging" vector, the negative sign must be used with this component. *This condition makes Q negative for the inductive case when V is used as reference.*

8.13. Graphical Summary of Relations for Series Parameters

Figure 8.13 shows in graphical form a summary of relations applying to simple series combinations of "pure" circuit parameters. Because parameters in series carry a common current, current has been drawn horizontally and serves as reference in all diagrams. A thorough study of this chart is recommended. Note particularly the progression from *impedance diagram* through *vector diagram* to *power diagram* for each element. Each diagram may be obtained from the one above it by multiplying all sides by *effective* current, I. The corresponding complex expressions for the quantities represented by the hypotenuses of these triangular diagrams are given. Our later detailed analysis of the series circuit will be based on a similar chart.

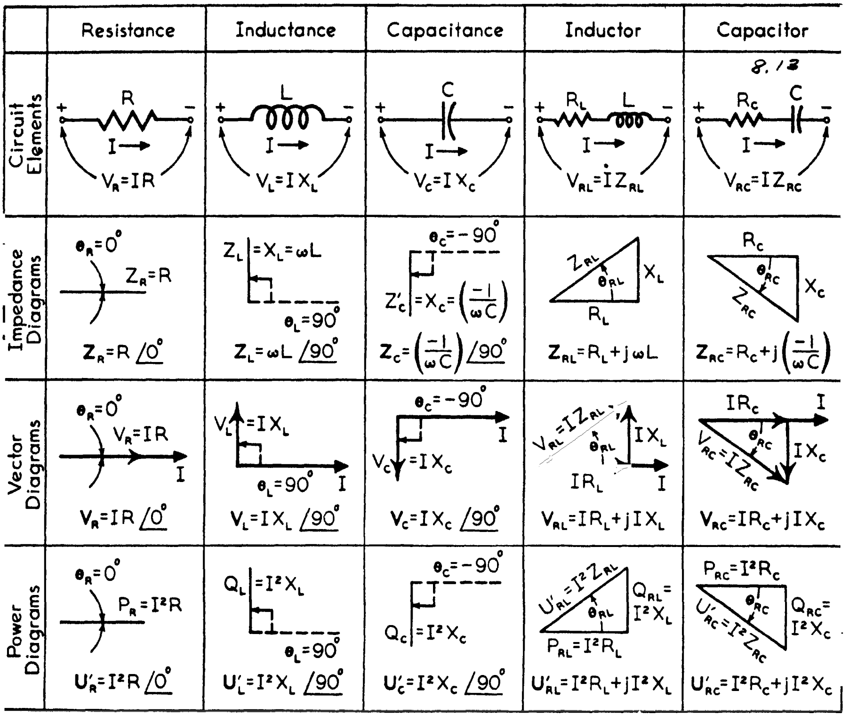


Fig. 8.13. Graphical summary of relations applying to typical series circuit elements.

8.14. Graphical Summary of Relations for Parallel Parameters

Figure 8.14 shows in graphical form a summary of relations applying to simple parallel combinations of "pure" circuit parameters. Because parameters in parallel experience a common potential drop, potential has been drawn horizontally and serves as reference in all diagrams. Here the progression from *admittance diagram* through *vector diagram* to *power diagram* should be studied for each element. Each diagram may be obtained from the one above it by multiplying all sides by effective potential drop, *V*. The corresponding complex expressions for the quantities represented by the hypotenuses of these triangular diagrams are given. Our later detailed analysis of the parallel circuit will be based on a similar chart. *This figure should be compared with Fig.* 8.13 *with an eye to noting duality.*

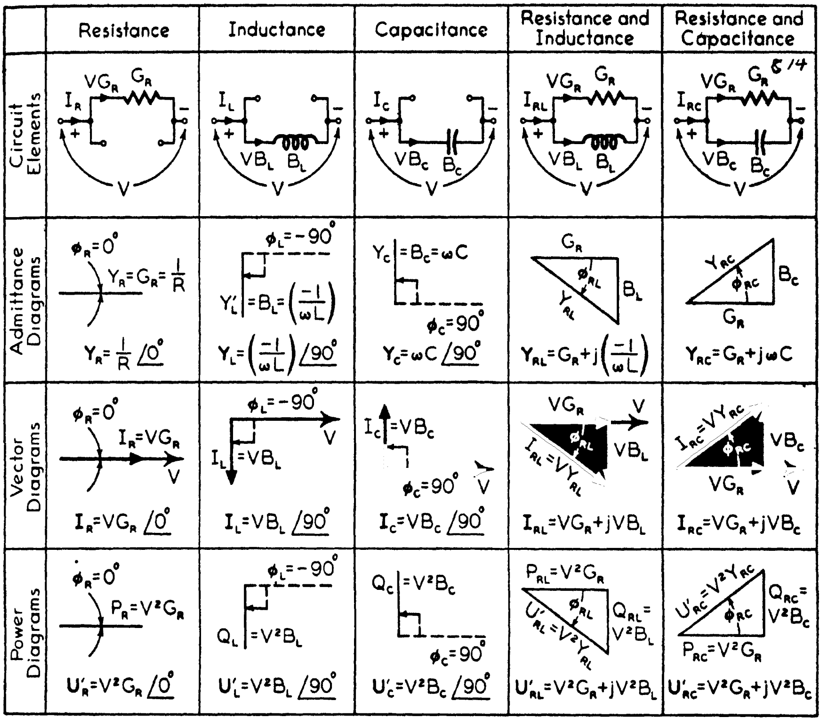


Fig. 8.14. Graphical summary of relations applying to typical parallel circuit elements.

PROBLEMS

8-1. (a) What is the dual of a circuit consisting of a series combination of the following parameters?

$$R = 2 \text{ ohms}, L = 10 \text{ millihenrys, and } C = 10 \text{ microfarads.}$$

(b) Determine the equation of the wave of current resulting if the following emf is applied to the original series circuit of (a):

$$e = 100 \sin \omega t \text{ volts.}$$

(c) Determine the equation of the wave of total current resulting if the same emf is applied to the dual of the original series circuit, and if the members of the dual are assigned the following values: $R = 2$ ohms, $L = 10$ millihenrys, and $C = 10$ microfarads.

(d) Is the dual of a circuit the electrical equivalent of that circuit? Explain.

8-2. Equation (17-34) gives the following expression for current flowing through a short circuit placed across a series combination of an initially charged capacitor and a resistor:

$$i = -\frac{E_0}{R}\epsilon^{-\frac{t}{RC}},$$

where:

i = *instantaneous value of current at time t,*

E_0 = *initial potential across capacitor terminals due to initial charge of capacitor,*

R = *series resistance,*

C = *capacitance,*

ϵ = *natural logarithmic base.*

Using the principle of duality,

(a) Write the expression for instantaneous potential appearing at the terminals of a parallel combination of inductance and resistance whose external circuit, containing the source of emf, has suddenly been opened.

(b) Write the equation for the decay of current in the inductance under this circumstance. This current, as it decays, circulates through the shunt resistance. The solution to this part of the problem is, therefore, the equation for decay of current in a series R-L circuit whose emf has suddenly been replaced by a short circuit!

8-3. The current in a certain circuit starts at zero and increases linearly to a value of 5 amperes in 1 millisecond It remains constant thereafter for 3 milliseconds, after which it drops to zero in an additional 2 milliseconds. It remains at zero for 4 milliseconds, after which the cycle repeats

(a) Determine the indication of an a-c ammeter in this circuit.

(b) Determine the indication of a d-c ammeter in this circuit.

(c) What is the form factor of this wave?

8-4. A certain potential wave has positive and negative loops which are semicircles. Determine its crest factor.

8-5. A motor of a certain hoist takes 40 amperes for 15 seconds, after which the power is off for 50 seconds It then draws 30 amperes for 10 seconds, after which the power is off for 90 seconds. If this duty cycle is repeated continuously, what must be the continuous current rating of the motor?

8-6. The current of a given circuit starts at 0 1 ampere and increases linearly to 0.15 ampere, after which it drops in negligible time back to 0.1 ampere, and the cycle repeats.

(a) If the current passes through a d-c milliammeter in series with an a-c milliammeter, what should be the indication of each instrument?

(b) If the resistance of the circuit is constant, which of the above indications should be employed in determining the power of the circuit by the formula $P = I^2R$?

8-7. Determine the indication of an a-c voltmeter across whose terminals is applied a potential wave,

$$e = 141.4 \sin \omega t - 70.7 \sin 3\omega t \text{ volts.}$$

8-8. The currents in the two branches of a parallel circuit are: $i_1 = 10 \sin (\omega t - 10°)$ and $i_2 = 5 \sin (\omega t + 20°) + 2 \sin (3\omega t - 30°)$ amperes.

(a) Write the equation for the wave of total current approaching the parallel combination.

(b) Find the indication of an a-c ammeter carrying the total current

(c) Find the indication of a d-c ammeter carrying the total current.

8-9. Find the real power and the reactive power associated with each of the following pairs of vector potentials and vector currents, and sketch the power triangles: (a) $\mathbf{V}_m = 100 + j50$ volts; $\mathbf{I}_m = 10 - j2$ amperes. (b) $\mathbf{V} = -10 - j30$ volts; $\mathbf{I} = -7 + j2$ amperes.

8-10. Find the real power and the reactive power associated with each of the following pairs of vector potentials and vector currents, and sketch the power triangles: (a) $\mathbf{V} = 110/\underline{120°}$ volts; $\mathbf{I} = 2/\underline{90°}$ amperes. (b) $\mathbf{V} = 120/\underline{240°}$ volts; $\mathbf{I} = 5/\underline{180°}$ amperes.

8-11. A certain 220-volt, 5-horsepower motor draws 25 amperes at a lagging power factor of 0.8 under rated load conditions.

(a) Find the active, reactive, and vector power taken by the motor under rated conditions.

(b) What is the reactive factor?

(c) What is the efficiency of the motor?

8-12. (a) Using potential as reference, draw a vector diagram showing the in-phase and quadrature components of current associated with the condition of Prob. 8-11.

(b) Employing numerical coefficients, write as functions of ω and t the equations of all quantities represented by the vectors of (a).

8-13. Repeat Prob. 8-12, except use current as reference.

8-14. A voltmeter, an ammeter, and a wattmeter are employed to obtain the following information concerning the behavior of a certain coil to which a 60-cycle sinusoidal potential is applied: $V = 102$ volts; $I = 3.6$ amperes; $W = 180$ watts.

(a) Draw to scale the impedance triangle for this coil, clearly indicating the numerical value of each part of the diagram.

(b) Using current as reference, draw to scale a vector diagram showing in-phase and quadrature components of potential, clearly indicating the numerical value of each component.

(c) Draw to scale a power triangle for this coil.

(d) What is the value of the reactive parameter?

(e) What are the over-all power factor and the reactive factor?

8-15. Repeat Prob. 8-14 for a circuit element containing capacitance and resistance in series, for which the same test results were obtained.

8-16. A voltmeter, an ammeter, and a wattmeter are employed to obtain the following information concerning the behavior of a certain inductive circuit element to which a 60-cycle sinusoidal potential is applied: $V = 102$ volts; $I = 3.6$ amperes; $W = 180$ watts.

Assuming the element to consist of pure R in parallel with pure L,

(a) Draw to scale the admittance triangle for this element, clearly indicating the numerical value of each part of the diagram.

(b) Using potential as reference, draw to scale a vector diagram showing clearly in-phase and quadrature components of current, clearly indicating the numerical value of each component.

(c) Draw to scale a power triangle for this element.

(d) What is the value of the reactive parameter?

(e) What are the over-all power factor and reactive factor?

8-17. Repeat Prob. 8-16 for a circuit element containing capacitance and resistance in parallel, for which the same test results were obtained.

CHAPTER 9

COMPLEX IMPEDANCES IN SERIES

9.01. Impedances in Series

Figure 9.01 shows complex impedances, $\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \ldots, \mathbf{Z}_n$, so connected that all carry the same current. This type is known as a *series combination* of complex impedances. The net impedance of the series combination is found by application of Kirchhoff's potential law and Ohm's law. By Kirchhoff's potential law,

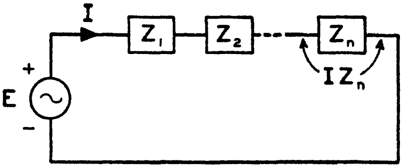


Fig. 9.01. Complex impedances in series.

▶ $$\mathbf{V}_s = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 + \cdots + \mathbf{V}_n. \quad (9\text{-}1)$$

By Ohm's law, dividing by the complex current will yield the complex impedance. The net impedance of the series combination of impedances is, then,

$$\mathbf{Z}_s = \frac{\mathbf{V}_s}{\mathbf{I}_s} = \frac{\mathbf{V}_1}{\mathbf{I}_1} + \frac{\mathbf{V}_2}{\mathbf{I}_2} + \frac{\mathbf{V}_3}{\mathbf{I}_3} + \cdots + \frac{\mathbf{V}_n}{\mathbf{I}_n}, \qquad (9\text{-}2)$$

or

▶ $$\mathbf{Z}_s = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \cdots + \mathbf{Z}_n, \qquad (9\text{-}3)$$

where:

$\mathbf{Z}_s$ = *net impedance of series combination*, in **complex ohms,**
$\mathbf{Z}_n$ = *impedance of the nth series element*, in **complex ohms.**

We have found that *the net impedance of a series combination of impedances is the complex sum of the individual impedances.* This statement is true for *any* given frequency, including zero cycles per second.

9.02. Resistances in Series

For the case of a series of pure resistances, equation (9-3) may be written:

▶ $$R_s = R_1 + R_2 + R_3 + \cdots + R_n, \tag{9-4}$$

where all quantities are in **ohms.**

9.03. Inductances in Series

For the case of a series of pure self inductances, equation (9-3) may be written:

$$jX_{L_s} = jX_{L_1} + jX_{L_2} + jX_{L_3} + \cdots + jX_{L_n}, \tag{9-5a}$$

or

$$j\omega L_s = j\omega L_1 + j\omega L_2 + j\omega L_3 + \cdots + j\omega L_n. \tag{9-5b}$$

Multiplying through by $(1/j\omega)$ produces the following rule for combining self inductances in series:

▶ $$L_s = L_1 + L_2 + L_3 + \cdots + L_n, \tag{9-6}$$

where all quantities are in **henrys.**

9.04. Capacitances in Series

For the case of a series of pure capacitances, equation (9-3) may be written:

$$jX_{C_s} = jX_{C_1} + jX_{C_2} + jX_{C_3} + \cdots + jX_{C_n}, \tag{9-7a}$$

or

$$j\frac{-1}{\omega C_s} = j\frac{-1}{\omega C_1} + j\frac{-1}{\omega C_2} + j\frac{-1}{\omega C_3} + \cdots + j\frac{-1}{\omega C_n}\cdot \tag{9-7b}$$

Multiplying through by $(-\omega/j)$ produces the following rule for combining capacitances in series:

▶ $$\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_n}\cdot \tag{9-8}$$

where all quantities are in **farads.**

9.05. Admittance of Parameters in Series

Chapter 7 developed potential, current, and power relations applying to various parallel combinations of *pure parameters.* Complex admittance, conductance, and susceptance were defined for such special cases of pure parameters. At best, in practice, circuit elements only approximate pure parameters. A completely resistanceless inductor, for example, has never been built; all capacitors have some power loss, although it may be small; at some frequency value, the capacitance of an inductor may prove highly significant, to cite further examples. Many practical circuit elements must be treated as *combinations* of pure parameters. At a given frequency, it is possible to represent a particular practical circuit element as a series combination of resistance and reactance—either inductive or capacitive. As a result, when we deal with parallel combinations of circuit elements, we find ourselves dealing with parallel combinations of complex impedances, not of pure parameters. For this reason, it is important that we be able to determine the *complex admittance of a series combination of parameters.*

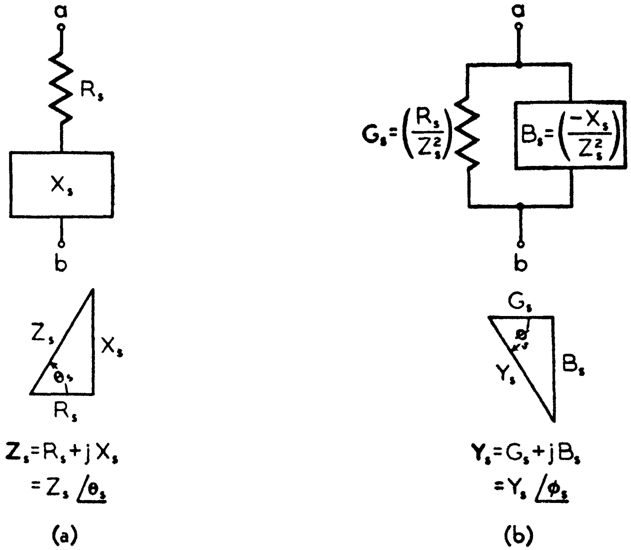


Fig. 9.05. Admittance of parameters in series. (a) Parameters in series. (b) Corresponding admittance.

Referring to Fig. 9.05, let us denote the complex impedance of a series combination of parameters as:

$$\blacktriangleright \qquad \mathbf{Z}_s = R_s + jX_s = Z_s\underline{/\theta_s}, \qquad (9\text{-}9)$$

where:

$\mathbf{Z}_s$ = *complex impedance of series combination of parameters*, in complex ohms,
R_s = ΣR = *total series resistance*, in ohms, and
X_s = $\Sigma X = \Sigma X_L + \Sigma X_C$ = *total series reactance*, in ohms.

By equation (7-13), complex admittance is defined as:

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}_m}{\mathbf{V}_m}. \tag{9-10}$$

The complex admittance of a series combination of parameters is, then,

▶ $$\mathbf{Y}_s = \frac{1}{\mathbf{Z}_s} = \frac{1}{R_s + jX_s} = \frac{1}{Z_s\underline{/\theta_s}}. \tag{9-11a}$$

This equation may be rationalized as follows:

$$\mathbf{Y}_s = \frac{1}{\mathbf{Z}_s} = \frac{1}{R_s + jX_s} \cdot \frac{R_s - jX_s}{R_s - jX_s} = \frac{R_s - jX_s}{R_s^2 + X_s^2} = \frac{R_s - jX_s}{Z_s^2}, \tag{9-11b}$$

or $$\mathbf{Y}_s = \frac{1}{\mathbf{Z}_s} = \left(\frac{R_s}{Z_s^2}\right) + j\left(\frac{-X_s}{Z_s^2}\right) = \left(\frac{1}{Z_s}\right)\underline{/-\theta_s}. \tag{9-11c}$$

We learned, however, that the *real component* of complex admittance is *conductance*, and that the *imaginary component* is *susceptance*. Equation (9-11) may then be written:

▶ $$\mathbf{Y}_s = \frac{1}{\mathbf{Z}_s} = Y_s\underline{/\phi_s} = \sqrt{G_s^2 + B_s^2}\,\underline{/\tan^{-1}\left(\frac{B_s}{G_s}\right)} = G_s + jB_s, \tag{9-12}$$

where:

$\mathbf{Y}_s$ = *complex admittance of series parameter combination,* in **complex ohms,**

$G_s = \left(\frac{R_s}{Z_s^2}\right)$ = *conductance of series parameter combination,* in **mhos,** and

$B_s = \left(\frac{-X_s}{Z_s^2}\right)$ = *susceptance of series parameter combination,* in **mhos.**

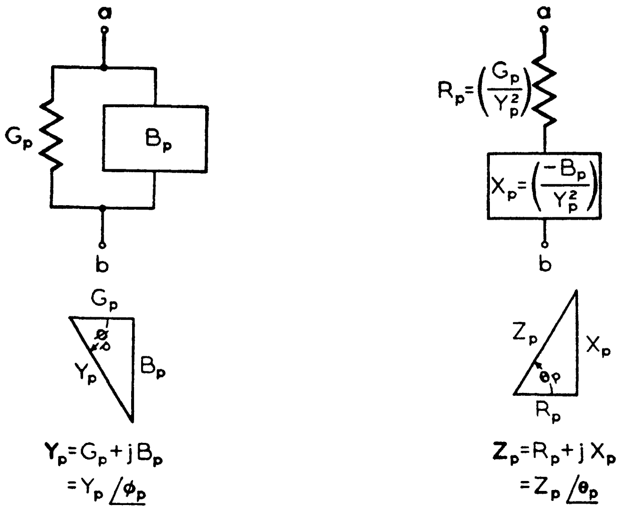
$\phi_s = (-\theta_s)$ = *admittance angle of series parameter combination.*

Graphically, equation (9-12) represents an admittance triangle similar to those of Chapter 7.

It is highly important for the reader to note that equation (9-12) *may be used to provide a* **general definition of complex admittance,** *and of its components,* **conductance,** *and* **susceptance.** The relations of Chapter 7 are merely special cases, which rarely, if ever, occur in practice.

9.06. Impedance of Parameters in Parallel

Just as a practical circuit element may be represented, at a given frequency, by a series combination of resistance and reactance, so it may also be represented by a parallel combination of pure circuit parameters. In making computations involving circuit elements which are so represented, we frequently find it necessary to be able to write expressions for their impedances. The *complex impedance of a parallel combination of parameters* may readily be found as follows.



(a) (b)

Fig. 9.06. Impedance of parameters in parallel. (a) Parameters in parallel. (b) Corresponding admittance impedance.

Referring to Fig. 9.06, let us denote the complex admittance of a parallel combination of parameters as:

$$\blacktriangleright \qquad \mathbf{Y}_p = G_p + jB_p = Y_p\angle\phi_p, \qquad (9\text{-}13)$$

where:

$\mathbf{Y}_p$ = *complex admittance of parallel combination of parameters*, in **complex mhos**,

$G_p = \Sigma G$ = *total parallel conductance*, in **mhos**, and

$B_p = \Sigma B = \Sigma B_L + \Sigma B_C$ = *total parallel susceptance*, in **mhos**.

By equation (7-13), complex impedance may be expressed as:

$$\mathbf{Z} = \frac{1}{\mathbf{Y}} = \frac{\mathbf{V}_m}{\mathbf{I}_m}. \tag{9-14}$$

The complex impedance of a parallel combination of parameters, is, then,

▶ $$\mathbf{Z}_p = \frac{1}{\mathbf{Y}_p} = \frac{1}{G_p + jB_p} = \frac{1}{Y_p\underline{/\phi_p}}. \tag{9-15a}$$

This equation may be rationalized as follows:

$$\mathbf{Z}_p = \frac{1}{\mathbf{Y}_p} = \frac{1}{G_p + jB_p} \cdot \frac{G_p - jB_p}{G_p - jB_p} = \frac{G_p - jB_p}{G_p^2 + B_p^2} = \frac{G_p - jB_p}{Y_p^2}, \tag{9-15b}$$

or

▶ $$\mathbf{Z}_p = \frac{1}{\mathbf{Y}_p} = \left(\frac{G_p}{Y_p^2}\right) + j\left(\frac{-B_p}{Y_p^2}\right) = \left(\frac{1}{Y_p}\right)\underline{/-\phi_p} \tag{9-15c}$$

The *real component* of this equation must be the *equivalent resistance* of the parallel combination, and the *imaginary component* must be the *equivalent reactance*. This requirement leads us to write

▶ $$\mathbf{Z}_p = \frac{1}{\mathbf{Y}_p} = Z_p\underline{/\theta_p} = \sqrt{R_p^2 + X_p^2}\,\underline{/\tan^{-1}\left(\frac{X_p}{R_p}\right)} = R_p + jX_p, \tag{9-16}$$

where:

$\mathbf{Z}_p$ = *complex impedance of parallel combination of parameters*, in **complex ohms,**

$R_p = \left(\frac{G_p}{Y_p^2}\right)$ = *equivalent resistance of parallel combination*, in **ohms,**

$X_p = \left(\frac{-B_p}{Y_p^2}\right)$ = *equivalent reactance of parallel combination*, in **ohms,**

$\theta_p = (-\phi_p)$ = *equivalent impedance angle of parallel combination of parameters.*

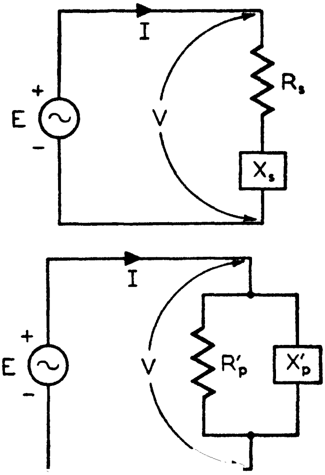
The dualism which exists between equations (9-12) and (9-16) should be noted.

9.07. Equivalent Simple Series and Parallel Combinations

Figure 9.07A shows two simple combinations of resistance and reactance which may be made *electrically equivalent.* By this statement we mean that the substitution of one of these combinations for another in an electric circuit would in no way affect the over-all current, potential, and power relations. This electrical equivalence must not be confused with the dualism which has been stressed thus far. The two impedance combinations are not duals, as we shall soon be able to see.

$$R_s = \frac{(X'_p)^2}{(X'_p)^2 + (R'_p)^2} R'_p$$

$$X_s = \frac{(R'_p)^2}{(X'_p)^2 + (R'_p)^2} X'_p$$



$$R'_p = R_s + \frac{X_s^2}{R_s}$$

$$X'_p = X_s + \frac{R_s^2}{X_s}$$

Fig. 9.07A. Equivalent series and parallel impedance combinations.

If the two impedance combinations are to be electrically equivalent, we must be able to write, using the notation of the previous two articles

$$\mathbf{Z}_s = R_s + jX_s = \mathbf{Z}_p = R_p + jX_p, \tag{9-17}$$

or
$$R_s + jX_s = \left(\frac{G_p}{Y_p^2}\right) + j\left(\frac{-B_p}{Y_p^2}\right). \tag{9-18}$$

Equating separately the real terms, we find that

$$R_s = \frac{G_p}{Y_p^2} = \frac{G_p}{G_p^2 + B_p^2}. \tag{9-19}$$

Equating separately the imaginary terms yields, similarly,

$$X_s = \frac{-B_p}{Y_p^2} = \frac{-B_p}{G_p^2 + B_p^2}. \tag{9-20}$$

Let us now denote

$R'_p = \left(\frac{1}{G_p}\right)$ = *resistance of conductive branch of simple parallel combination,*

$X'_p = \left(\frac{-1}{B_p}\right)$ = *reactance of susceptive branch of simple parallel combination.*

Equation (9-19) then becomes

$$\blacktriangleright \quad R_s = \frac{\frac{1}{R'_p}}{\left(\frac{1}{R'_p}\right)^2 + \left(\frac{-1}{X'_p}\right)^2} = \frac{1}{\frac{1}{R'_p} + \frac{R'_p}{(X'_p)^2}} = \frac{(X'_p)^2}{(X'_p)^2 + (R'_p)^2} R'_p, \tag{9-21}$$

and equation (9-20) becomes

▶ $$X_s = \frac{-\left(\dfrac{-1}{X'_p}\right)}{\left(\dfrac{1}{R'_p}\right)^2 + \left(\dfrac{-1}{X'_p}\right)^2} = \frac{1}{\dfrac{X'_p}{(R'_p)^2} + \dfrac{1}{X'_p}} = \frac{(R'_p)^2}{(X'_p)^2 + (R'_p)^2} X'_p. \tag{9-22}$$

By these two equations we may convert a simple parallel combination of resistance, R'_p, and reactance, X'_p, into an equivalent series combination of resistance, R_s, and reactance, X_s. We note that if the reactance, X'_p, of the reactive parallel branch is inductive, it will have a positive value which, by (9-22), will similarly provide a positive value of equivalent reactance, X_s. This result is to be expected, since if equivalence is to exist, the potential drop must lead the current for both combinations in the inductive case. The value of equivalent resistance, R_s, is unaffected by the type of reactance in the parallel combination. It is always positive, as a study of (9-21) shows.

We may wish to substitute an equivalent simple parallel combination of resistance and reactance for a given series combination. Derivation of the following equations by which this may be accomplished is left as a student exercise:

▶ $$R'_p = R_s + \frac{X_s^2}{R_s}, \tag{9-23}$$

and

▶ $$X'_p = X_s + \frac{R_s^2}{X_s}, \tag{9-24}$$

where all quantities are as defined above. The series circuit reactance, X_s, and the reactance, X'_p, of the reactive branch of the equivalent simple parallel combination are observed to have the same algebraic sign, as was the case when the conversion was made the other way. Resistance, again, is seen to be positive under all circumstances.

Figure 9.07B shows the circuit diagrams and vector diagrams for a particular case of inductive reactance. We see that in the series combination there is only one current vector but that there are resistive and reactive (in-phase and quadrature) components of potential drop. In the equivalent parallel combination, there are conductive and susceptive (in-phase and quadrature) components of current but only one vector of potential drop.

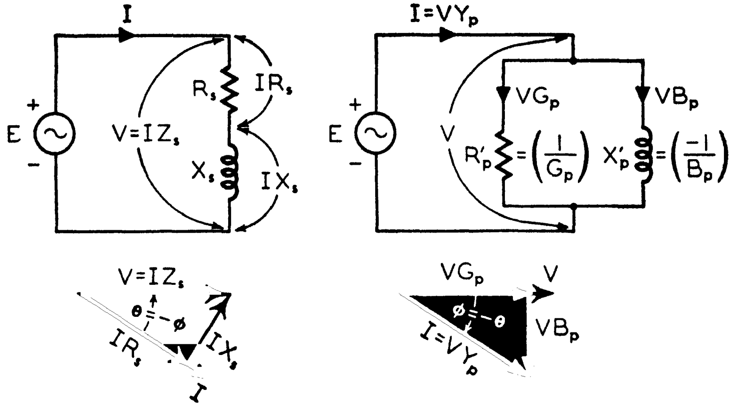


Fig. 9.07B. Equivalent inductive circuits.

Example 9.07. When a current $i = 0.02 \sin 377t$ amperes is sent through a certain capacitor, the resulting potential drop is $v = 250 \sin (377t - 80°)$ volts. (a) Find the values of pure circuit parameters in simple series combination which may be used to represent this capacitor in circuit computations at the given frequency. (b) Repeat (a) for parameters in simple parallel combination.

(a)

$$\mathbf{I}_m = 0.02\underline{/0°}; \qquad \mathbf{V}_m = 250\underline{/-80°}.$$

$$\mathbf{Z}_s = \frac{\mathbf{V}_m}{\mathbf{I}_m} = \frac{250\underline{/-80°}}{0.02\underline{/0°}} = 12{,}500\underline{/-80°} = 2175 - j12{,}300.$$

$$R_s = 2175 \text{ ohms}; \qquad C_s = \frac{-1}{\omega X_s} = \frac{-1}{377 \times (-12{,}300)} = 0.216 \text{ microfarad.}$$

(b) We have a choice of ways for solving this part of the problem. Let us solve it by employing equations (9-23) and (9-24), then checking by a direct solution based on Ohm's law.

$$R'_p = (2175) + \frac{(-12{,}300)^2}{2175} = 2175 + 69{,}500 = 71{,}675 \text{ onms,}$$

$$X'_p = (-12{,}300) + \frac{(2175)^2}{(-12{,}300)} = -12{,}300 - 385 = -12{,}685 \text{ ohms} = \frac{-1}{\omega C'_p},$$

$$C'_p = \frac{-1}{377(-12{,}685)} = 0.209 \times 10^{-6} = 0.209 \text{ microfarad.}$$

Checking, by application of Ohm's law

$$\mathbf{Y}_p = \frac{\mathbf{I}_m}{\mathbf{V}_m} = \frac{0.02\underline{/0^\circ}}{250/-80^\circ} = 0.00008\underline{/80^\circ} = 0.0000139 + j0.0000786.$$

$$G_p = 0.0000139 \text{ mho}; \qquad B_p = 0.0000786 \text{ mho}.$$

$$R_p' = \frac{1}{G_p} = \frac{1}{0.0000139} = 71{,}800 \text{ ohms}; \qquad X_p' = \left(\frac{-1}{B_p}\right) = \frac{-1}{0.0000786}$$

$$= -12{,}700 \text{ ohms}$$

$$C_p' = \frac{-1}{\omega X_p'} = \frac{-1}{377 \times (-12{,}700)} = 0.209 \times 10^{-6} = 0.209 \text{ microfarad}.$$

This checks the answer obtained by the other method within slide-rule precision.

9.08. The Series Circuit

"*A series circuit is a circuit supplying energy to a number of devices connected in series, i.e., the same current passes through each device in completing its path to the source of supply.*"[1] We saw in the first part of this chapter that the total impedance of a number of elements connected in series is the complex sum of the individual impedances, or that

$$▶ \qquad \mathbf{Z}_s = R_s + jX_s = \Sigma(R + jX) = \Sigma R + j\Sigma X. \qquad (9\text{-}25)$$

It is apparent that the graphical interpretation of this equation may be arrived at by "adding end to end" individual impedance triangles to obtain the resultant impedance triangle of the series combination. Figure 9.08 illustrates this procedure for a particular series circuit.

Multiplying each of the terms of (9-25) by the effective value of the common current, I, results in one form of Kirchhoff's potential law, as it applies to the series circuit:

$$▶ \qquad I\mathbf{Z}_s = \mathbf{V}_s = IR_s + jIX_s = \Sigma IR + j\Sigma IX. \qquad (9\text{-}26)$$

This equation, too, has a graphical interpretation which is shown by Fig. 9.08. In this figure all component potential drops are referred to the common current, **I**, which is drawn horizontally. The resistive and reactive drops constitute, for a given element, the legs of a right triangle whose hypotenuse is the drop across the

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[1] *American Standard Definitions of Electrical Terms*, No. 35.40.045. New York: American Institute of Electrical Engineers, 1942.

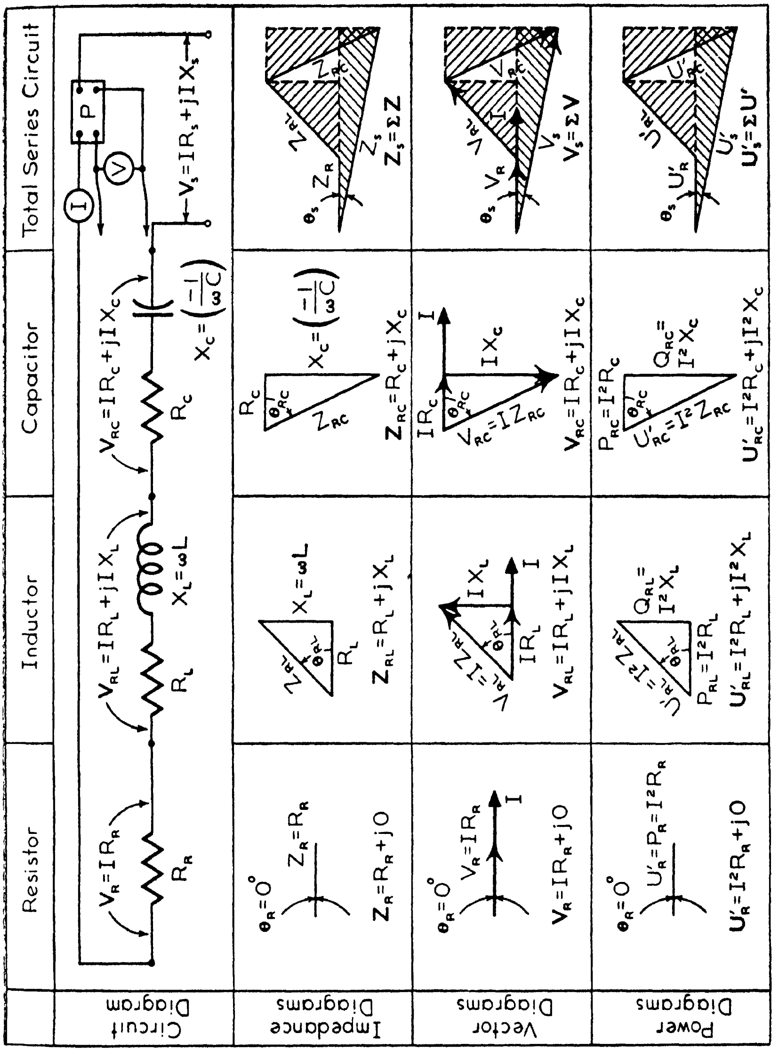


Fig. 9.08. Graphical summary of relations applying to a particular series circuit.

impedance of that element. Adding these potential triangles end to end produces the resultant triangle of total equivalent potential drop. Since the sides of these potential triangles are all rotating time vectors, it is proper to place arrowheads on them. This is not the case with either the impedance triangles or with the power triangles which are to be considered next.

Multiplying each of the terms of equation (9-26) by the effective value of the common current, I, results in complex power

$$\blacktriangleright \qquad I^2\mathbf{Z}_s = \mathbf{U}'_s = I^2R_s + jI^2X_s = \Sigma I^2R + j\Sigma I^2X$$
$$= \Sigma P + j\Sigma Q = P_s + jQ_s = U'_s\underline{/\theta_s}. \qquad (9\text{-}27)$$

We see that, for a series circuit, the total active power is the sum of the individual active powers and that the total reactive power is the algebraic sum of the individual reactive powers. We note also that the resultant total power triangle may be thought of as the end-to-end summation of the individual power triangles.

Figure 9.08 shows us, further, that the power factor angle, θ, of a given element (or of the total circuit) carries on through with the same value in the impedance triangle, the vector diagram, and the power triangle. The *power factor* may be found from any one of the three types of triangle:

$$\blacktriangleright \qquad \text{P.F.} = \cos\theta = \frac{P}{U'} = \frac{IR}{IZ} = \frac{R}{Z}. \qquad (9\text{-}28)$$

Similarly, the *reactive factor* is

$$\blacktriangleright \qquad \text{R.F.} = \sin\theta = \frac{Q}{U'} = \frac{IX}{IZ} = \frac{X}{Z}. \qquad (9\text{-}29)$$

Example 9.08. The following series-connected impedances carry a current of 5 amperes: $\mathbf{Z}_A = 5 - j6$; $\mathbf{Z}_B = 3 + j4$; $\mathbf{Z}_C = 0 - j3$ complex ohms. Determine: (a) The total impedance. (b) The applied potential. (c) The potential drops across the individual impedances. (d) The individual and total complex vector powers. (e) The over-all power factor. (f) The over-all reactive factor.

(a) $\mathbf{Z}_s = \mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C = (5 + 3 + 0) + j(-6 + 4 - 3)$
$= 8 - j5$ complex ohms.

(b) Using current as reference, $\mathbf{I} = 5 + j0$ complex amperes, and
$\mathbf{V}_s = \mathbf{I}\mathbf{Z}_s = (5 + j0)(8 - j5) = 40 - j25$ complex volts.

(c) $\mathbf{V}_A = \mathbf{I}_A\mathbf{Z}_A = (5 + j0)(5 - j6) = 25 - j30$,
$\mathbf{V}_B = \mathbf{I}_B\mathbf{Z}_C = (5 + j0)(3 + j4) = 15 + j20$,
$\mathbf{V}_C = \mathbf{I}_C\mathbf{Z}_C = (5 + j0)(0 - j3) = 0 - j15$ complex volts.

(d) $\mathbf{U}_A' = \hat{\mathbf{I}}\mathbf{V}_A = (5 - j0)(25 - j30) = 125 - j150,$
$\mathbf{U}_B' = \hat{\mathbf{I}}\mathbf{V}_B = (5 - j0)(15 + j20) = 75 + j100,$
$\mathbf{U}_C' = \hat{\mathbf{I}}\mathbf{V}_C = (5 - j0)(0 - j15) = 0 - j75,$
$\mathbf{U}_s' = \hat{\mathbf{I}}\mathbf{V}_s = (5 - j0)(40 - j25) = 200 - j125$ volt-amperes.

(e) $$(\text{P.F.})_s = \frac{P_s}{U_s'} = \frac{200}{\sqrt{(200)^2 + (-125)^2}} = \frac{200}{235.5} = 0.85 \text{ (leading).}$$

(f) $$(\text{R.F.})_s = \frac{Q_s}{U_s'} = \frac{-125}{\sqrt{(200)^2 + (-125)^2}} = \frac{-125}{235.5}$$
$$= -0.53 \text{ (capacitive).}$$

9.09. Locus Diagrams

Our study thus far has dealt with the steady-state impedance, admittance, potential, current, and power relations associated with simple series and parallel combinations of circuit parameters. We have drawn vector diagrams and impedance, admittance, and power triangles to aid in visualizing the relations described analytically by the equations we have developed. These diagrams are said to be drawn in the *complex plane*, because they contain both real and imaginary rectangular components. A given diagram of this type is able to picture only one set of steady-state conditions. Another slightly different diagram of the same type would be needed to picture the new condition brought about by changing, for example, the value of one of the parameters, or the value of the frequency.

Frequently it is desirable to be able to picture graphically the condition existing over a whole range of steady-state conditions. This may be accomplished by letting a vector (or complex quantity) of a given diagram sweep out a locus on the complex plane as either a parameter or the frequency is varied. This type of extended vector diagram may be termed a *locus diagram*. It enables us to visualize and analyze circuit behavior quantitatively for a whole range of steady-state conditions. A locus diagram enables us to plot three-dimensional variation on a two-dimensional (plane) graph. Each locus involves the two dependent variables, *magnitude* and *phase angle* of the complex quantity, in addition to the independent variable—the value of variable circuit parameter or frequency. A study of the locus diagrams which follow will confirm this statement.

Locus diagrams are especially convenient and useful in practice in cases where the loci prove to be circles (or straight lines, the limiting case of circles whose radii approach infinity in value). Only a small amount of computation is required to establish and draw the loci for such cases. Fortunately, most practical loci are of this type, and *circle diagrams*, as they are commonly called, find wide application in both power and communications studies.

Just as we draw the various types of complex triangles, so we may draw locus diagrams of impedance, admittance, potential and current, and power. The following tabulation shows the most commonly drawn types of locus diagram:

(1) Locus of **impedance** as either R, L, C, or f is varied.

(2) Locus of **admittance** as either R, L, C, or f is varied.

(3) Locus of **current** as either R, L, C, or f is varied, but V is held constant.

(4) Locus of **potential** as either R, L, C, or f is varied, but I is held constant.

(5) Locus of **vector power** as either R, L, C, or f is varied, but either V or I is held constant.

Locus diagrams may be derived analytically in either the rectangular or the polar complex forms. In illustration of the mathematical procedure for establishing a locus, we may list the following steps for obtaining loci of types (3) and (4) in the rectangular complex form:

1. *Write, in the rectangular complex form, the a-c version of Ohm's law as it applies to the circuit under consideration.*

2. *Eliminate the independent variable.* (*If, for example, R is to be the variable parameter, eliminate R.*)

3. *Arrange the components of the quantity whose locus is desired in such form as to resemble the equation of a standard geometric figure. A circle* (*or a straight line*) *is hoped for.*

4. *Plot the curve represented by the equation.*

5. *Determine from a consideration of the physical nature of the circuit, what portion of the curve represents the physically possible locus.*

9.10. Series R-L Circuit Locus Diagram

Part (a) of Fig. 9.10 shows a particular series circuit in which an emf of constant frequency is applied to a series combination of constant inductance and variable resistance. Let us consider first the case in which applied emf is constant but current is free to vary. We are interested in knowing how the vector current varies with a change in resistance.

For convenience, let us assume the constant vector potential drop to coincide with the reference axis. Then

$$\mathbf{V}_s = V_s + j0. \tag{9-30}$$

The general expression for the circuit impedance is

▶ $$\mathbf{Z} = R + jX_L, \tag{9-31}$$

and the general expression for the dependently variable current is

▶ $$\mathbf{I} = I_x + jI_y, \tag{9-32}$$

where I_y will possess a negative value because the circuit is inductive.

Writing the a-c version of Ohm's law,

$$\begin{aligned}\mathbf{V}_s = V_s + j0 &= (R + jX_L)(I_x + jI_y)\\ &= (RI_x - X_LI_y) + j(X_LI_x + RI_y).\end{aligned} \tag{9-33}$$

Two simultaneous equations may be obtained by equating separately the real terms and the imaginary terms:

$$RI_x - X_LI_y = V_s, \tag{9-34}$$

and

$$X_LI_x + RI_y = 0. \tag{9-35}$$

To eliminate the variable, R, we multiply (9-34) by I_y and (9-35) by I_x and subtract the results, yielding

$$X_L(I_x^2 + I_y^2) = -V_sI_y. \tag{9-36}$$

Rearranging,

$$I_x^2 + I_y^2 + \frac{V_sI_y}{X_L} = 0. \tag{9-37}$$

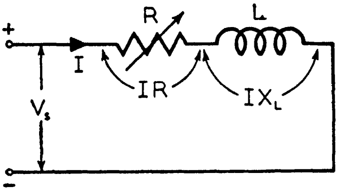
Completing squares by adding $(V_s^2/4X_L^2)$ to each side of the equation,

$$I_x^2 + \left(I_y^2 + \frac{V_s}{X_L}I_y + \frac{V_s^2}{4X_L^2}\right) = \frac{V_s^2}{4X_L^2}, \tag{9-38}$$

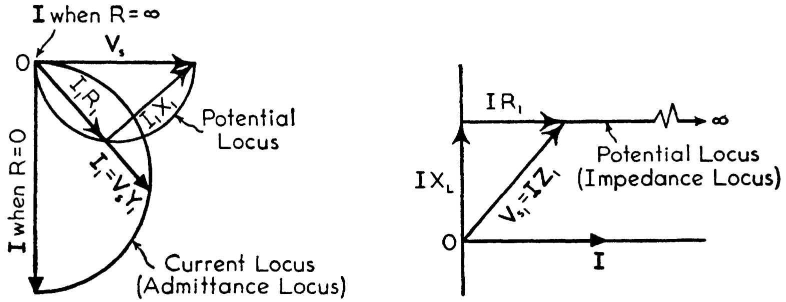
or

▶ $$(I_x - 0)^2 + \left(I_y + \frac{V_s}{2X_L}\right)^2 = \left(\frac{V_s}{2X_L}\right)^2. \tag{9-39}$$

Equation (9-39) is of the form $(x - h)^2 + (y - k)^2 = r^2$, the general equation of a circle. The locus is, then, a circle with the center located 0 units along the axis of reals and $(-V_s/2X_L)$ units along the j-axis. The diameter is (V_s/X_L) units, which is the value for the current when $R = 0$. Since, physically, the current cannot lag the potential drop by more than 90°, the true locus is the semicircle shown in part (b) of the figure. The IR and IX_L potential drops must always combine vectorially at right angles to equal the constant applied potential, V_s. A second semicircle, which is the locus of potential drops, can therefore be drawn, as shown. From this figure, it is possible to predict the changes in all dependent variables over the total possible range of resistance adjustment with constant applied emf. Since, from the a-c version of Ohm's law, $\mathbf{I} = \mathbf{V}/\mathbf{Z} = \mathbf{V} \cdot \mathbf{Y}$, it is evident that total circuit admittance, Y, is always directly proportional to current and can be represented by the same locus.



(a) Circuit diagram.



(b) Constant applied potential. (c) Constant current.

Fig. 9.10. Locus diagrams for variable-resistance R-L series circuit.

Let us now consider the alternative case where, as R is varied, the applied emf must be varied correspondingly in order to maintain a constant current. The locus diagram of part (c) of the figure should prove self-explanatory.

The foregoing discussion, although devoted solely to a particular series circuit and to independent variation of only one of the several circuit quantities which might be used as independent variables, should serve to illustrate the method by which locus diagrams for any series circuit can be obtained.

9.11. Series R-C Circuit Locus Diagram

The development of the locus diagram for the series *R-C* circuit is very similar to that for the series *R-L* circuit and will be left as a student exercise.

9.12. Series R-L-C Circuit Locus Diagram

A circuit of particular interest and importance is that which contains resistance, inductance, and capacitance in series. In Article 6.11 the following equation was developed for the impedance of this circuit:

$$\mathbf{Z}_{RLC} = \sqrt{R^2 + \left(\omega L + \frac{-1}{\omega C}\right)^2} \Big/ \underline{\tan^{-1}\left(\frac{\omega L + \frac{-1}{\omega C}}{R}\right)}. \qquad (9\text{-}40)$$

Since admittance is the reciprocal of impedance,

$$\mathbf{Y}_{RLC} = \frac{1}{\mathbf{Z}_{RLC}}$$

$$= \frac{1}{\sqrt{R^2 + \left(\omega L + \frac{-1}{\omega C}\right)^2}} \Big/ \underline{\tan^{-1}\left(\frac{-\omega L + \frac{1}{\omega C}}{R}\right)}. \qquad (9\text{-}41)$$

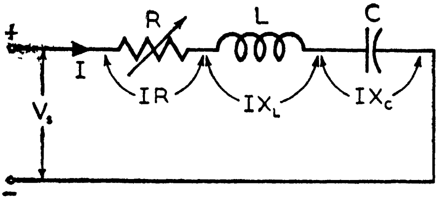
Inasmuch as the current locus is similar to the admittance locus for constant applied potential, and the potential locus is similar to the impedance locus for a constant current, in this article only the loci of impedances and admittances will be discussed.

An examination of equations (9-40) and (9-41) reveals that the values of both the impedance and the admittance can be changed by varying any one of the following quantities:

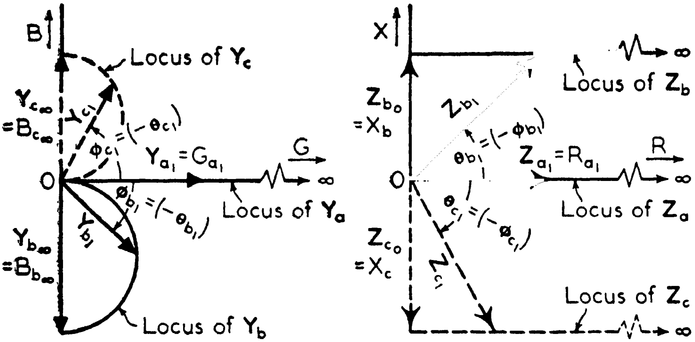
(1) Resistance.
(2) Inductance.
(3) Capacitance.
(4) Frequency of the supply.

We shall now consider, in turn, each of the above cases.

1. *Variable R.* Part (a) of Fig. 9.12A shows a series *R-L-C* circuit in which the resistance, *R*, is the variable parameter. There are three cases to be considered, depending upon the relative values of the reactances. They occur when:



(a) Circuit diagram.



(b) Admittance loci. (c) Impedance loci.

Fig. 9.12A. Locus diagrams for variable-resistance R-L-C series circuit.

(a) $$\omega L = \frac{1}{\omega C}.$$

(b) $$\omega L > \frac{1}{\omega C}.$$

(c) $$\omega L < \frac{1}{\omega C}.$$

(a) Let us first consider the case in which $\omega L = 1/\omega C$. Under this condition,

$$\mathbf{Z}_a = R\underline{/0^\circ}, \tag{9-42}$$

and

$$\mathbf{Y}_a = \frac{1}{R}\underline{/0^\circ}. \tag{9-43}$$

The impedance is always resistive, and its locus is a horizontal line in the positive direction. The admittance locus also is a horizontal line in the positive direction.

(b) When $\omega L > 1/\omega C$, the impedance has a value, when $R = 0$. of

$$\mathbf{Z}_b = \left(\omega L + \frac{-1}{\omega C}\right) \underline{/90^\circ}, \tag{9-44}$$

and the corresponding admittance is

$$\mathbf{Y}_b = \frac{1}{\left(\omega L + \frac{-1}{\omega C}\right)} \underline{/-90^\circ}. \tag{9-45}$$

As resistance is added to the circuit, impedance increases and admittance decreases, both of them moving into a more nearly horizontal position. The impedance is infinite and the admittance zero, both at an angle of zero degrees, as R approaches an infinite value.

(c) When $\omega L < 1/\omega C$, the impedance has a value, when $R = 0$, of

$$\mathbf{Z}_c = \left(\omega L + \frac{-1}{\omega C}\right) \underline{/90^\circ} = \left(\frac{1}{\omega C} - \omega L\right) \underline{/-90^\circ}, \tag{9-46}$$

and the corresponding admittance is

$$\mathbf{Y}_C = \frac{1}{\left(\frac{1}{\omega C} - \omega L\right)} \underline{/90^\circ}. \tag{9-47}$$

A bit of reflection will confirm the statement that this case is the negative of case (b), as the locus diagrams indicate.

2. *Variable L.* Part (a) of Fig. 9.12B shows a series *R-L-C* circuit in which the self inductance, L, is the variable element. When $L = 0$, the impedance possesses the value

$$\mathbf{Z}_0 = \sqrt{R^2 + \left(\frac{-1}{\omega C}\right)^2} \underline{/\tan^{-1}\left(\frac{-1}{\omega CR}\right)}. \tag{9-48}$$

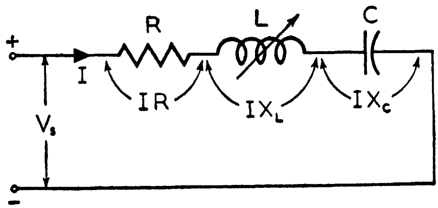
When $\omega L = 1/\omega C$, the impedance has a minimum value

$$\mathbf{Z}_1 = R/0^\circ. \tag{9-49}$$

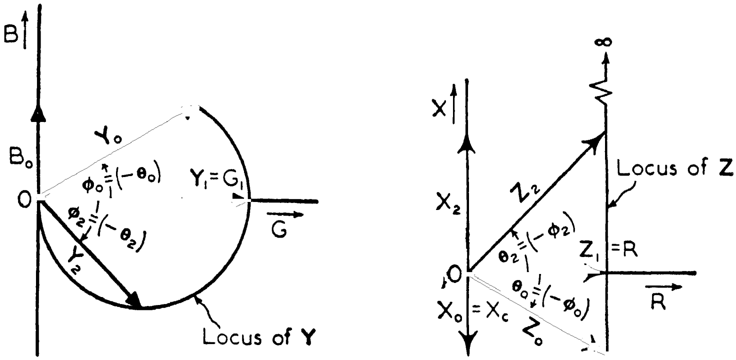
As L is increased to infinity, the impedance attains the value

$$\mathbf{Z}_\infty = \infty\underline{/90^\circ}. \qquad (9\text{-}50)$$

The corresponding admittances are the reciprocals of the impedances.



(a) Circuit diagram.



(b) Admittance locus.

(c) Impedance locus.

Fig. 9.12B. Locus diagrams for variable-inductance R-L-C series circuit.

3. *Variable C.* Part (a) of Fig. 9.12C shows a series *R-L-C* circuit in which the capacitance, C, is the variable element. When $C = 0$ (the open-circuit condition), the impedance has the value

$$\mathbf{Z}_0 = \infty\underline{/-90^\circ}. \qquad (9\text{-}51)$$

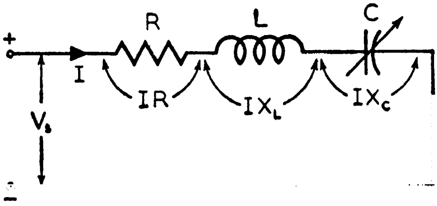
When $\omega L = 1/\omega C$, the impedance has a minimum value

$$\mathbf{Z}_1 = R\underline{/0^\circ}. \qquad (9\text{-}52)$$

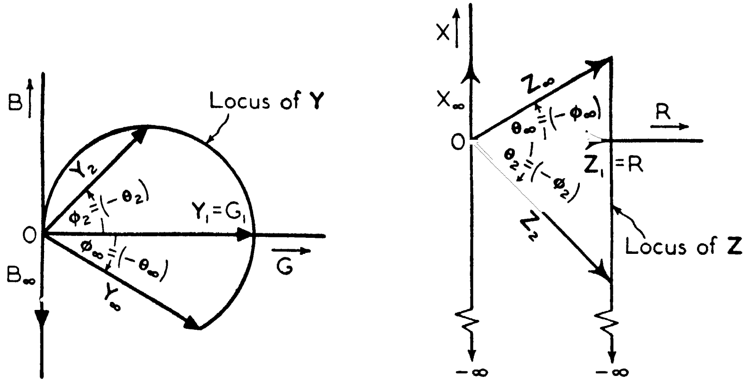
As C is increased to infinity (the condition of short circuit across the capacitor), the impedance attains the value

$$\mathbf{Z}_\infty = \sqrt{R^2 + (\omega L)^2}\,\underline{/\tan^{-1}\left(\frac{\omega L}{R}\right)}. \tag{9-53}$$

The corresponding admittances are the reciprocals of the impedances.



(a) Circuit diagram.



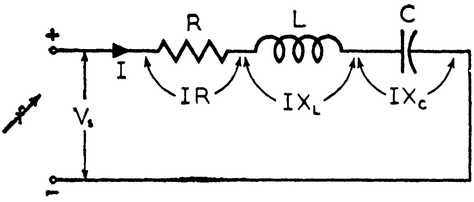
(b) Admittance locus. (c) Impedance locus.

Fig. 9.12C. Locus diagrams for variable-capacitance R-L-C series circuit.

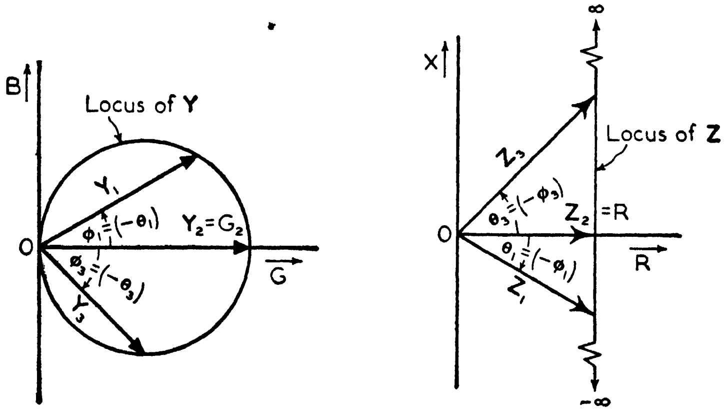
4. *Variable f.* Part (a) of Fig. 9.12D shows a series *R-L-C* circuit in which it is assumed that *R*, *L*, and *C* are constants independent of frequency, and frequency is the independent variable. Since $X_L = \omega L$ and $X_C = (-1/\omega C)$, inductive reactance will vary directly with frequency, and capacitive reactance will vary inversely with frequency. When $f = 0$, inductive reactance is zero and capacitive reactance is infinite, yielding:

$$\mathbf{Z}_0 = \infty\,\underline{/-90°} = -\infty\,\underline{/90°}. \tag{9-54}$$

As frequency is gradually increased in value, the absolute values of inductive and capacitive reactance approach each other, the inductive reactance gradually increasing and the capacitive reactance gradually decreasing. When the absolute values of inductive and capacitive reactance become equal



(a) Circuit diagram.



(b) Admittance locus.

(c) Impedance locus.

Fig. 9.12D. Locus diagrams for variable-frequency R-L-C series circuit.

$$\omega L = \frac{1}{\omega C}, \tag{9-55}$$

and

$$\omega^2 = \frac{1}{LC}, \tag{9-56}$$

or

$$f = \frac{1}{2\pi\sqrt{LC}}. \tag{9-57}$$

At this frequency, the impedance is a minimum and the power-factor angle of the circuit is zero degrees, with the result that the power factor is unity:

$$\mathbf{Z}_2 = R\underline{/0^\circ}. \tag{9-58}$$

As the frequency is increased to infinity, the inductive reactance becomes infinite in value, whereas the capacitive reactance diminishes to zero:

$$\mathbf{Z}_\infty = \infty\underline{/90^\circ}. \tag{9-59}$$

From the impedance locus diagram it is evident that *above* the frequency defined by equation (9-57), $\omega L > 1/\omega C$ and the circuit is predominately inductive; that *at* that frequency the circuit behaves like a resistive circuit; and that *below* that frequency the circuit is predominately capacitive. As before, admittance under any condition is the reciprocal of impedance under that condition.

9.13. Series Resonance

"*Series (phase) resonance is the steady-state condition which exists in a circuit comprising inductance and capacitance connected in series, when the current in the circuit is in phase with the voltage across the circuit.*"[2]

Reference to Fig. 9.08 makes it clear that, although the AIEE defines series resonance as above, this condition may, with equal accuracy, be defined by any one of the following relations in an *R-L-C* series circuit:

▶ $$(\text{P.F.})_s = \cos\theta_s = 1, \tag{9-60}$$

▶ $$Q_s = Q_L + Q_C = 0, \tag{9-61}$$

▶ $$IX_s = IX_L + IX_C = 0, \tag{9-62}$$

and

▶ $$X_s = X_L + X_C = 0. \tag{9-63}$$

It suits the purpose of this text to employ (9-63) in the analysis of series resonance.

Substituting $X_L = \omega L$ and $X_C = -1/\omega C$ in (9-63), we obtain

$$\omega L + \frac{-1}{\omega C} = 0, \tag{9-64}$$

or

$$\omega L = \frac{1}{\omega C}, \tag{9-65}$$

and

$$\omega^2 = \frac{1}{LC}, \tag{9-66}$$

---

[2] *Ibid.*, No. 65.10.530.

which reduces to

▶ $$f = \frac{1}{2\pi\sqrt{LC}}. \tag{9-67}$$

Equation (9-67) is sometimes used to define the condition of series resonance.

As (9-67) reveals, any one of the three quantities, L, C, or f, may be varied to *tune* a given series R-L-C circuit to series resonance. In the previous article, locus diagrams were drawn for a series R-L-C circuit for the three cases in which L, C, and f, respectively, were the variables. The condition of series resonance can readily be recognized on each of these diagrams as the condition in which the variable was so adjusted as to make equation (9-65) hold.

Reference to the locus diagrams, as well as to Fig. 9.08 and to equations (9-60) through (9-64), indicates that series resonance is, in many applications, an extremely desirable condition of operation. It represents a condition of minimum impedance which is equal to merely the resistance of the circuit. With a given applied potential, a maximum current can be obtained. Conversely, any given current can be obtained by application of a minimum potential. The advantage of series resonance in circuits designed to transmit power is obvious. A potential danger in attaining series resonance unexpectedly lies in the possibility of excessive current, leading to excessive IX_L and IX_C drops of such magnitude as to damage the insulation of the reactive elements.

Under the condition of series resonance, since unity power factor prevails, the power source need supply only real power to the circuit. This condition makes the most effective use of the power source, since no extra capacity is needed to handle reactive power. The reactive power of the circuit is handled entirely by the capacitance and the inductance. During the quarter cycles of current increase in either a positive or a negative direction, the inductance absorbs reactive energy from the capacitance; during the alternate quarter cycles in which current is decreasing toward a value of zero from either a positive or a negative direction, the capacitance receives the same reactive energy back from the inductance. At resonance, then, the reactive energy oscillates with double frequency between the inductance and the capacitance. Meanwhile, a series of double-frequency sinusoidal surges of active power is being supplied by the power source to the load. These relations are made clear by Fig. 9.13A, which can easily be derived from Figs. 4.32B, 4.33, and 4.39, using the common current as a reference.

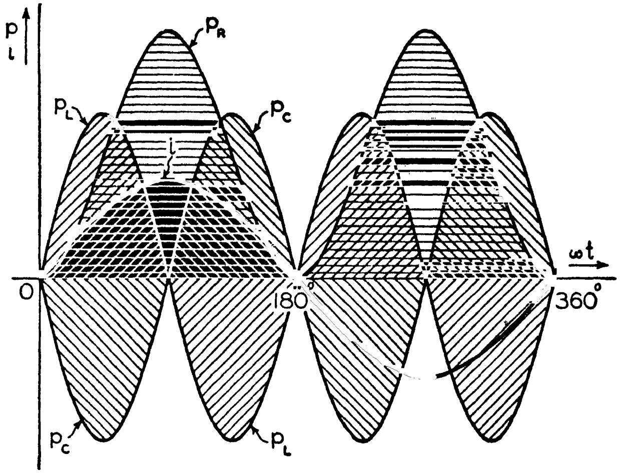


Fig. 9.13A. Current and power relations in a series-resonant circuit.

In Fig. 9.13A, it is seen that at every instant of time in a series-resonant circuit,

$$p_L = -p_C, \qquad (9\text{-}68)$$

and

▶ $$p_s = p_R + p_L + p_C = p_R. \qquad (9\text{-}69)$$

Since the area under any power curve represents energy, it is apparent that energy absorbed by one of the reactive elements equals that given up by the other, and that the energy to the resistance is delivered in the form of double-frequency surges, one for each half cycle of current.

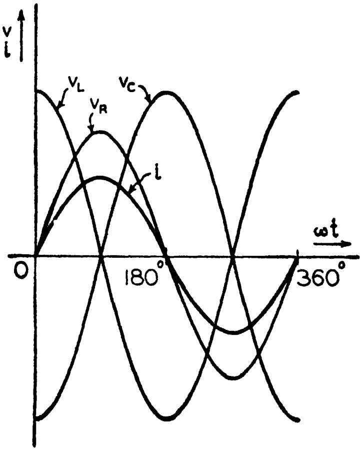


Fig. 9.13B. Current and potential relations in a series-resonant circuit.

Figure 9.13B shows the relationship among the various potential drops of a series-resonant circuit.

We note that at every instant of time in a series-resonant circuit

$$v_L = -v_C \tag{9-70}$$

and

▶ $$v_s = v_R + v_L + v_C = v_R. \tag{9-71}$$

Equation (9-70) is justified by recalling that inductive current lags its potential drop by 90°, and that capacitive current leads its potential drop by 90°. If the two elements carry the same current, it is apparent that the inductive potential drop leads this common current by 90°, while the capacitive potential drop lags this common current by 90°, and the two drops are equal in absolute value. The power supply must provide, then, only enough emf to balance the *iR* drop of the circuit. The capacitance and the inductance are completely cared for by each other.

9.14. Tuning for Series Resonance

In the previous article, a summary and discussion of circuit relationships precisely at the condition of series resonance was presented. It is the purpose of the present article to investigate the corresponding relationships in the vicinity of series resonance. This can best be done by reference to curves. We shall treat, in order, the three methods of tuning to series resonance: by varying L, by varying C, and by varying f. The curves which we shall draw will show in each case the variation of inductive, capacitive, and total reactances, of effective value of current, and of effective values of potential drops across the three circuit parameters, R, L, and C, as the independent variable is changed in value. A constant effective value of applied potential will be assumed throughout. The equations of the curves to be plotted have been previously derived and can be summarized for convenient reference, as follows:

▶ $$X_s = X_L + X_C = 2\pi fL + \frac{-1}{2\pi fC}, \tag{9-72}$$

▶ $$I = \frac{V_s}{Z} = \frac{V_s}{\sqrt{R^2 + (X_L + X_C)^2}} = \frac{V_s}{\sqrt{R^2 + \left(2\pi fL + \frac{-1}{2\pi fC}\right)^2}}, \tag{9-73}$$

▶ $$V_R = IR = \frac{V_sR}{\sqrt{R^2 + \left(2\pi fL + \frac{-1}{2\pi fC}\right)^2}}, \tag{9-74}$$

▶
$$V_L = IX_L = \frac{V_s(2\pi fL)}{\sqrt{R^2 + \left(2\pi fL + \dfrac{-1}{2\pi fC}\right)^2}}, \tag{9-75}$$

▶
$$V_C = IX_C = \frac{V_s(-1/2\pi fC)}{\sqrt{R^2 + \left(2\pi fL + \dfrac{-1}{2\pi fC}\right)^2}}. \tag{9-76}$$

9.15. Inductance Tuning

When L is varied to tune for series resonance, the series of curves shown in Fig. 9.15 is obtained. The general equations above are the equations of the current and potential drop curves shown. It will be noted that V_C becomes a maximum at resonance, whereas the maximum value of V_L occurs at a value of inductance above that giving resonance. The reason for this difference is apparent when one considers that $V_L = IX_L$ will continue to increase as L is increased as long as the rate of increase of X_L is greater than the rate of decrease of I. Although I is maximum at resonance, the rate of decrease afterward is small enough at first to permit a peak of V_L to the right of resonance.

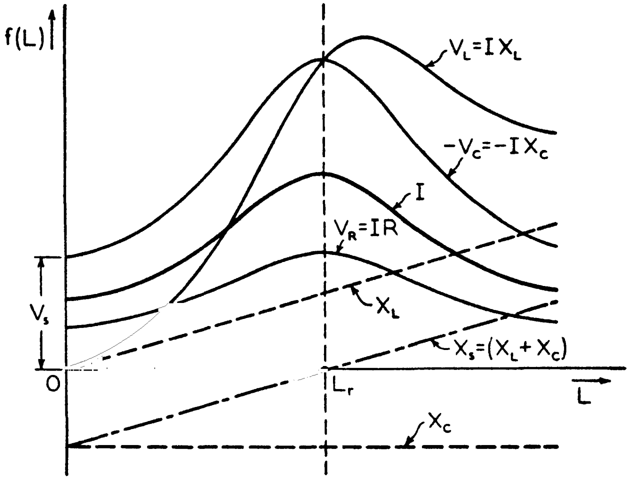


Fig. 9.15. Variable-inductance series-resonant curves.

The value of inductance for maximum V_L can be obtained by setting $(dV_L/dX_L) = 0$. Differentiating equation (9-75) and equating the result to zero yield the following condition for maximum V_L when L is varied:

$$X_L = \frac{-(R^2 + X_C^2)}{X_C} \tag{9-77a}$$

or

▶ $$L_{(\text{max.}V_L)} = C(R^2 + X_C^2) \tag{9-77b}$$

Example 9.15. (a) What value of inductance must be placed in series with a 50-microfarad capacitor to produce resonance at 60 cycles? (b) If the resistance of the circuit is 10 ohms, what is the total circuit impedance at resonance? (c) What value of inductance will produce maximum inductive reactance drop?

(a) *At series resonance:*

$$\omega L + \frac{-1}{\omega C} = 0; \qquad \omega = 2\pi f = 377 \text{ cycles per second,}$$

$$L = \frac{1}{\omega^2 C} = \frac{10^6}{(377)^2 \times 50} = 0.1408 \text{ henry} = 140.8 \text{ millihenrys.}$$

(b) *At series resonance:*

$$Z_s = \sqrt{R_s^2 + X_s^2} = \sqrt{R_s^2 + 0} = R_s = 10 \text{ ohms.}$$

(c) *Maximum inductive drop:*

$$L = C(R^2 + X_C^2) = CR^2 + \frac{1}{\omega^2 C}$$

$$= 50 \times 10^{-6} \times 10^2 + \frac{10^6}{(377)^2 \times 50} = 0.1458 \text{ henry} = 145.8 \text{ millihenrys.}$$

9.16. Capacitance Tuning

When C is varied to tune for series resonance, the curves of Fig. 9.16 are obtained. As in the previous case, equations (9-73) through (9-76) are the equations of these curves. Since X_L is constant, the maximum value of V_L occurs at resonance, where I is maximum. V_C, on the other hand, is maximum to the left of resonance. This difference is accounted for by a process of reasoning similar to that employed in explaining the position of the V_L peak in the previous article.

The value of capacitance for maximum V_C can be obtained by setting $(dV_C/dX_C) = 0$. This problem is left as a student exercise.

Example 9.16. (a) What value of capacitance should be added in series with a coil of 2 ohms resistance and 10 millihenrys inductance to obtain a maximum effective value of current for a 25-cycle total applied potential of fixed value? (b) What is V_C at resonance?

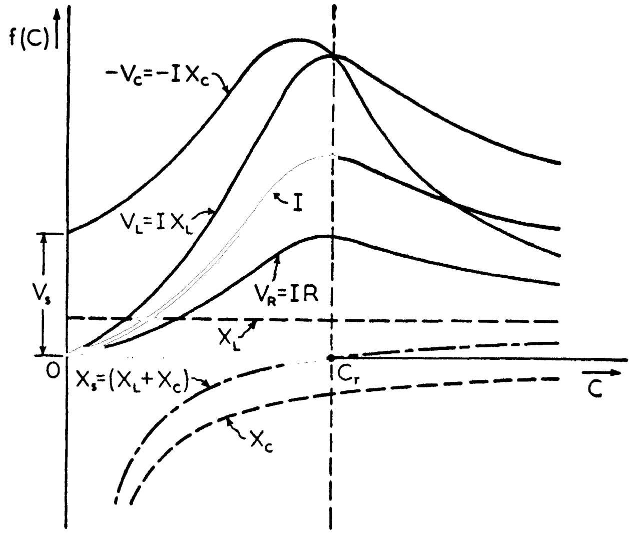


Fig. 9.16. Variable-capacitance series-resonance curves.

(a) Maximum effective current occurs at resonance, where $\omega L + \dfrac{-1}{\omega C} = 0$.

$$C = \frac{1}{\omega^2 L} = \frac{10^3}{(2\pi \times 25)^2 \times 10} = 0.00405 \text{ farad} = 4050 \text{ microfarads.}$$

$$\text{(b)} \quad V_C = IX_C = \frac{V_s}{R} \times \frac{1}{\omega C} = \frac{10^6}{2 \times 157 \times 4050} V_s = 0.785V$$

9.17. Variable-frequency Series Resonance Curves

When frequency, rather than inductance or capacitance, is the independent variable in a series *R-L-C* circuit with constant effective value of applied emf, the resulting "resonance curves" resemble those of Fig. 9.17. Inasmuch as a variation of f produces a directly proportional variation of X_L and an inversely proportional variation of X_C, it is clear that this type of variation is somewhat like a combination of inductance tuning and capacitance tuning. Here, however, neither V_L nor V_C is maximum at resonance, one peak falling on each side of resonance. Otherwise, the curves are rather similar to those of the preceding two articles. A thoughtful study of these curves in the light of our previous discussions should obviate the necessity for a more complete discussion of this figure.

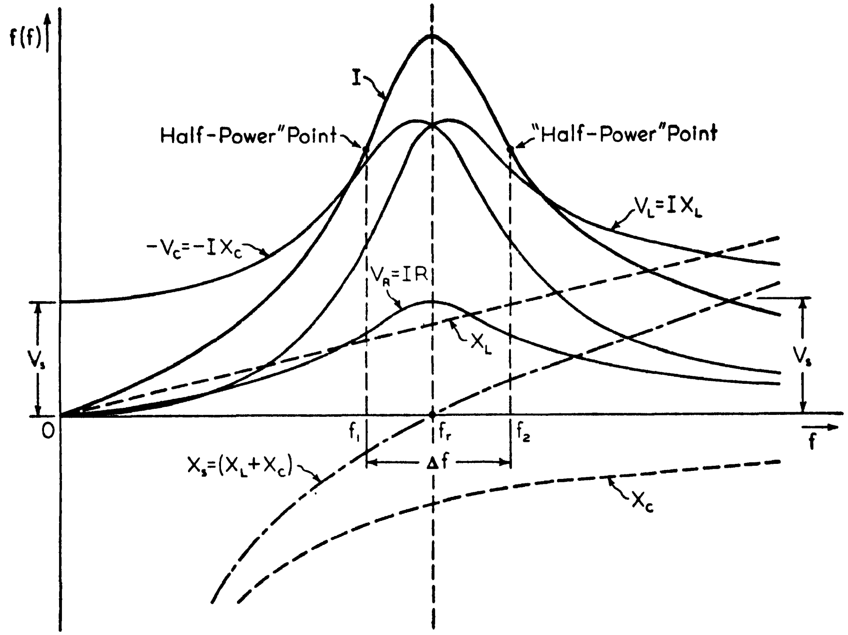


Fig. 9.17. Variable-frequency series-resonance curves.

EXAMPLE 9.17. A 10-microfarad capacitor and a 10-ohm, 10-millihenry coil are connected in series. At what frequency will this combination be resonant?

By equation (9-67),

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{10 \times 10^{-6} \times 10 \times 10^{-3}}} = \frac{10^4}{2\pi\sqrt{10}}$$

$$= 504 \text{ cycles per second.}$$

9.18. Current Response as a Function of Resonant Frequency

The preceding articles have shown that if an applied emf of constant effective value but variable frequency is applied to a series combination of R, L, and C, the effective value of the current is greatest at frequencies in the immediate vicinity of resonance.

Although such a circuit "passes" to some degree waves of all frequencies between zero and infinity, it passes most readily waves of frequencies in the vicinity of resonance. Such a circuit is said to possess *frequency selectivity.*

The current response of a series *R-L-C* circuit at any frequency, f, can be expressed in terms of the resonant frequency, f_r, or of the corresponding resonant angular velocity, $\omega_r = 2\pi f_r$. We know from the a-c version of Ohm's law that

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}_s} = \frac{\mathbf{V}_s}{R + j\left(\omega L + \dfrac{-1}{\omega C}\right)}. \tag{9-78}$$

This equation may be manipulated as follows:

$$\mathbf{I} = \frac{\mathbf{V}_s}{R} \cdot \frac{1}{1 + j\dfrac{1}{R}\left(\omega L - \dfrac{\omega L}{\omega^2 LC}\right)} = \frac{\mathbf{V}_s}{R} \cdot \frac{1}{1 + j\dfrac{L}{R}\left(\omega - \dfrac{1}{\omega LC}\right)}. \tag{9-79}$$

At resonance, however,

$$\omega_r = \frac{1}{\sqrt{LC}}, \qquad \text{and} \qquad \omega_r^2 = \frac{1}{LC}. \tag{9-80}$$

By substitution, then, we obtain:

$$▶ \quad \mathbf{I} = \frac{\mathbf{V}}{R} \cdot \frac{1}{1 + j\dfrac{L}{R}\left(\omega - \dfrac{\omega_r^2}{\omega}\right)} = \frac{\mathbf{V}}{R} \cdot \frac{1}{1 + j\dfrac{\omega_r L}{R}\left(\dfrac{\omega}{\omega_r} - \dfrac{\omega_r}{\omega}\right)}. \tag{9-81}$$

This is a convenient form for comparing the current response at off-resonant frequencies with the resonant value, $\mathbf{I}_r = \mathbf{V}/R$.

Equation (9-81) may be rewritten:

$$\mathbf{I} = \frac{\mathbf{V}}{R} \cdot \frac{1}{1 + j\dfrac{\omega_r L}{R}\left(\dfrac{\omega}{\omega_r} - \dfrac{1}{\omega/\omega_r}\right)}, \tag{9-82}$$

where: $\omega_r = 2\pi f_r$ = *resonant angular velocity,* in **radians per second.** This form shows that *the dependence of current on frequency is wholly relative to resonant frequency:* the current response at a given angular velocity, $\omega = 2\pi f$, depends only upon the value of this angular velocity relative to the resonant angular velocity, $\omega_r = 2\pi f_r$.

The percentage departure from resonance is easily found from the ratio (ω/ω_r). For example, if $\omega_1/\omega_r = 1.2$ or if $\omega_2/\omega_r = 0.8$, the frequencies in question are 20 per cent above and below resonance, respectively, whereas the ratio $(\omega/\omega_r = 1)$ refers to resonance itself.

The magnitude of the current response of equation (9-81) is given by

$$\blacktriangleright \qquad I = \frac{V}{R} \cdot \frac{1}{\sqrt{1 + \left(\frac{\omega_r L}{R}\right)^2 \left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)^2}}. \qquad (9\text{-}83)$$

9.19. Quality, Q, of a Coil

Electric communication engineers have adopted the following symbolism for the "figure of merit," or *quality* of a practical coil to be used in resonant circuits:

$$\blacktriangleright \qquad Q \equiv \frac{X_L}{R} = \frac{\omega L}{R} = \frac{2\pi f L}{R}, \qquad (9\text{-}84)$$

where:

Q = *quality of the coil* (a dimensionless ratio),
$\omega = 2\pi f$ = *angular velocity for which the value of Q is defined,* in **radians per second,**
L = *effective self inductance of coil,* in **henrys,** and
R = *effective resistance of coil,* in **ohms.**

The symbol Q as used here must not be confused with the symbol as used to represent reactive power. The two quantities are not equivalent. As used here, the Q of a coil is a measure of its ability to produce resonant effects. We shall investigate this aspect of Q in the next article.

The resonant-frequency value, $Q_r = (\omega_r L/R)$, can be observed in equations (9-82) and (9-83). If appreciable resistance and inductance exist in circuit elements other than the coil, it may be desirable to utilize equations (9-84) to define the Q of the entire circuit, in which case R and L symbolize, respectively, the total resistance and the total inductance of the circuit. The quantity Q is commonly used in specifying coil or circuit characteristics at high frequencies because, as a result of skin effect and similar influences, it is more nearly constant than the value of effective resistance at those frequencies.

9.20. Frequency Selectivity of a Series R-L-C Circuit

We have already seen that a series *R-L-C* circuit possesses the property of *frequency selectivity*—the ability to discriminate among waves of differing frequencies, as exemplified by Fig. 9.20. We note that the current response as a function of frequency rises rapidly in the vicinity of resonance to a prominent peak at resonance. In most applications of resonant circuits, it is important to have a measure of the "sharpness" of resonance, or the width of the current peak.

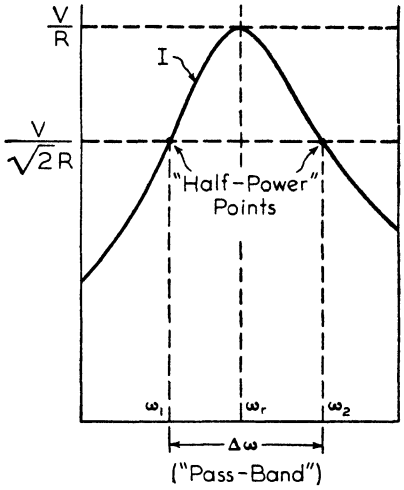


Fig. 9.20. Determination of series R-L-C circuit selectivity.

The most commonly used measure is the width of the peak at the so-called "*half-power*" *points*, at which the power is one half of its value at the resonance peak. At the resonant frequency,

$$I_r = \frac{V}{R} \qquad \text{and} \qquad P_r = \frac{V^2}{R}. \tag{9-85}$$

At the half-power points, then,

$$▶ \qquad P = \frac{V^2}{2R} = I^2R \qquad \text{and} \qquad I = \frac{V}{\sqrt{2}\,R} = 0.707\,\frac{V}{R}. \tag{9-86}$$

Let us now denote ω_x = *angular velocities at the half-power points*. We may then write, for the half-power points,

$$I = \frac{V}{\sqrt{2}\,R} = \frac{V}{\sqrt{R^2 + \left(\omega_x L + \dfrac{-1}{\omega_x C}\right)^2}}. \tag{9-87}$$

From this, by equating the denominators and solving for ω_x, we may find the half-power angular velocities. Thus:

$$\sqrt{2}\,R = \sqrt{R^2 + \left(\omega_x L + \frac{-1}{\omega_x C}\right)^2} \tag{9-88}$$

and
$$2R^2 = R^2 + \left(\omega_x L + \frac{-1}{\omega_x C}\right)^2. \tag{9-89}$$

Then
$$R^2 = \left(\omega_x L + \frac{-1}{\omega_x C}\right)^2, \tag{9-90}$$

and
$$\pm R = \omega_x L + \frac{-1}{\omega_x C}. \tag{9-91}$$

Rearranging terms, we obtain the quadratic equation

$$\omega_x^2 LC \pm \omega_x RC - 1 = 0. \tag{9-92}$$

Solution of this quadratic gives

$$\omega_x = \mp \frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}}. \tag{9-93}$$

In a practical selective circuit, $(R/2L) \ll (1/LC)$. To a good approximation, equation (9-93) becomes

$$\omega_x = \mp \frac{R}{2L} \pm \frac{1}{\sqrt{LC}}, \tag{9-94}$$

or
$$\omega_x = \mp \frac{R}{2L} \pm \omega_r. \tag{9-95}$$

Inasmuch as only positive values of resonant angular velocity have any significance here, this equation may be rewritten:

$$\omega_x = \omega_r \pm \frac{R}{2L}. \tag{9-96}$$

The two values of half-power angular velocity are, then,

▶
$$\omega_1 = \omega_r - \frac{R}{2L} \quad \text{and} \quad \omega_2 = \omega_r + \frac{R}{2L}. \tag{9-97}$$

The width of the peak between the half-power points is

$$\Delta\omega = (\omega_2 - \omega_1) = \frac{R}{2L} - \left(-\frac{R}{2L}\right), \tag{9-98}$$

or

▶
$$\Delta\omega = \frac{R}{L}, \tag{9-99}$$

where:

$\Delta\omega$ = *width of peak*, in **radians per second,**
R = *resistance of circuit*, in **ohms, and**
L = *inductance of circuit*, in **henrys.**

Dividing equation (9-99) through by 2π gives the *frequency pass band,* in cycles per second,

▶ $$\Delta f = \frac{\Delta\omega}{2\pi} = \frac{R}{2\pi L}. \qquad (9\text{-}100)$$

It is often convenient to express the ratio of this pass band to the resonant frequency:

▶ $$\frac{\Delta f}{f_r} = \frac{R}{2\pi f_r L} = \frac{R}{\omega_r L} = \frac{R}{X_{L_r}} = \frac{1}{Q_r}, \qquad (9\text{-}101)$$

where: Q_r is the *quality of the circuit at resonant frequency.*

Equation (9-101), together with Fig. (9-20), shows why Q is employed as a measure of the degree of selectivity, or sharpness of tuning of a circuit. Obviously, the larger the value of Q, the thinner the pass band and the sharper the peak of the resonance curve. For the coils usually employed in resonant communication circuits at frequencies of 10^4 to 10^7 cycles per second, the value of Q is of the order of 100 to 300. At frequencies below this range, the values of Q for such coils are lower.

EXAMPLE 9.20. (a) How wide is the pass band of a circuit having a value of $Q_r = 100$ at a resonant frequency of 10 kilocycles? (b) What are the half-power frequencies of this circuit?

(a) $$\Delta f = \frac{f_r}{Q_r} = \frac{10^4}{100} = 100 \text{ cycles per second.}$$

(b) $$f_x = f_r \pm \frac{R}{4\pi L} = f_r \pm \frac{R}{4\pi f_r L} f_r = f_r\left(1 \pm \frac{1}{2Q}\right)$$

$$= 10^4\left(1 \pm \frac{1}{200}\right) = 10^4(1 \pm 0.005),$$

$$f_1 = 10^4(0.995) = 9950 \text{ cycles per second,}$$
$$f_2 = 10^4(1.005) = 10{,}050 \text{ cycles per second.}$$

PROBLEMS

9-1. A coil having a resistance $R_L = 5$ ohms and an inductance $L = 5$ millihenrys is connected in series with a resistor having a resistance of 10 ohms. Upon application of a 50-cycle emf, a current of 10 amperes is observed to flow.

(a) Write the complex expressions for the individual impedances and for the total impedance.

(b) Write the complex expressions for the individual potential drops and for the total potential drop.

(c) Write the complex expressions for the individual vector powers and for the total vector power.

(d) Prepare a chart similar to Fig. 9.08 to show in summary form the correspondence between the impedance, vector, and vector power diagrams.

(e) Compute the over-all power factor and reactive factor.

(f) What current will flow if the frequency is reduced to zero (the d-c case)?

9-2. Substitute a capacitor of negligible resistance but 10 microfarads capacitance for the coil of Prob. 9-1, and repeat the procedure outlined for that problem.

9-3. Place the capacitor of Prob. 9-2 in series with the coil and resistor of Prob. 9-1, and repeat the procedure outlined for those problems.

9-4. A coil having an impedance $\mathbf{Z}_L = 2 + j4$ ohms, a capacitor having an impedance $\mathbf{Z}_C = 0 - j10$ ohms, and a resistor having an impedance $\mathbf{Z}_R = 4 + j0$ ohms, all at a frequency of 60 cycles per second, are connected in series across a 60-cycle, 110-volt potential source. Using complex quantities wherever possible,

(a) Draw to scale an impedance diagram showing how the individual impedances combine vectorially to form the total impedance.

(b) Compute the total impedance.

(c) Compute the current, using total potential as reference.

(d) Compute the potential drops across the individual impedances.

(e) Draw to scale a vector diagram showing how the individual potentials combine to form the total potential.

(f) Redraw this diagram, using current as reference.

(g) Using the proper rotational operator, convert the complex expressions of all potentials and currents to the new reference.

(h) Determine the complex expression for vector power for each element, as well as for the total circuit.

(i) Draw to scale a power diagram showing how the individual vector powers combine to form the total vector power.

(j) If a 110-volt direct emf is substituted for the 60-cycle emf, what potential drops will appear across the various circuit elements?

9-5. A coil having an impedance at 60 cycles per second of $\mathbf{Z}_a = 3 + j4$ ohms, and a capacitor having an impedance at 60 cycles per second of $\mathbf{Z}_b = 1 - j10$ ohms, are connected in series. At the following frequencies, what vector current will flow upon application of an emf having an rms value of 100 volts? (a) 60 cycles per second. (b) 40 cycles per second. (c) 0 cycles per second.

9-6. The following measurements were made on a certain circuit element:

f, in Cycles per Second	*V, in Volts*	*I, in Amperes*	*P, in Watts*
60	100	8.47	287
30	100	2.90	33.6

What parameters in series make up this element?

9-7. An electric arc has a characteristic such that when connected to a constant-potential circuit, as the value of arc current increases, the value of arc resistance decreases. This condition tends to result in a virtually unlimited increase in arc current. To prevent the arc current from thus "running away," a *ballast* must be placed in series to stabilize the current at a reasonable value. In a d-c circuit, this ballast must take the form of a series resistor; whereas in an a-c circuit, it may consist of either a resistor, an inductor, a capacitor, or a suitable combination.

A given arc lamp which requires 8 amperes at 15 volts is to be operated from a 110-volt supply line.

(a) If the ballast is to be purely resistive, determine:
  (1) The value of ballast resistance required.
  (2) The efficiency with which the circuit delivers power to the arc lamp.
(b) If a 60-cycle a-c supply is to be used, determine:
  (1) The complex expression for the impedance of an inductive reactor having a 60-cycle $Q(=\omega L/R)$ of 10 which will provide suitable ballast.
  (2) The efficiency with which the circuit delivers power to the arc lamp.
(c) Draw to scale the complete vector diagram associated with (b).

9-8. (a) Determine the consequences of inserting sufficient capacitance in series with the circuit of Prob. 9-7(b) to obtain unity power factor. Can you suggest an alternative?

(b) Determine the consequences of using the inductive ballast with the arc in a d-c circuit.

9-9. A particular series circuit which dissipates 600 watts also requires 800 vars when a potential of 100 volts is applied. Find the possible circuit parameters.

9-10. A certain series circuit consists of the following parameters: $R = 2$ ohms, $L = 10$ millihenrys, and $C = 100$ microfarads.

(a) What is the simplest parallel combination of pure parameters which will be the electrical equivalent of this circuit at 60 cycles per second?

(b) Is this parallel combination of pure parameters the electrical equivalent of the original series combination at 120 cycles per second?

9-11. Derive equations (9-23) and (9-24) from equations (9-16) and (9-17).

9-12. What is the simplest parallel combination (two branches) of pure parameters which will be the electrical equivalent of the series combination of $R = 2$ ohms, $L = 10$ millihenrys, and $C = 100$ microfarads at 15.9 kilocycles per second?

9-13. (a) Prove analytically that the impedance locus of Fig. 9.12B(c) is a straight line.

(b) Prove analytically that the admittance locus of Fig. 9.12B(b) lies on a circle through the origin.

9-14. (a) Prove analytically that the impedance locus of Fig. 9.12C(c) is a straight line.

(b) Prove analytically that the admittance locus of Fig. 9.12C(b) lies on a circle through the origin.

9-15. (a) Prove analytically that the impedance locus of Fig. 9.12D(c) is a straight line.

(b) Prove analytically that the admittance locus of Fig. 9.12D(b) lies on a circle through the origin.

9-16. At what frequency will resonance occur in a series circuit in which $R = 4$ ohms, $L = 100$ millihenrys, and $C = 100$ microfarads?

9-17. (a) At what frequency would the circuit of Prob. 9-3 be resonant?

(b) To what value must the inductance be adjusted to obtain series resonance at 50 cycles per second?

(c) To what value must the capacitance of the original circuit be adjusted to obtain series resonance at 100 cycles per second?

9-18. (a) Find the resonant frequency of a series circuit whose inductance is 400 microhenrys, whose capacitance is 200 microfarads, and whose resistance is 1000 ohms.

(b) Find the resonant current of this circuit if a potential of 100 volts is applied.

9-19. (a) What is the value of the quality, Q, of the coil of Prob. 9-3 at a frequency of 50 cycles per second?

(b) What is the value of the quality, Q, of the circuit of Prob. 9-3 at a frequency of 50 cycles per second?

(c) What is the value of Q at resonance?

9-20. Under what conditions does the ratio (IX_C/V_s) of a series R-L-C circuit equal Q for the circuit?

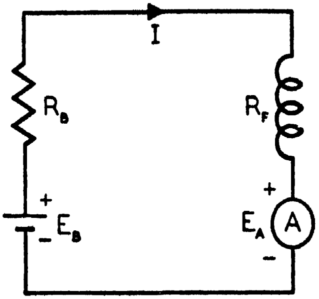
9-21. At 1500 cycles per second, the inductance coil of a certain R-L-C series circuit has a Q of 70. If all the resistance is concentrated in the inductance coil,

(a) Find the width of the pass band of the circuit, in cycles per second.

(b) Find the width of the pass band if Q is doubled in value.

(c) Find the ratio of V_C to V_s for each of the above cases.

9-22. An automobile starter is, in reality, a d-c series motor supplied through the starting switch by a 6-volt automobile storage battery. The accompanying diagram represents such a starting circuit. The series field, F, of the motor has a resistance of 0.06 ohm, and the battery has an internal resistance of 0.008 ohm. If the armature of the motor provides a back emf which is proportional to both the speed of rotation and the armature current, find the maximum power that can be converted to the mechanical form by the armature.



Prob. 9-22.

9-23. A resistor, an inductor, and a capacitor are connected in series, and a 60-cycle, 100-volt emf is applied to the series combination. The following measurements are made: $P_{\text{total}} = 300$ watts; $V_C = 90$ volts. The capacitor nameplate specifies a rating of 100 microfarads. Assuming capacitor resistance to be negligible, what two values might the inductance have? (*Hint:* Sketch a vector diagram.)

CHAPTER 10

COMPLEX IMPEDANCES IN PARALLEL

10.01. Impedances in Parallel

Figure 10.01 shows complex impedances $\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3, \cdots, \mathbf{Z}_n$, so connected that all experience the same applied potential difference. This type is known as a *parallel combination* of complex impedances. The net impedance of the parallel combination is found by application of Kirchhoff's current law and Ohm's law. By Kirchhoff's current law,

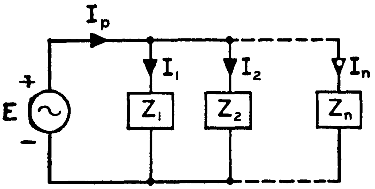


Fig. 10.01. Complex impedances in parallel.

▶ $$\mathbf{I}_p = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \cdots + \mathbf{I}_n. \quad (10\text{-}1)$$

By Ohm's law, dividing by the complex potential drop will yield the reciprocal complex impedance

$$\frac{1}{\mathbf{Z}_p} = \frac{\mathbf{I}_p}{\mathbf{V}} = \frac{\mathbf{I}_1}{\mathbf{V}} + \frac{\mathbf{I}_2}{\mathbf{V}} + \frac{\mathbf{I}_3}{\mathbf{V}} + \cdots + \frac{\mathbf{I}_n}{\mathbf{V}}, \qquad (10\text{-}2)$$

or

▶ $$\frac{1}{\mathbf{Z}_p} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \cdots + \frac{1}{\mathbf{Z}_n}, \qquad (10\text{-}3)$$

where:

$\mathbf{Z}_p$ = *net impedance of parallel combination,* in **complex ohms,** and

$\mathbf{Z}_n$ = *impedance of the nth parallel element,* in **complex ohms.**

We have shown that *the reciprocal of the net impedance of a parallel combination of impedances is the complex sum of the reciprocals of the individual impedances.*

10.02. Equivalent Parallel Complex Admittances

In work with parallel combinations of practical complex impedances, it is often convenient to rewrite equation (10-3) in terms of equivalent complex admittances. In Article 9.05 we did this and attached the subscript s to the two admittance components, conductance and susceptance, to remind us that they are not the reciprocals, respectively, of pure resistance and a paralleling pure reactance, but that they are derived from the resistance and reactance whose series combination they are to replace in a circuit. The last paragraph of Article 9.05 stated that the general equation for complex admittance may be written in the following form:

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = Y\underline{/\phi} = \sqrt{G^2 + B^2}\underline{\Big/\tan^{-1}\left(\frac{B}{G}\right)} = G + jB, \quad (10\text{-}4)$$

where:

$\mathbf{Z} = Z\underline{/\theta} = (R + jX)$ = *complex impedance of a circuit element,* in **complex ohms,**

$\mathbf{Y}$ = *corresponding complex admittance of the circuit element,* in **complex mhos,**

$G = \left(\frac{R}{Z^2}\right)$ = *corresponding conductance of the circuit element,* in **mhos,**

$B = \left(\frac{-X}{Z^2}\right)$ = *corresponding susceptance of the circuit element,* in **mhos,** and

$\phi = (-\theta)$ = *corresponding admittance angle of the circuit element.*

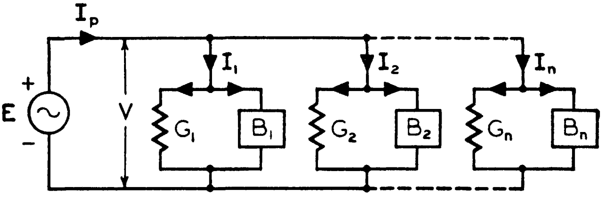


Fig. 10.02. Equivalent parallel circuit to that of Fig. 10.01.

The subscript s which appeared in equation (9-12) has been omitted here, in view of the general interpretation of this equation. It must be understood that *the admittances, conductances, and susceptances encountered hereafter in this text are defined by equation* (10-4). With this specification in mind, we may rewrite equation (10-3) as follows:

▶ $$\mathbf{Y}_p = \frac{1}{\mathbf{Z}_p} = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \cdots + \mathbf{Y}_n, \quad (10\text{-}5)$$

where:

$\mathbf{Y}_p$ = *net equivalent admittance of parallel combination,* in **complex mhos,** and

$\mathbf{Y}_n$ = *equivalent admittance of the nth parallel element,* in **complex mhos.**

In words, we may state that *the net equivalent admittance of a parallel combination of impedances is the complex sum of the equivalent admittances of the individual impedances.* Figure 10.02 shows the equivalent circuit to that of Fig. 10.01.

10.03. Resistances in Parallel

For the case of a parallel combination of pure resistances, equation (10-3) assumes the form

▶
$$\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots + \frac{1}{R_n}. \qquad (10\text{-}6)$$

10.04. Inductances in Parallel

For the case of a parallel combination of pure self inductances, equation (10-3) may be written

$$\frac{1}{jX_{L_p}} = \frac{1}{jX_{L_1}} + \frac{1}{jX_{L_2}} + \frac{1}{jX_{L_3}} + \cdots + \frac{1}{jX_{L_n}}. \qquad (10\text{-}7a)$$

or
$$\frac{1}{j\omega L_p} = \frac{1}{j\omega L_1} + \frac{1}{j\omega L_2} + \frac{1}{j\omega L_3} + \cdots + \frac{1}{j\omega L_n}. \qquad (10\text{-}7b)$$

Multiplying through by $j\omega$ produces the following rule for combining inductances in parallel:

▶
$$\frac{1}{L_p} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_n}, \qquad (10\text{-}8)$$

where all quantities are in **henrys.**

10.05. Capacitances in Parallel

For the case of a parallel combination of pure capacitances, equation (10-3) becomes

$$\frac{1}{jX_{C_p}} = \frac{1}{jX_{C_1}} + \frac{1}{jX_{C_2}} + \frac{1}{jX_{C_3}} + \cdots + \frac{1}{jX_{C_n}}, \qquad (10\text{-}9a)$$

or
$$\frac{1}{j\dfrac{-1}{\omega C_p}} = \frac{1}{j\dfrac{-1}{\omega C_1}} + \frac{1}{j\dfrac{-1}{\omega C_2}} + \frac{1}{j\dfrac{-1}{\omega C_3}} + \cdots + \frac{1}{j\dfrac{-1}{\omega C_n}}. \qquad (10\text{-}9b)$$

Multiplying through by $\left(-j\frac{1}{\omega}\right)$ results in the following rule for combining capacitances in parallel:

▶ $$C_p = C_1 + C_2 + C_3 + \cdots + C_n. \qquad (10\text{-}10)$$

10.06. Impedance and Admittance Quantities for Individual Parameters

Through the use of equation (10-4), we may write the following convenient summary of impedance and admittance expressions applying to individual pure parameters.

▶ (1) *Pure resistance:*

$$\left.\begin{aligned}&\mathbf{Z}_R = R\underline{/0^\circ} = R + j0.\\&Z_R = R; \qquad R_R = R; \qquad X_R = 0; \qquad \theta_R = 0^\circ.\end{aligned}\right\}\qquad (10\text{-}11)$$

$$\left.\begin{aligned}&\mathbf{Y}_R = \frac{1}{\mathbf{Z}_R} = \frac{1}{R}\underline{/0^\circ} = \left(\frac{1}{R}\right) + j0.\\&Y_R = \left(\frac{1}{R}\right); \qquad G_R = \left(\frac{1}{R}\right); \qquad B_R = 0; \qquad \phi_R = 0^\circ.\end{aligned}\right\}\qquad (10\text{-}12)$$

▶ (2) *Pure inductance:*

$$\left.\begin{aligned}&\mathbf{Z}_L = \omega L\underline{/90^\circ} = 0 + j\omega L.\\&Z_L = \omega L; \qquad R_L = 0; \qquad X_L = \omega L; \qquad \theta_L = 90^\circ.\end{aligned}\right\}\qquad (10\text{-}13)$$

$$\left.\begin{aligned}&\mathbf{Y}_L = \frac{1}{\mathbf{Z}_L} = \left(\frac{1}{\omega L}\right)\underline{/-90^\circ} = \left(\frac{-1}{\omega L}\right)\underline{/90^\circ} = 0 + j\left(\frac{-1}{\omega L}\right).\\&Y_L = \frac{1}{\omega L}; \qquad G_L = 0; \qquad B_L = \frac{-1}{\omega L}; \qquad \phi_L = 90^\circ.\end{aligned}\right\}\qquad (10\text{-}14)$$

▶ (3) *Pure capacitance:*

$$\left.\begin{aligned}&\mathbf{Z}_C = \frac{1}{\omega C}\underline{/-90^\circ} = \left(\frac{-1}{\omega C}\right)\underline{/90^\circ} = 0 + j\left(\frac{-1}{\omega C}\right).\\&Z_C = \left(\frac{1}{\omega C}\right); \qquad R_C = 0; \qquad X_C = \left(\frac{-1}{\omega C}\right); \qquad \theta_C = 90^\circ.\end{aligned}\right\}\qquad (10\text{-}15)$$

$$\left.\begin{aligned}&\mathbf{Y}_C = \frac{1}{\mathbf{Z}_C} = -\omega C\underline{/90^\circ} = \omega C\underline{/90^\circ} = 0 + j\omega C.\\&Y_C = \omega C; \qquad G_C = 0; \qquad B_C = \omega C; \qquad \phi_C = 90^\circ.\end{aligned}\right\}\qquad (10\text{-}16)$$

10.07. The Parallel Circuit

A parallel circuit is a circuit supplying energy to a number of devices connected in parallel, i.e., the same potential difference is

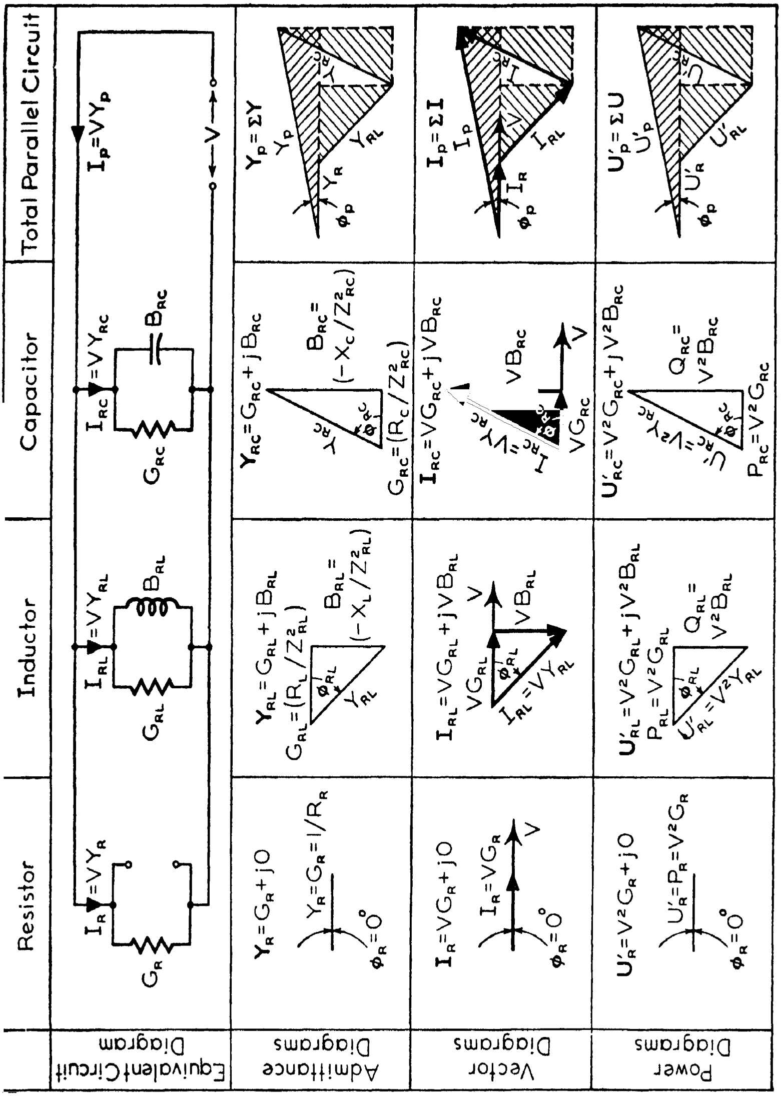


Fig. 10.07. Graphical summary of relations applying to a particular parallel circuit.

applied across each device by the source of supply, and the summation of the currents taken by the individual devices constitutes the supply current. We saw in the first part of this chapter that the total admittance of a number of elements connected in parallel is the complex sum of the individual admittances, or that

$$\blacktriangleright \qquad \mathbf{Y}_p = G_p + jB_p = \Sigma(G + jB) = \Sigma G + j\Sigma B. \qquad (10\text{-}17)$$

It is apparent that the graphical interpretation of this equation may be arrived at by "adding end to end" individual admittance triangles to obtain the resultant admittance triangle of the parallel combination. Figure 10.07 illustrates this procedure for a particular parallel circuit.

Multiplying each of the terms of (10-17) by the effective value of the common potential drop, V, results in one form of Kirchhoff's current law, as it applies to the parallel circuit:

$$\blacktriangleright \qquad V\mathbf{Y}_p = \mathbf{I}_p = VG_p + jVB_p = \Sigma VG + j\Sigma VB. \qquad (10\text{-}18)$$

This equation, too, has a graphical interpretation which is shown by Fig. 10.07. In this figure, all component currents are referred to the common potential drop, **V**, which is drawn horizontally. The conductive and susceptive current components constitute, for a given element, the legs of a right triangle whose hypotenuse is the current taken by that element. Adding these current triangles end to end produces the resultant triangle of total equivalent current. Since the sides of these current triangles are all rotating time vectors, it is proper to place arrowheads on them. This is not the case with either the admittance triangles or the power triangles which are to be considered next.

Multiplying each of the terms of equation (10-18) by the effective value of the common potential drop, V, results in complex power

$$\blacktriangleright \qquad V^2\mathbf{Y}_p = \mathbf{U}'_p = V^2G_p + jV^2B_p = \Sigma V^2G + j\Sigma V^2B$$
$$= \Sigma P + j\Sigma Q = P_p + jQ_p. \qquad (10\text{-}19)$$

We see that for a parallel circuit, too, the total active power is the sum of the individual active powers and that the total reactive power is the algebraic sum of the individual reactive powers. We note also that the resultant total power triangle may be thought of as the end-to-end summation of the individual power triangles.

Figure 10.07 shows us, further, that the admittance angle ϕ (which equals the negative of the power factor angle θ of a given element or of the total circuit) carries on through with the same value in the admittance triangle, the vector diagram, and the power triangle. The *power factor* may be found from any one of the three types of triangles:

$$\blacktriangleright \qquad \text{P.F.} = \cos\phi = \frac{P}{U'} = \frac{VG}{VY} = \frac{G}{Y}. \tag{10-20}$$

Similarly, the *reactive factor* is

$$\blacktriangleright \qquad \text{R.F.} = \sin\phi = \frac{Q}{U'} = \frac{VB}{VY} = \frac{B}{Y}. \tag{10-21}$$

Comparison of equations (10-21) and (9-29) reveals that reactive factor carries the algebraic sign of reactive power. In the inductive case, as a result, both reactive power and reactive factor are positive for the series circuit, and negative for the parallel circuit. The old standard of the AIEE arbitrarily called for all inductive reactive power (and factors) to be considered negative. This convention had the effect of arbitrarily reversing the vertical directions of the series circuit power triangles in Fig. 9.08. By the time of this writing, however, the AIEE had reversed this standard, and now arbitrarily considers all capacitive reactive power to be negative. Such a decision renders correct the series circuit power triangles of Fig. 9.08, but necessitates arbitrarily reversing the vertical directions of the parallel circuit power triangles of Fig. 10.07. As the reader can see, there is little choice between the two schemes; as long as one is used consistently throughout a given solution, it does not matter which one it is, provided the resulting vars are properly labeled either "inductive" or "capacitive." In a series-parallel circuit or in a network, where neither potential nor current is common to all elements, a standard such as that of the AIEE is useful.

Example 10.07. The following impedances are connected in parallel across a 100-volt source: $\mathbf{Z}_A = 5 - j6$; $\mathbf{Z}_B = 3 + j4$; $\mathbf{Z}_C = 0 - j3$ complex ohms. Using the admittance method, determine: (a) The total admittance. (b) The individual branch currents and the total current. (c) The individual and total complex vector powers. (d) The over-all power factor. (e) The over-all reactive factor. (f) The equivalent total impedance.

$$\mathbf{Y}_A = \frac{1}{\mathbf{Z}_A} = \frac{1}{5 - j6} = \frac{5}{25 + 36} + j\frac{-(-6)}{25 + 36} = 0.082 + j0.0983,$$

$$\mathbf{Y}_B = \frac{1}{\mathbf{Z}_B} = \frac{1}{3+j4} = \frac{3}{9+16} + j\frac{-(4)}{9+16} = 0.120 - j0.160,$$

$$\mathbf{Y}_C = \frac{1}{\mathbf{Z}_C} = \frac{1}{0-j3} = \frac{0}{0+9} + j\frac{-(-3)}{0+9} = 0 + j0.333 \text{ complex mhos.}$$

(a) $\mathbf{Y}_p = \mathbf{Y}_A + \mathbf{Y}_B + \mathbf{Y}_C = 0.202 + j0.271$ complex mhos.

(b) Using potential as reference, $\mathbf{V} = 100 + j0$ complex volts, and

$$\begin{aligned}
\mathbf{I}_A &= \mathbf{V}\mathbf{Y}_A = (100 + j0)(0.082 + j0.0983) = 8.2 + j9.83,\\
\mathbf{I}_B &= \mathbf{V}\mathbf{Y}_B = (100 + j0)(0.12 - j0.16) = 12.0 - j16.0,\\
\mathbf{I}_C &= \mathbf{V}\mathbf{Y}_C = (100 + j0)(0 + j0.333) = 0 + j33.3,\\
\mathbf{I}_p &= \mathbf{V}\mathbf{Y}_p = (100 + j0)(0.202 + j0.271) = 20.2 + j27.1 \text{ complex amperes.}
\end{aligned}$$

(c)
$$\begin{aligned}
\mathbf{U}'_A &= \mathbf{V}\hat{\mathbf{I}}_A = (100 + j0)(8.2 - j9.83) = 820 - j983,\\
\mathbf{U}'_B &= \mathbf{V}\hat{\mathbf{I}}_B = (100 + j0)(12.0 + j16.0) = 1200 + j1600,\\
\mathbf{U}'_C &= \mathbf{V}\hat{\mathbf{I}}_C = (100 + j0)(0 - j33.3) = 0 - j3333,\\
\mathbf{U}'_p &= \mathbf{V}\hat{\mathbf{I}}_p = (100 + j0)(20.2 - j27.1)\\
&= 2020 - j2710 \text{ complex volt-amperes.}
\end{aligned}$$

(d) $$(\text{P.F.})_p = \frac{P_p}{U'_p} = \frac{2020}{\sqrt{(2020)^2 + (-2710)^2}} = \frac{2020}{3380} = 0.597 \text{ (leading).}$$

(e) $$(\text{R.F.})_p = \frac{Q_p}{U'_p} = \frac{-2710}{\sqrt{(2020)^2 + (-2710)^2}} = \frac{-2710}{3380} = -0.802 \text{ (capacitive).}$$

(f) $$\mathbf{Z}_p = \frac{1}{\mathbf{Y}_p} = \frac{1}{0.202 + j0.271} = 1.763 - j2.32 \text{ complex ohms.}$$

10.08. Locus Diagrams for Parallel Impedances

Inasmuch as circuit elements consisting of individual "pure" parameters may be treated as resistanceless or reactanceless impedances, we shall study locus diagrams applying to impedances in parallel, and shall consider circuits consisting of pure parameters in parallel to be special cases of this general case.

The methods of developing locus diagrams for impedances in parallel are similar to those used in developing series-circuit locus diagrams. We already know how to draw impedance and admittance triangles, current and potential vector diagrams, and power triangles for impedances in parallel, as well as for impedances in series. Locus diagrams are merely developments of such vector diagrams and triangles, whether the circuit is a series or a parallel circuit. We are not surprised to learn that the parallel-circuit loci, like the series-circuit loci, usually turn out to be circles or straight lines.

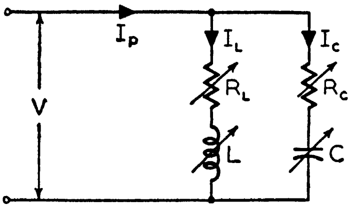


Fig. 10.08. Two-branch parallel-tuning circuit.

Figure 10.08 shows a parallel circuit of two branches, one consisting of resistance and inductance in series, and the other consisting of resistance and capacitance in series. Many other types of parallel circuits are possible, but this one finds particular application and consequently is a good example to employ in studying parallel-circuit locus diagrams.

Referring to the figure, we may write the following equation for the admittance of this particular parallel combination:

$$\mathbf{Y}_p = G_p + jB_p = \left[\frac{R_L}{Z_{RL}^2} + \frac{R_C}{Z_{RC}^2}\right] + j\left[\frac{-X_L}{Z_{RL}^2} + \frac{-X_C}{Z_{RC}^2}\right]. \tag{10-22}$$

In terms of the basic parameters, this equation becomes

$$▶\quad \mathbf{Y}_p = G_p + jB_p = \left[\frac{R_L}{R_L^2 + \omega^2L^2} + \frac{R_C}{R_C^2 + \dfrac{1}{\omega^2C^2}}\right] + j\left[\frac{-\omega L}{R_L^2 + \omega^2L^2} + \frac{\dfrac{1}{\omega C}}{R_C^2 + \dfrac{1}{\omega^2C^2}}\right]. \tag{10-23}$$

An examination of this equation reveals a total of five different quantities which may be used as the independent variable in constructing a locus diagram: frequency, R_L, R_C, L, and C.

The a-c version of Ohm's law states that the total current taken by the parallel combination is

$$\mathbf{I}_p = \mathbf{I}_L + \mathbf{I}_C = \mathbf{V}\mathbf{Y}_p = VY_p\underline{/(\theta_v + \phi_p)} = I_p\underline{/\theta_{ip}}. \tag{10-24}$$

From equations (10-22) and (10-24), we may write the following expression for the *magnitude* of the resultant current:

$$▶\quad I_p = VY_p = V\sqrt{\left[\frac{R_L}{Z_L^2} + \frac{R_C}{Z_C^2}\right]^2 + \left[\frac{-X_L}{Z_L^2} + \frac{-X_C}{Z_C^2}\right]^2}. \tag{10-25a}$$

The resultant power factor angle is

$$\blacktriangleright \quad \theta_p = (-\phi_p) = -\tan^{-1}\left[\frac{-X_L Z_C^2 - X_C Z_L^2}{R_L Z_C^2 + R_C Z_L^2}\right]$$

$$= \tan^{-1}\left[\frac{X_L Z_C^2 + X_C Z_L^2}{R_L Z_C^2 + R_C Z_L^2}\right]. \qquad (10\text{-}25b)$$

These equations form a convenient basis for studying the circuit consisting of $\mathbf{Z}_{RL}$ in parallel with $\mathbf{Z}_{RC}$.

10.09. Locus Diagrams for $\mathbf{Z}_{RL}$ and $\mathbf{Z}_{RC}$ in Parallel

As was pointed out in the previous article, there are five possible independent variables involved in the circuit consisting of $\mathbf{Z}_{RL}$ in parallel with $\mathbf{Z}_{RC}$. This means five sets of locus diagrams, one for each variable. We shall treat them in turn, assuming the applied potential, V, to be constant.

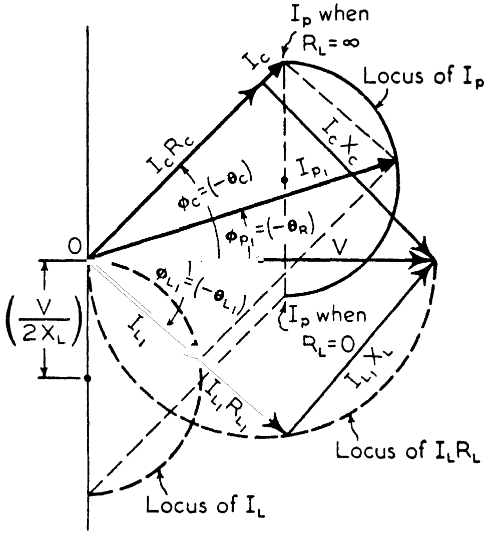


Fig. 10.09A. Locus diagram for variable R_L parallel-tuning circuit.

1. *R_L Variable.* Let us suppose that the resistance of the inductive branch of the parallel circuit is varied from zero to infinity (open circuit). Since the capacitive branch is unaffected by R_L, the capacitive current may be represented by a constant vector such as $\mathbf{I}_C$ of Fig. 10.09A. The vector representing the inductive current, $\mathbf{I}_L$, varies along a semicircle obtained as in

Article (9.10), here shown as a dashed line. The locus of total current, $\mathbf{I}_p$, is found to be the solid semicircle obtained by adding the semicircular locus of $\mathbf{I}_L$ to the fixed vector, $\mathbf{I}_C$, in accordance with Kirchhoff's current law. The maximum value of the total current is, of course, the value of $\mathbf{I}_p$ when the vector $\mathbf{I}_p$ passes through the center of the locus curve of $\mathbf{I}_p$. Depending on θ_C and the value of I_C relative to the diameter, (V/X_L), of the locus curve of $\mathbf{I}_p$, the locus curve may either contact or lie completely above the horizontal axis. If it lies above, no amount of adjusting of R_L alone could put $\mathbf{I}_p$ in phase with $\mathbf{V}$. The reasoning behind the potential locus should be obvious.

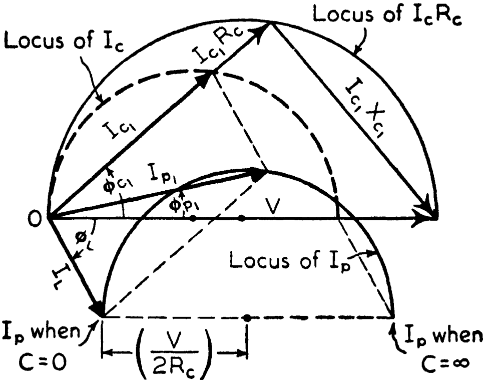


Fig. 10.09B. Locus diagram for variable C parallel-tuning circuit.

2. *R_C Variable.* This case is so similar to the preceding case as to need no further discussion. The construction of a locus diagram for this case is left as a student exercise.

3. *L Variable.* This case is so similar to the following case as to need no discussion. The construction of a locus diagram for this case is left as a student exercise.

4. *C Variable.* If the capacitance of the capacitive branch of Fig. 10.08 is considered as the only variable, a locus diagram similar to that of Fig. 10.09B may be established. In this case, $\mathbf{I}_L$ is fixed. As C is varied from zero (an open circuit) to infinity (a short circuit), I_C varies from zero to a maximum value equal to (V/R_C), and θ_C varies from 90° leading to zero. From a thoughtful consideration of Fig. 9.12C for the case where L is zero, we may see that the locus

of $\mathbf{I}_C$ in the present case is the semicircle shown in Fig. 10.09B. From Kirchhoff's current law, we find the locus of $\mathbf{I}_p$ also to be a semicircle. Similarly, the potential locus formed by the intersection of the $\mathbf{I}_C\mathbf{R}_C$ and the $\mathbf{I}_C\mathbf{X}_C$ drops is a locus. We note here that, depending on the values of θ_L and I_L, and on the radius, $(V/2R_C)$, of the locus curve of $\mathbf{I}_p$, the locus of $\mathbf{I}_p$ may either lie entirely below, be tangent to, or intersect the horizontal axis. Consequently, there are three possibilities concerning the attaining of over-all unity power factor by the adjustment of C alone. First, there is the possibility that no amount of adjusting of C will put $\mathbf{I}_p$ in phase with $\mathbf{V}$ because the semicircular locus of $\mathbf{I}_p$ lies entirely below $\mathbf{V}$. Second, there is the possibility that $\mathbf{I}_p$ may be placed in phase with $\mathbf{V}$ at *one* particular value of C because the semicircular locus of $\mathbf{I}_p$ is tangent to $\mathbf{V}$. Finally, there is the possibility that $\mathbf{I}_p$ may be placed in phase with $\mathbf{V}$ at *two* distinct values of C because the semicircular locus of $\mathbf{I}_p$ intersects $\mathbf{V}$. The minimum value of $\mathbf{I}_p$ occurs when $\mathbf{I}_p$ lies along the line from the origin through the center of the locus curve.

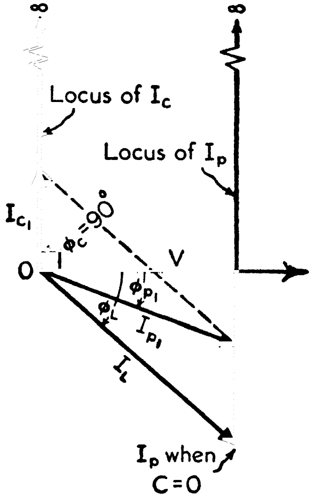


Fig. 10.09C. Locus diagram for variable C parallel-tuning circuit in which $R_C = 0$.

It is instructive to note that as R_C is decreased toward zero, the radius $(V/2R_C)$ of the locus curve of $\mathbf{I}_p$ approaches infinity, and the locus itself approaches the straight line shown in Fig. 10.09C. Minimum current then occurs at unity power factor.

5. *f Variable.* In the case where frequency is the independent variable, the locus diagram for a circuit consisting of $\mathbf{Z}_{RL}$ in parallel with $\mathbf{Z}_{RC}$ is not simple. An analytical treatment of this case shows that, depending on the relative values of the circuit parameters, five different types of variation of total admittance magnitude are possible: an increasing function, a decreasing function, a function which attains a maximum, a function which attains a minimum, and an admittance which is constant at unity over-all power factor for

all values of frequency.[1] Because of the evident relative complexity of this case, a representative general locus diagram cannot be drawn. If a locus diagram for a specific circuit of this type is desired, it may be obtained by combining graphically plots of the loci for the individual branches of that circuit, as determined by the methods of Article 9.12 and by calculation of specific points.

10.10. Parallel Resonance

"Parallel (phase) resonance is the steady-state condition which exists in a circuit comprising inductance and capacitance connected in parallel, when the current entering the circuit from the supply lines is in phase with the voltage across the circuit."[2]

Reference to Fig. 10.07 makes it clear that, although the AIEE defines parallel resonance as above, this condition may, with equal accuracy, be defined by any one of the following relations in a parallel circuit consisting of $\mathbf{Z}_{RL}$ in parallel with $\mathbf{Z}_{RC}$:

▶ $$(\text{P.F.})_p = \cos \phi_p = 1, \tag{10-26}$$

▶ $$Q_p = Q_L + Q_C = 0, \tag{10-27}$$

▶ $$VB_p = VB_L + VB_C = 0, \tag{10-28}$$

and

▶ $$B_p = B_L + B_C = 0. \tag{10-29}$$

It suits the purpose of this text to employ (10-29) in the analysis of parallel resonance.

Substituting $B_L = (-X_L/Z_L^2)$ and $B_C = (-X_C/Z_C^2)$ in (10-29) we obtain

$$\frac{+X_L}{Z_L^2} + \frac{+X_C}{Z_C^2} = 0. \tag{10-30}$$

Realizing that $X_L = \omega L$ and $X_C = (-1/\omega C)$, we may rewrite this as

$$\frac{\omega L}{R_L^2 + \omega^2 L^2} + \frac{\dfrac{-1}{\omega C}}{R_C^2 + \dfrac{1}{\omega^2 C^2}} = 0. \tag{10-31}$$

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[1] For a more complete discussion see Reed, M. B., *Alternating-Current Circuit Theory,* pages 162–168. New York: Harper & Brothers, 1948.

[2] *American Standard Definitions of Electrical Terms,* No. 65.10.535. American Institute of Electrical Engineers, 1942.

This reduces to

$$\omega^2 LC(L - CR_C^2) = (L - CR_L^2), \tag{10-32}$$

from which we may write

$$\omega_{\text{p.r.}} = \frac{1}{\sqrt{LC}}\sqrt{\frac{L - CR_L^2}{L - CR_C^2}}, \tag{10-33}$$

and

▶ $$f_{\text{p.r.}} = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{L - CR_L^2}{L - CR_C^2}}, \tag{10-34}$$

where the subscript "p.r." denotes "parallel-resonant." This equation also may be used to define the condition of parallel resonance.

A study of equation (10-34) reveals a number of interesting facts about the possibility of tuning a circuit for parallel resonance. First of all, we observe that if a given circuit is parallel-resonant to a particular frequency, variation of any of the four quantities under the second radical will "detune" the circuit. This fact suggests the possibility that variation of any one of the four quantities R_L, R_C, L, or C, or of the fifth quantity, frequency, may be employed to tune a circuit to parallel resonance. Further study reveals that this is so—with certain qualifications. We note that whenever the "two-story" quantity within the large radical is negative, parallel resonance could occur only at an "imaginary" frequency and is impossible, practically. This statement would be true whenever *either* $CR_L^2 > L$ *or* $CR_C^2 > L$. If *both* of these conditions are simultaneously true, parallel resonance again becomes possible.

A second interesting fact gained from a study of equation (10-34) is that whenever $R_L = R_C \neq \sqrt{L/C}$, the large radical reduces to unity, and the equation itself becomes

▶ $$f_{\text{p.r.}} = \frac{1}{2\pi\sqrt{LC}} \qquad \left(\text{when } R_L = R_C \neq \sqrt{\frac{L}{C}}\right). \tag{10-35}$$

This formula is the same as for the condition of series resonance. A third and related fact is that frequently, in practice, R_L and R_C are so small relative to L that the negative terms of the large radical are negligible, and equation (10-35) becomes a very good approximation of the parallel-resonant condition and consequently, it is sometimes used in communications work to define both series and (approximately) parallel resonance.

The fourth interesting fact gleaned from a study of equation (10-34) is that if $R_L = R_C = \sqrt{L/C}$, this equation is indeterminate and becomes unable either to specify a particular frequency at which the circuit will be resonant or to specify any frequencies at which the circuit will not be resonant! This state of affairs suggests that perhaps under this condition the circuit will resonate all frequencies! We shall investigate to see.

Let us begin with equation (10-31), which must always hold at parallel resonance:

$$\frac{\omega L}{R_L^2 + \omega^2 L^2} = \frac{\dfrac{1}{\omega C}}{R_C^2 + \dfrac{1}{\omega^2 C^2}}. \tag{10-36}$$

As this equation stands, only particular combinations of R_L, R_C, L, C, and ω will allow the equation to hold. Let us insert $R_L = R_C = \sqrt{L/C}$ and see whether this equation then holds, regardless of frequency. If it does, our speculation will have been proved correct; we shall have shown that under a particular condition a circuit may be parallel-resonant at all frequencies. Making the substitution, we have

$$\frac{\omega L}{\dfrac{L}{C} + \omega^2 L^2} = \frac{\dfrac{1}{\omega C}}{\dfrac{L}{C} + \dfrac{1}{\omega^2 C^2}}. \tag{10-37}$$

Cross multiplication yields the following identity, which does hold regardless of frequency:

$$\frac{\omega L^2}{C} + \frac{L}{\omega C^2} = \frac{\omega L^2}{C} + \frac{L}{\omega C^2}. \tag{10-38}$$

We have discovered that if $R_L = R_C = \sqrt{L/C}$, parallel resonance exists at all frequencies!

It will prove profitable to pursue this study a bit further, evaluating the total admittance of the circuit under this condition. Since, at parallel resonance, $B_L + B_C = 0$, we conclude that the total admittance of this circuit is

$$\mathbf{Y}_p = \mathbf{Y}_L + \mathbf{Y}_C = (G_L + G_C) + j0$$

$$= \left[\frac{R_L}{Z_L^2} + \frac{R_C}{Z_C^2}\right]\underline{/0^\circ} = \left[\frac{\sqrt{\frac{L}{C}}}{\frac{L}{C} + \omega^2 L^2} + \frac{\sqrt{\frac{L}{C}}}{\frac{L}{C} + \frac{1}{\omega^2 C^2}}\right]\underline{/0^\circ}. \qquad (10\text{-}39)$$

This simplifies to the following relation:

▶ $$\mathbf{Y}_p = Y_p\underline{/\phi_p} = \sqrt{\frac{C}{L}}\,\underline{/0^\circ} \qquad \left(\text{when } R_L = R_C = \sqrt{\frac{L}{C}}\right). \qquad (10\text{-}40)$$

We see, then, that under the special condition where $R_L = R_C = \sqrt{L/C}$, the parallel circuit is resonant at all frequencies and has a total admittance of constant value. This is one of the conditions mentioned in part 5 of Article 10.09.

Equation (10-32) may be solved not only for frequency but also for any one of the four parameters. These solutions involve only simple algebraic manipulations and are left as student exercises. The results of these solutions are presented in the following convenient summary of general conditions for parallel resonance in a two-branch parallel circuit:

▶ $$f_{\text{p.r.}} = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{L - CR_L^2}{L - CR_C^2}}, \qquad (10\text{-}41)$$

▶ $$R_{L\text{p.r.}} = \sqrt{R_C^2 LC\omega^2 - L^2\omega^2 + \frac{L}{C}}, \qquad (10\text{-}42)$$

▶ $$R_{C\text{p.r.}} = \sqrt{\frac{R_L^2}{\omega^2 LC} - \frac{1}{\omega^2 C^2} + \frac{L}{C}}, \qquad (10\text{-}43)$$

▶ $$L_{\text{p.r.}} = \frac{\omega^2 C^2 R_C^2 + 1}{2\omega^2 C} \pm \sqrt{\left[\frac{\omega^2 C^2 R_C^2 + 1}{2\omega^2 C}\right]^2 - \frac{R_L^2}{\omega^2}}, \qquad (10\text{-}44)$$

and

▶ $$C_{\text{p.r.}} = \frac{R_L^2 + \omega^2 L^2}{2\omega^2 R_C^2 L} \pm \sqrt{\left[\frac{R_L^2 + \omega^2 L^2}{2\omega^2 R_C^2 L}\right]^2 - \frac{1}{\omega^2 R_C^2}}. \qquad (10\text{-}45)$$

Equation (10-41) needs no further discussion.

We note that, by equations (10-42) and (10-43), if either one of the resistances is used as the independent variable in tuning for parallel resonance, no more than one value of that resistance will produce resonance in a given circuit. Furthermore, there is the distinct possibility of an imaginary solution to the equation, in which case no physically possible value of that resistance could produce resonance. This analysis verifies the conclusions which we reached graphically in cases 1 and 2 of our study of locus diagrams in the preceding article.

We note, further, by equations (10-44) and (10-45) that if *either L or C* is used as the independent variable in tuning for parallel resonance, there are three distinct possibilities for a given circuit: two different values of the variable can produce resonance (the radical is positive), only one value of the variable can produce resonance (the radical is zero), or no physically possible value of the variable can produce resonance (the radical is imaginary). The locus diagrams of cases 3 and 4 of the previous article illustrate this graphically.

Example 10.10. A coil having a resistance of 2 ohms and an inductance of 10 millihenrys is placed in parallel with a capacitor having a capacitance of 50 microfarads and negligible resistance. Can parallel resonance occur, and if so, at what frequency?

$$
\begin{aligned}
f_{\text{p.r.}} &= \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{L - CR_L^2}{L - CR_C^2}} \\
&= \frac{1}{2\pi\sqrt{10 \times 10^{-3} \times 10 \times 10^{-6}}}\sqrt{\frac{10 \times 10^{-3} - 10 \times 10^{-6} \times 2^2}{10 \times 10^{-3} - 0}} \\
&= \frac{1}{2\pi\sqrt{10} \times 10^{-4}}\sqrt{\frac{0.010 - 0.000040}{0.010}} \approx \frac{10^4}{2\pi\sqrt{10}} \\
&= 504 \text{ cycles per second.}
\end{aligned}
$$

Even the 2-ohm resistance of the coil does not exert appreciable influence in determining the resonant frequency in this case. Inasmuch as the answer is not an imaginary number, parallel resonance can exist at a frequency of 504 cycles per second.

10.11. Parallel Resonance Curves

In parallel circuit studies, it is often desirable to obtain curves of magnitude of effective currents, potential drops, and power factor as functions of the independent circuit variable. Such curves are particularly useful in the vicinity of resonance and therefore are frequently called *parallel resonance curves.*

Equation (10-25a) gives us the expression for the magnitude of effective total current for a two-branch circuit comprised of $\mathbf{Z}_{RL}$ in parallel with $\mathbf{Z}_{RC}$, and (10-25b) gives the expression for power-factor angle. The expressions for the individual branch currents are obtainable, of course, from Ohm's law, as are the expressions for any potential drops which may be desired. Following is a summary of such relations:

▶ $$I_p = VY_p = V\sqrt{\left[\frac{R_L}{Z_L^2}+\frac{R_C}{Z_C^2}\right]^2+\left[\frac{-X_L}{Z_L^2}+\frac{-X_C}{Z_C^2}\right]^2}, \qquad (10\text{-}46)$$

▶ $$\theta_p = \tan^{-1}\left[\frac{X_LZ_C^2+X_CZ_L^2}{R_LZ_C^2+R_CZ_L^2}\right], \qquad (10\text{-}47)$$

▶ $$I_L = VY_L = \frac{V}{Z_L} \quad \text{and} \quad I_C = VY_C = \frac{V}{Z_C}, \qquad (10\text{-}48)$$

▶ $$V_p = I_pZ_p = \frac{I_p}{Y_p}. \qquad (10\text{-}49)$$

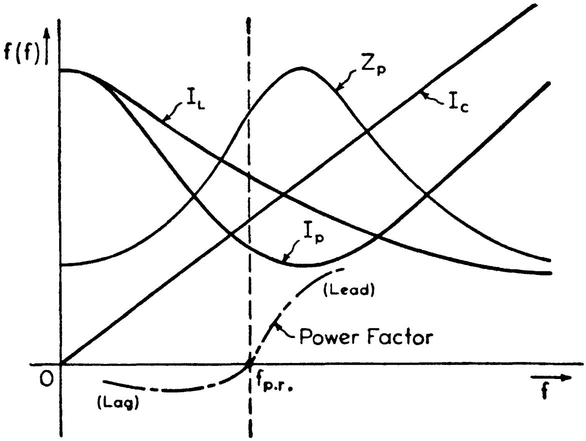


Fig. 10.11. Variable-frequency parallel-resonance curves.

The curves which may be plotted are usually based on either the assumption of constant total current or the assumption of constant applied potential. Since $V = IZ$, the former corresponds to plotting dependent variation of total impedance and the component admittance quantities; since $I = VY$, the latter corresponds to plotting variations of total admittance and the component admittance quantities. In either case, any one of the five quantities, R_L, R_C, L, C, or f, may be used as the independent variable.

Figure 10.11 shows parallel resonance curves for variable frequency and constant applied potential for a particular circuit. The reader is cautioned to recall that even when frequency is the independent variable, it is generally true that minimum total current occurs precisely at parallel resonance only when $R_L = R_C$. A review of the preceding article should be undertaken if this point is not clearly in mind. The sketching of typical parallel resonance curves for the cases of other independent variables is left for student exercises.

In Chapter 9 we saw that series-resonant circuits possess the property of frequency selectivity and are, therefore, very important in communications applications. This property arises from the fact that the impedance of a series-resonant circuit is minimum, with the result that a given applied potential supplies a maximum value of effective current.

We note that a parallel-resonant circuit exhibits characteristics which are very nearly the inverse of those displayed by the series-resonant circuit. Minimum current—and therefore maximum impedance—of a parallel combination occurs either close to or exactly at parallel resonance, depending on the values of resistance in the parallel branches. Whereas the series-resonant circuit is employed as a frequency discriminator to pass current of a wanted frequency in maximum amount, the parallel-resonant circuit is employed as a "wave trap" to block as completely as possible an unwanted frequency, passing current of that frequency in very nearly or exactly minimum amount. Example 10.12 illustrates both of these principles.

In another application, the property of virtually maximum impedance at parallel resonance is used to make a parallel-resonant load absorb maximum power from a high-impedance source by "matching" load and source impedances at resonant frequency. The principle of impedance matching will be taken up in Article 12.11.

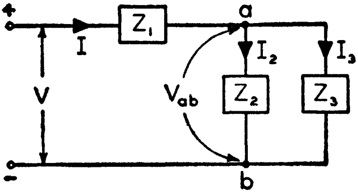


Fig. 10.12A. A series-parallel circuit.

10.12. Impedances in Series-parallel

When a circuit consists of a number of series elements some of which are composed of two or more impedances in parallel, the parallel combinations may be replaced by their equivalent series combinations, and the resulting circuit may be handled by the series-circuit methods outlined in the preceding chapter.

The circuit of Fig. 10.12A, for example, may be solved as follows.

The total impedance of the circuit is

$$Z = Z_1 + Z_{23} = Z_1 + \frac{1}{Y_{23}} \tag{10-50}$$

or

▶ $$Z = Z_1 + \frac{1}{Y_2 + Y_3} = Z_1 + \frac{1}{\frac{1}{Z_2} + \frac{1}{Z_3}} = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3}. \tag{10-51}$$

The current supplied to the circuit by the source is

▶ $$I = VY = \frac{V}{Z}. \tag{10-52}$$

Then

▶ $$V_1 = IZ_1, \tag{10-53}$$

and

▶ $$V_{23} = V - V_1 = IZ_{23}. \tag{10-54}$$

Also,

▶ $$I_2 = \frac{V_{23}}{Z_2}$$

and

$$I_3 = \frac{V_{23}}{Z_3} = I - I_2. \tag{10-55}$$

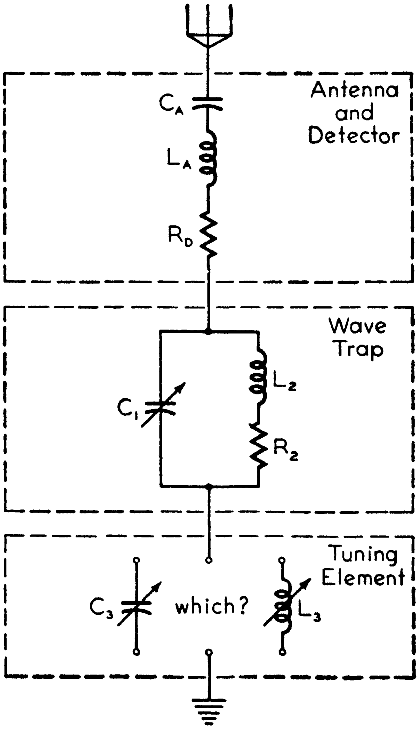


Fig. 10.12B. A particular radio antenna circuit.

Example 10.12. The antenna circuit of a certain radio receiver may be represented, as in Fig. 10.12B, by a capacitance, C_A, of 200 micromicrofarads in series with an inductance, L_A, of 20.5 microhenrys and a detector having a resistance, R_D, of 10 ohms. It is desired to place in series with this antenna a tuning element and a wave trap so designed that a 1490-kilocycle signal can be tuned in but an interfering 1510-kilocycle signal will be suppressed. The two signals can be considered to appear as superimposed potential differences between the top of the antenna and the ground. (a) Making use of an available inductor of 1 microhenry inductance and 0.10 ohm resistance, design a wave trap to suppress the unwanted 1510-kilocycle signal. (b) Determine the value of capacitance or inductance of the tuning element to tune the system to 1490 kilocycles. (c) Compute the ratio of the total impedance at 1510 kilocycles to that at 1490 kilocycles.

(a) *Wave-Trap Design.* Make the wave trap parallel-resonant to the unwanted 1510 kilocycles.

$$\begin{aligned}
\omega &= 2\pi f = 2\pi \times 1510 \times 10^3 = 9.46 \times 10^6 \text{ radians per second.}\\
X_2 &= \omega L_2 = 9.46 \times 10^6 \times 1 \times 10^{-6} = 9.46 \text{ ohms.}\\
Z_2 &= \sqrt{R_2^2 + X_2^2} = \sqrt{(0.1)^2 + (9.46)^2} = 9.46 \text{ ohms.}\\
B_2 &= \frac{-X_2}{Z_2^2} = \frac{-9.46}{(9.46)^2} = -0.1058 \text{ mho.}
\end{aligned}$$

For parallel resonance at 1510 kilocycles

$$\begin{aligned}
B_1 &= -B_2 = +0.1058 = \frac{-X_1}{Z_1^2} = \frac{-X_1}{X_1^2} = \frac{-1}{X_1} = \frac{-1}{-1/\omega C_1} = +\omega C_1.\\
C_1 &= \frac{B_1}{\omega} = \frac{0.1058}{9.46 \times 10^6} = 0.01118 \times 10^{-6} = 11{,}180 \text{ micromicrofarads.}
\end{aligned}$$

(b) *Tuning-Element Design.* Make the entire circuit series resonant to 1490 kilocycles.

$$\begin{aligned}
\omega' &= 2\pi \times 1490 \times 10^3 = 9.35 \times 10^6 \text{ radians per second.}\\
X_2' &= 9.35 \times 10^6 \times 1 \times 10^{-6} = 9.35 \text{ ohms.}\\
Z_2' &= \sqrt{(0.1)^2 + (9.35)^2} = 9.35 \text{ ohms.}\\
B_2' &= \frac{-9.35}{(9.35)^2} = -0.107 \text{ mho.}\\
B_1' &= \frac{-X_1'}{(Z_1')^2} = \frac{-X_1'}{(X_1')^2} = \frac{-1}{X_1'} = \omega' C_1\\
&= 9.35 \times 10^6 \times 0.01118 \times 10^{-6} = 0.1042 \text{ mho.}\\
B_{12}' &= B_1' + B_2' = 0.1042 - 0.107 = -0.0028 \text{ mho.}\\
G_1' &= 0 \text{ mho.}\\
G_2' &= \frac{R_2}{(Z_2')^2} = \frac{0.10}{(9.35)^2} = 0.001143 \text{ mho.}\\
G_{12}' &= G_1' + G_2' = 0.001143 \text{ mho.}\\
\mathbf{Y}_{12}' &= G_{12}' + jB_{12}' = 0.001143 - j0.0028 \text{ complex mho.}\\
\mathbf{Z}_{12}' &= \frac{1}{\mathbf{Y}_{12}'} = 125 + j305.5 \text{ complex ohms.}\\
X_{C_A}' &= \frac{-1}{\omega' C_A} = \frac{-1}{9.35 \times 10^6 \times 200 \times 10^{-12}} = -534.5 \text{ ohms.}\\
X_{L_A}' &= \omega' L_A = 9.35 \times 10^6 \times 20.5 \times 10^{-6} = 191.8 \text{ ohms.}
\end{aligned}$$

For series resonance at 1490 kilocycles

$$\begin{aligned}\Sigma X &= 0 = -534.5 + 191.8 + 305.5 + X_3' \\ X_3' &= +37.2 \text{ ohms} = \omega' L_3 \text{ (inductive, since positive).} \\ L_3 &= \frac{37.2}{9.35 \times 10^6} = 3.98 \times 10^{-6} \text{ henry} = 3.98 \text{ microhenrys.}\end{aligned}$$

(c) *Ratio of Impedances*

(1) *At* 1490 *kilocycles*, the circuit is series-resonant, and only resistance contributes to the net impedance:

$$Z' = R' = R_D + R_{12}' = 10 + 125 = 135 \text{ ohms.}$$

(2) *At* 1510 *kilocycles*

$$\begin{aligned}G_1 &= 0; \qquad G_2 = \frac{R_2}{Z_2^2} = \frac{0.1}{(9.46)^2} = 0.001117 \text{ mho.} \\ G_{12} &= G_1 + G_2 = 0.001117 \text{ mho.} \\ B_{12} &= 0 \text{ (since combination is parallel-resonant at 1510 kilocycles).} \\ \mathbf{Y}_{12} &= G_{12} + jB_{12} = 0.001117 + j0 \text{ complex mho.} \\ \mathbf{Z}_{12} &= \frac{1}{\mathbf{Y}_{12}} = 895 + j0 \text{ complex ohms.} \\ X_{C_A} &= \frac{-1}{\omega C_A} = \frac{-1}{9.46 \times 10^6 \times 200 \times 10^{-12}} = -529 \text{ ohms.} \\ X_{L_A} &= \omega L_A = 9.46 \times 10^6 \times 20.5 \times 10^{-6} = 194 \text{ ohms.} \\ X_3 &= \omega L_3 = 1510 \times 10^3 \times 3.98 \times 10^{-6} = 6 \text{ ohms.} \\ X_{12} &= 0 \text{ (since power factor is unity at parallel resonance).} \\ X &= X_{C_A} + X_{L_A} + X_{12} + X_3 = -329 \text{ ohms.} \\ R_{12} &= 895 \text{ ohms [from part (a)].} \\ R &= R_D + R_{12} = 10 + 895 = 905 \text{ ohms.} \\ Z &= \sqrt{R^2 + X^2} = \sqrt{(905)^2 + (-329)^2} = 963 \text{ ohms.}\end{aligned}$$

Then

$$\frac{Z}{Z'} = \frac{963}{135} = 7.14.$$

This figure compares favorably with the value of about 10 that is normally encountered for such a ratio in the broadcast band.

10.13. Multiple-Resonance Curves

The preceding example dealt with a circuit which was parallel-resonant to one frequency and series-resonant to another. Suitable combinations of additional impedances could have rendered this circuit series-resonant at several distinct frequencies and parallel-resonant at several other distinct frequencies. Such circuits are said to possess the property of *multiple resonance*.

In the preceding example, as in most communications circuits, resistance may be neglected in many types of computations without appreciably affecting the results. In other cases the design engineer may make rapid preliminary calculations of an exploratory nature under the assumption of negligible resistance and then alter his computations to include the parameter where necessary. Such computations, of course, deal only with reactances and consider susceptance to be the negative reciprocal of reactance.

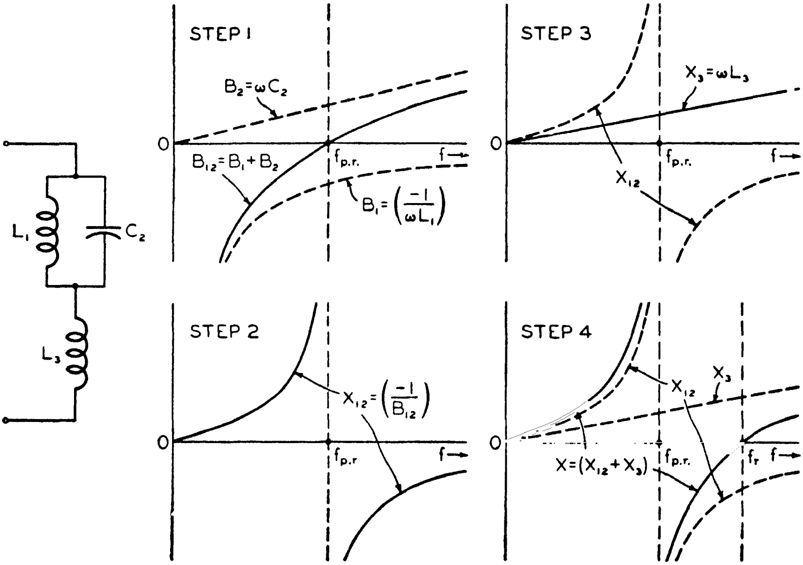


Fig. 10.13. Steps in graphical study of multiple resonance in a particular series-parallel circuit.

The phenomenon of multiple resonance can be studied quite easily and profitably by sketching curves of reactance and susceptance based on a qualitative study of the circuit. Figure 10.13 illustrates this process for a particular circuit.

The steps shown in the qualitative graphical circuit analysis of Figure 10.13 may be outlined as follows:

1. Sketch the susceptance curves of L_1 and L_2 and add, to obtain the resultant curve of susceptance of the parallel combination.

2. Sketch the reciprocal of the resultant susceptance curve. This is the curve of reactance, X_{12}, of the parallel combination.

3. Sketch the curve of reactance, X_3, associated with L_3.

4. Add curves of reactances X_{12} and X_3 to obtain the curve of total reactance, X. Parallel resonance exists wherever this curve shoots off to infinity, and series resonance wherever it crosses the axis through zero.

Similar procedures may be carried out in analyzing other cases. It is instructive to note that replacing L_3 by a capacitance C_3 in the example presented by Fig. 10.13 results in a shift of the series-resonance point from the right to the left of the parallel-resonance point.

10.14. Power Quantities in Series-parallel Circuits

Figures 9.08 and 10.07, which summarize relations for the series circuit and the parallel circuit, respectively, possess power triangles that are entirely similar except for the algebraic sign attached to reactive power. If we were to reverse arbitrarily the algebraic sign of reactive power for one, but not the other, of these two types of circuit, the power triangles for the series and the parallel circuits would be entirely similar. We would note, for example, that positive active power, in watts, would always be drawn horizontally to the right, and that reactive power, in vars, would always be drawn vertically, with inductive vars always in the same direction and capacitive vars always in the opposite direction. These are the conventions because, under steady-state conditions, the quantities represented by the sides of a power triangle are not time-varying but are constant.

As a consequence, if we agree that all inductive vars shall have a certain algebraic sign and all capacitive vars the opposite sign, we may combine the power triangles of both series and parallel elements of a series-parallel combination, just as we combined those of an exclusively series combination. Mathematically, then, for any circuit, regardless of the manner of connection,

$$\blacktriangleright \quad P_T = \Sigma P = \Sigma VI \cos \theta, \tag{10-56}$$

$$\blacktriangleright \quad Q_T = \Sigma Q = \pm \, \Sigma VI \sin \theta, \tag{10-57}$$

$$\blacktriangleright \quad \mathbf{U}'_T = \Sigma \mathbf{U}' = P_T + jQ_T = U'_T\underline{/\theta_T}. \tag{10-58}$$

PROBLEMS

10-1. Three impedances, $\mathbf{Z}_a = 4 + j3$, $\mathbf{Z}_b = 2 - j7$, and $\mathbf{Z}_c = 6 + j8$, are connected in parallel across a 200-volt a-c supply line.

(a) Compute the complex admittances of the individual branches and of the entire circuit.

(b) Write the complex expressions for the individual currents and the total current.

(c) Write the complex expressions for the individual vector powers and for the total vector power.

(d) Prepare a chart similar to Fig. 10.07 to show in summary form the correspondence between the admittance, vector, and vector power diagrams.

(e) Compute the over-all power factor and reactive factor.

(f) What will be the individual branch currents and the total current if the frequency is reduced to zero cycles per second (the d-c case)?

10-2. Repeat Prob. 10-1 for a parallel combination of two branches, the first of which consists of 5 ohms of resistance in series with 10 millihenrys of inductance, and the second of which consists of 2 ohms of resistance in series with 100 microfarads of capacitance. The frequency of the 110-volt applied emf is 159 cycles per second.

10-3. Two impedances, $\mathbf{Z}_a$ and $\mathbf{Z}_b$, are connected in parallel. $\mathbf{Z}_a = 4 - j3$, and $\mathbf{Z}_b$ has such value that its removal reduces the magnitude of the total current to 40 per cent of its former value without affecting its phase angle. What is the value of $\mathbf{Z}_b$?

10-4. Repeat Prob. 10-3, but determine $\mathbf{Z}_b$ such that its removal not only reduces the value of total current to 40 per cent of its former value but also advances the phase position of the total current through an angle of 30°.

10-5. What value of capacitance placed in parallel with a 220-volt, 60-cycle, 5-horsepower, 0.8-efficient, 0.8 (lagging) power-factor induction motor will result in unity power factor in the supply line under rated conditions?

10-6. A certain industrial load of 500 kilowatts and 0.6 lagging power factor is supplied by a 2300-volt supply line. It is planned to install a bank of capacitors to improve the power factor to 0.9 lagging. What total value of capacitance connected across the 2300-volt lines would be needed?

10-7. Draw a typical set of locus diagrams for $\mathbf{Z}_{RL}$ and $\mathbf{Z}_{RC}$ in parallel, with R_C variable.

10-8. Draw a typical set of locus diagrams for $\mathbf{Z}_{RL}$ and $\mathbf{Z}_{RC}$ in parallel, with L variable.

10-9. Draw a typical set of locus diagrams for $\mathbf{Z}_{RL}$ and $\mathbf{Z}_{RC}$ in parallel with R_C equal to zero, and L variable.

10-10. Draw a typical set of locus diagrams for $\mathbf{Z}_{RL}$ and $\mathbf{Z}_{RC}$ in parallel, with R_C equal to zero, and C variable. Discuss the occurrence of maximum, minimum, and resonance value for I_p with constant applied emf.

10-11. (a) Design a circuit consisting of $\mathbf{Z}_{RL}$ and $\mathbf{Z}_{RC}$ in parallel such that resonance and minimum admittance occur at a frequency of 1 megacycle per second. Use a 100-microhenry coil having a Q of 80.

(b) What is the value of minimum admittance?

10-12. Derive equations (10-42) through (10-45) from equation (10-32).

10-13. Branch A of a two-branch parallel impedance combination consists of a coil having a resistance of 10 ohms and an inductance of 10 millihenrys. Branch B consists of a capacitor having an equivalent series resistance of 0.5 ohm and a capacitance of 10 microfarads. A potential of 100 volts is applied.

(a) Employing the admittance method, find the total current taken by the circuit at a frequency of 100 cycles per second.

(b) Draw the admittance triangles for the individual branches and show how the individual admittances combine to equal the total admittance.

(c) Draw a diagram for the equivalent series circuit.

(d) At what frequency will parallel resonance occur?

10-14. What type and value of circuit parameter must be placed in series with the parallel impedance combination of Prob. 10-13 to produce an over-all power factor of 0.6 leading at a frequency of 100 cycles per second?

10-15. The branch impedances of a two-branch parallel circuit are, at a frequency of 60 cycles per second: $\mathbf{Z}_A = 8 - j6$, and $\mathbf{Z}_B = 3 + j4$ ohms. If L is varied,

(a) What are the maximum and minimum values of Y_p?

(b) How many resonant points are there?

(c) Proper variation of one of the quantities R_L, R_C, or C, will make it possible to obtain only one resonant point by varying L. What is the value of this quantity?

(d) What value of L will produce resonance under the condition of (c)?

10-16. Branch A of a two-branch parallel circuit consists of a series combination of the following parameters: $R_A = 4$ ohms, $L_A = 0.2$ henry, and $C_A = 4$ microfarads. Branch B consists of the following parameters in series: $R_B = 20$ ohms, $L_B = 0.4$ henry, $C_B = 20$ microfarads.

(a) Using frequency as abscissa, plot the total current of this circuit as a function of frequency for an applied potential maintained at 100 volts.

(b) How many maximum and minimum points are there?

(c) Are these points resonant points? Explain.

10-17. Assume the impedances of Fig. 10.12A to have the following values: $\mathbf{Z}_1 = 2 + j8$, $\mathbf{Z}_2 = 10 + j3$, $\mathbf{Z}_3 = 2 - j10$ complex ohms. If $V_T = 100$ volts,

(a) Find $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$.

(b) Find $\mathbf{V}_{ab}$.

(c) Repeat (a) and (b) for double the frequency.

10-18. Repeat Prob. 10-17 for the following initial values of impedance: $\mathbf{Z}_1 = 8 - j2$, $\mathbf{Z}_2 = 3 - j10$, $\mathbf{Z}_3 = 10 + j2$ complex ohms.

10-19. In the antenna circuit of Fig. 10.12B, $C_A = 200$ micromicrofarads, $L_A = 20.5$ microhenrys, $R_D = 10$ ohms, $R_2 = 0.10$ ohm, and $L_2 = 1$ microhenry.

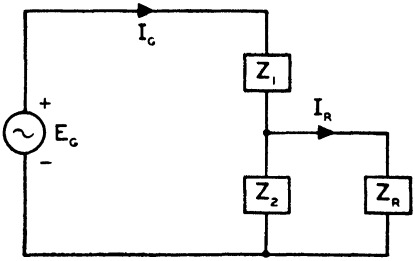
(a) Design a wave trap and a tuning element which will tune in a frequency of 620 kilocycles per second while suppressing a frequency of 600 kilocycles per second.

(b) Determine the ratio of 620-kilocycle current to 600-kilocycle current, assuming the same value of applied potential appears across the circuit at each frequency.

10-20. Referring to the circuit obtained in Example 10.12, on the same graph plot the following quantities as functions of frequency: (a) Total impedance, Z_T. (b) Total admittance, Y_T. (c) Over-all power factor angle, θ_T.

10-21. Repeat Prob. 10-20 for the circuit of Prob. 10-19.

10-22. In practice, it is often convenient to employ in analyses the concept of the "infinite bus," or constant-potential power source of unlimited power capacity. Such a source may be pictured as a generator with no internal impedance. In the accompanying figure, the generator developing $\mathbf{E}_g$ is such a source. The impedance $\mathbf{Z}_1 = jX_L$ and the impedance $\mathbf{Z}_2 = jX_C$.



Prob. 10-22

(a) Show that if $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are equal in *magnitude*, $\mathbf{I}_R$ is independent of $\mathbf{Z}_R$.

(b) Explain what happens to the potential drops across $\mathbf{Z}_1$ and $\mathbf{Z}_2$ as $\mathbf{Z}_R$ is varied from zero to infinity.

10-23. (a) Ignoring resistance (assuming the impedances to contain reactance only), develop, through a series of sketches, the resonant curves applying to the circuit obtained in Example 10.12. Indicate series-resonant and parallel-resonant frequencies.

(b) Compare these curves with the graph obtained in Prob. 10-20.

10-24. Repeat Prob. 10-23 for the circuit of Prob. 10-19 and compare the resulting curves with the graph obtained in Prob. 10-21.

10-25. Two equal incandescent lamps are connected in series across a constant-potential a-c supply line. If one lamp is shunted by a suitable resistance and the other by a suitable inductive impedance, both lamps will glow more brightly. Explain how this may happen.

PART III

NETWORKS

CHAPTER 11

GENERAL SINUSOIDAL STEADY-STATE NETWORK ANALYSIS

11.01. Network Definitions

The preceding chapters of this book have developed concepts involving two-terminal electric circuit elements consisting of individual lumped parameters or simple combinations of lumped parameters, describable as impedances or as admittances. Simple series, parallel, and series-parallel circuits have also been presented. In practice, however, electric circuits commonly consist of many circuit elements, together with emf's, in much more complicated combinations than we have studied thus far. The entire aggregation of circuit elements and emf's in such a combination is termed an *electric network* (or simply *network*). It is the purpose of this article to present certain definitions which are basic to the study of networks carried out through the rest of this chapter.

"*An electric network is a combination of any number of electric elements, the impedances of which may be either lumped or distributed or both, which are connected in any manner, conductively, inductively or capacitatively.*"[1] A network which contains one or more energy sources, or emf's, is called an *active network*. On the other hand, a network which contains no energy source, or emf, is called a *passive network*. An electric network is generally understood to be passive unless the contrary is stated. Figure 11.01 represents in simplified form a portion of an active network.

Each element of this figure, such as *ab*, *bc*, or *d2e*, is termed a *branch*. Each junction point of two or more branches is termed a *node*. Sometimes it is convenient to apply the term "branch" only to combinations of elements having but two terminals, such as *cf*.

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[1] *American Standard Definitions of Electrical Terms*, No. 65.20.625. New York: American Institute of Electrical Engineers, 1942.

In such a case, nodes such as *d* and *e* drop out. The term *loop* is applied to a closed path such as *abcgha* or *abcd2efgha.* A *mesh* is a loop which cannot be subdivided into other loops, such as *afgha.* The circuit of Fig. 11.04B, for example, consists of two active and three passive branches, four nodes, three loops, and two meshes.

Circuit parameters and their impedances whose values are not functions of current or potential are said to be *linear,* and networks composed of such parameters are said to be *linear networks.* Circuit parameters and their impedances whose values are independent of the direction of current flow (that is, which display no rectifying action) are said to be *bilateral,* and networks composed of such elements are said to be *bilateral networks.* Unless otherwise specified, all the parameters and networks discussed in this book are both linear and bilateral. Also, unless otherwise specified, *lumped* parameters will be assumed.

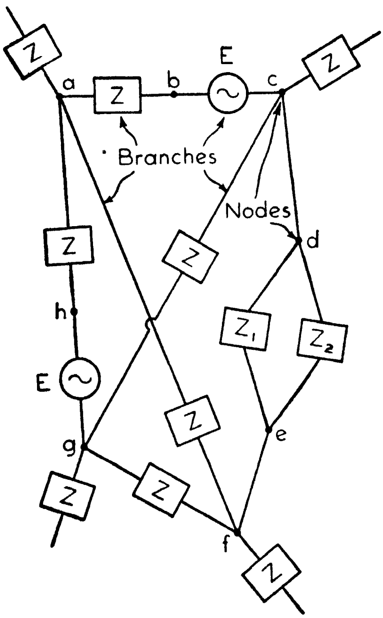

Fig. 11.01. A portion of an active network.

11.02. General Considerations Underlying Network Analysis

Since World War I, electric network theory has undergone a rapid development which is still continuing.[2] Many theorems which are widely useful in network analysis have been evolved, some general and some of only special application. Inasmuch as the complete analysis of the general network is beyond the scope of this book, discussion will be limited to those features which are of special utility in electrical engineering practice.

There are two major aspects of the study of electric network theory: the analysis of the behavior of a given network under all conditions, and the design of a network to perform specific functions. This book will be concerned largely with the former aspect, leaving the more complex problem of design to the more advanced texts in fields of special application. The present treatment will be limited also to the steady-state condition. Transient conditions are treated in Chapter 17.

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[2] For a general historical survey see Dietzold, R. L., "Network Theory Comes of Age," *Electrical Engineering,* September 1948, Vol. 67, No. 9, page 895.

As our previous study suggests, the simplest components into which a lumped linear bilateral network can be resolved are emf's and resistance, inductance, and capacitance elements, or their reciprocals. In the most general type of formulation, each of these components is considered to be a separate branch having its own pair of terminals. By the methods of Chapters 9 and 10 these components can often be so grouped as to reduce the number of effective elements that must be considered in writing equations and, as we shall shortly see, the number of equations that must be written.

Successful analysis of networks requires careful correlation of the algebraic signs attached to current, power, and potential quantities in circuit equations with the assumed or actual positive current and power directions and potential polarities of the actual circuit. The rules for such correlation were presented in Article 3.04 in terms of instantaneous values. If complex expressions for currents and potentials, and vector power computations by the method of conjugates, are substituted for the corresponding instantaneous-value expressions and computations, these same rules can be applied to the vector methods of handling alternating-current circuits. In this connection, the reader should review Article 3.04 carefully at this point.

11.03. Network Geometry and the Application of Kirchhoff's Laws

Network analysis is performed to determine the interrelationships and values of circuit quantities of the following types:

(1) Currents.
(2) Potential differences.
(3) Power quantities.
(4) Impedances (or their component parameters).
(5) Frequencies.

Physically, network analysis is based on Kirchhoff's two laws and Ohm's law, which were presented, respectively, in Articles 2.05 and 3.05. These articles should be reviewed at this point. For our present purposes, these laws may be stated as follows, for the general single-frequency a-c case:

(1) **Kirchhoff's current law:** *The vector sum of all currents approaching any node of a network is zero.*

(2) **Kirchhoff's potential law:** *The vector sum of all potential drops successively encountered in traversing any loop of a network is zero.*

Similar statements may be made, of course, for instantaneous values.

Mathematically, a network problem can be solved only when it is possible to write a number of independent equations equal to the number of unknown quantities, and to include all of the network potentials, currents, and parameters in so doing. An independent equation is one which cannot be obtained from any other or any combination of other independent equations. The information initially available in solving a network problem usually takes one or both of the following forms:

(1) Specification of emf's and circuit parameters.

(2) Specification of individual "loads" of specified powers, potentials, and power factors.

The geometry of a network circuit diagram indicates the manner in which the component branches of the network are connected. In developing general network theory it is fruitful to study the properties of this circuit diagram. The study of these properties of the network circuit diagram is sometimes called the *network geometry.* As was suggested in Article 11.02, the network geometry may sometimes profitably be altered by appropriate substitutions of equivalent circuit elements, thereby reducing the number and complexity of the necessary independent equations. For example, the substitution of a single equivalent impedance for the parallel branches *d1e* and *d2e* of Fig. 11.01 reduces by one the number of meshes and the number of currents.

With the aid of Fig. 11.02, we may readily verify the following statements concerning the writing of network equations:

1. *The total number of independent equations needed to solve a network in which the current of each branch is treated as a distinct dependent variable equals the number of major branches in the network.* (A "major branch" will be understood to be one path between major nodes, such as *aeb*. It may include several elements in series.)

2. *Under the above conditions, the number of independent equations obtainable by application of Kirchhoff's current law is one less than the number of major nodes.* (A "major node" will be understood to be the junction of more than two branches, such as *a*, *b*, *c*, and *d*.)

3. *The remaining independent equations needed for solution can be obtained by application of Kirchhoff's potential law to the appropriate number of loops.* From items (1) and (2) it is seen that the number of independent equations thus obtainable equals the number of major branches minus the number of applications of the current law.

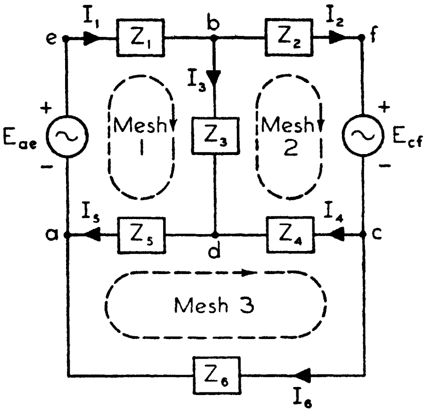


Fig. 11.02. Meshes and nodes of a particular network.

Referring to the figure, we see that there are *four* major nodes: *a*, *b*, *c*, and *d*, and that consequently the number of independent current equations should be $4 - 1 = 3$. The four possible current equations (both independent and dependent) are:

At node a: $I_6 + I_5 - I_1 = 0.$ (11-1)
At node b: $I_1 - I_2 - I_3 = 0.$ (11-2)
At node c: $I_2 - I_4 - I_6 = 0.$ (11-3)
At node d: $I_3 + I_4 - I_5 = 0.$ (11-4)

Any three of these equations are independent, but the fourth inevitably is dependent. For example, if the first three are considered to be independent, no one can be obtained from the other two. Each contains one current quantity not contained by either of the others, whereas together they contain all of the currents.

Thus, the first equation contains exclusively $\mathbf{I}_5$; the second, $\mathbf{I}_3$; the third, $\mathbf{I}_4$. Together, they contain all six currents. The fourth equation is dependent, however, since it contains no current quantity exclusively but contains the quantities $\mathbf{I}_3$, $\mathbf{I}_4$, and $\mathbf{I}_5$, all of which have been previously mentioned.

Similarly, there are *six* major branches: *aeb, bfc, bd, cd, da,* and *ca*; and the number of independent potential equations should be, consequently, $6 - (4 - 1) = 3$. The possible potential-drop equations (both independent and dependent) include the following:

Loop	*Equation*	
aebda:	$\mathbf{I}_1\mathbf{Z}_{eb} + \mathbf{I}_3\mathbf{Z}_{bd} + \mathbf{I}_5\mathbf{Z}_{da} - \mathbf{E}_{ae} = 0,$	(11-5)
bfcdb:	$\mathbf{I}_2\mathbf{Z}_{bf} + \mathbf{E}_{cf} + \mathbf{I}_4\mathbf{Z}_{cd} - \mathbf{I}_3\mathbf{Z}_{db} = 0,$	(11-6)
cadc:	$\mathbf{I}_6\mathbf{Z}_{ca} - \mathbf{I}_5\mathbf{Z}_{ad} - \mathbf{I}_4\mathbf{Z}_{dc} = 0,$	(11-7)
aebfca:	$-\mathbf{E}_{ae} + \mathbf{I}_1\mathbf{Z}_{eb} + \mathbf{I}_2\mathbf{Z}_{bf} + \mathbf{E}_{cf} + \mathbf{I}_6\mathbf{Z}_{ca} = 0,$	(11-8)
aebfcda:	$-\mathbf{E}_{ae} + \mathbf{I}_1\mathbf{Z}_{eb} + \mathbf{I}_2\mathbf{Z}_{bf} + \mathbf{E}_{cf} + \mathbf{I}_4\mathbf{Z}_{cd} + \mathbf{I}_5\mathbf{Z}_{da} = 0,$	(11-9)
aebdca:	$-\mathbf{E}_{ae} + \mathbf{I}_1\mathbf{Z}_{eb} + \mathbf{I}_3\mathbf{Z}_{bd} - \mathbf{I}_4\mathbf{Z}_{dc} + \mathbf{I}_6\mathbf{Z}_{ca} = 0,$	(11-10)
adbfca:	$-\mathbf{I}_5\mathbf{Z}_{ad} - \mathbf{I}_3\mathbf{Z}_{db} + \mathbf{I}_2\mathbf{Z}_{bf} + \mathbf{E}_{cf} + \mathbf{I}_6\mathbf{Z}_{ca} = 0.$	(11-11)

Any three of these equations are independent, but the others inevitably are dependent. For example, if the first three are considered independent, each must contain at least one impedance quantity or emf not contained by the other two, and together they must contain all the impedance quantities and emf's of the network. Thus, the first equation contains exclusively $\mathbf{Z}_{be}$ and $\mathbf{E}_{ae}$; the second, $\mathbf{Z}_{bf}$ and $\mathbf{E}_{cf}$; and the third, $\mathbf{Z}_{ca}$. Together, they contain all six impedances and both emf's. The last four equations cannot possibly contain any impedances or emf's not already accounted for by the first three equations. The last four equations therefore cannot be independent.

In solving the network of Fig. 11.02 we may, then, use any three of the four current equations, plus any three of the potential equations which, together, contain all impedances and all emf's. This procedure verifies for this case the three statements previously made concerning the relationship between network geometry and the writing of independent equations necessary for the solution of the network.

11.04. Network Solution by Direct Application of Kirchhoff's Laws

The following steps can be outlined to aid in solving a general network problem by direct application of Kirchhoff's laws:

1. *Draw a complete circuit diagram.*

(a) Indicate all known quantities, using arrows to indicate positive current directions, and (+) and (−) signs to indicate the polarities of potentials.

(b) Indicate by letters all unknown quantities, arbitrarily assuming polarities of unknown potentials, and positive directions of unknown currents and indicating these directions with arrows. Note that Kirchhoff's current law can be applied in this step in such a way as to reduce to a minimum the number of unknown quantities, thereby reducing the number of independent equations. As an example, if $\mathbf{I}_1 + \mathbf{I}_2 = \mathbf{I}_3$, do not write $\mathbf{I}_3$, but call that current $(\mathbf{I}_1 + \mathbf{I}_2)$. By this procedure the necessity for writing equations based on the current law can often be eliminated. Time will be saved if the first node chosen in applying this step is that which includes the least number of unknown currents.

2. *Begin sketch of vector diagram.*

If the mathematical solution is carried out by complex algebra, this step is not essential to obtaining a solution. However, even a roughly sketched vector diagram is so often helpful in keeping the goal in mind and in visualizing analytically obtained results that this step is included as a part of the standard procedure. Often such a sketch is helpful in detecting mathematical errors.

(a) Choose a convenient known quantity as reference. Sometimes a particular reference is already inferred by the initial information. Refer all other known quantities to this one, employing convenient scales.

(b) Use appropriate arrowheads to distinguish clearly between potentials, currents, and magnetic quantities. Suggested symbols are:

——➤	**E, V**	(potentials)
——►	**I**	(currents)
——◆	**Φ, B, H**	(magnetic quantities)

3. *Write necessary independent current equations.*

(a) Step 1(b), if properly carried out, may already have performed the equivalent operation of this step.

(b) Begin first at that node which involves the least number of unknown currents.

4. *Write necessary independent potential equations.*

(a) Decide whether potential rises or potential drops are to be considered positive.

(b) Employ the simplest possible loops involving the fewest possible different unknown quantities per path. The meshes are often the best loops to follow. (See Fig. 11.04A.)

5. *Solve independent equations simultaneously.*

(a) Use the simplest possible procedure.

(b) If no "simplest" procedure suggests itself in a complex case, eliminate current equations by substitutions in potential equations, and then solve potential equations by the method of determinants. (See any good algebra text.)

6. *Check solution.* Substitute answers in one of the dependent equations obtainable from Kirchhoff's laws.

7. *Complete sketch of vector diagram.* Even though only a free-hand sketch, the vector diagram can often serve as a convenient additional check on the correctness of the solution, and it is always an aid in understanding and evaluating the results.

Example 11.04A. Find the value of current I_5 through the central branch of the unbalanced Wheatstone bridge circuit shown in Fig. 11.04A.

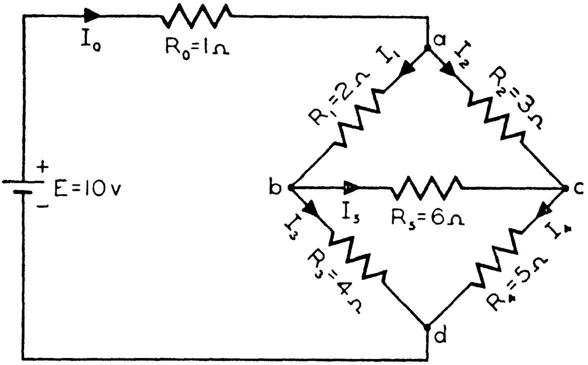


Fig. 11.04A. An unbalanced Wheatstone bridge circuit.

First, we shall assume the positive current directions and the potential polarities indicated on the circuit diagram. Since there are four major nodes, we may write (4 − 1 = 3) independent equations based on Kirchhoff's current law. There are six major branches and therefore six unknowns, so that we need (6 − 3 = 3) independent potential equations to achieve a solution. Six independent equations which would serve the purpose are, calling potential rises positive:

At node a: $I_0 = I_1 + I_2,$
At node b: $I_1 = I_3 + I_5,$
At node c: $I_4 = I_2 + I_5,$
Path abdea: $E - I_0R_0 - I_1R_1 - I_3R_3 = 0,$
Path acba: $-I_2R_2 + I_5R_5 + I_1R_1 = 0,$
Path bcdb: $-I_5R_5 - I_4R_4 + I_3R_3 = 0.$

These six equations can now be solved simultaneously by any one of several algebraic methods. Since only one of the currents is desired, the method of determinants might be as quick as any. First, the current equations must be incorporated into the potential equations.

With slight rearrangement, the current equations are

$$I_0 = I_1 + I_2, \qquad I_3 = I_1 - I_5, \qquad I_4 = I_2 + I_5.$$

Substituting these values in the potential equations gives

$$\begin{aligned} E - I_1R_0 - I_2R_0 - I_1R_1 - I_1R_3 + I_5R_3 &= 0, \\ - I_2R_2 + I_5R_5 + I_1R_1 &= 0, \\ -I_5R_5 - I_2R_4 - I_5R_4 + I_1R_3 - I_5R_3 &= 0. \end{aligned}$$

It will be noticed that only three unknown currents, I_1, I_2, and I_5, are present in these last three independent equations. A solution is therefore possible. Regrouping the terms of these equations to obtain a cyclical order, we have

$$\begin{aligned} (R_0 + R_1 + R_3)I_1 + (R_0)I_2 + (-R_3)I_5 &= R_AI_1 + R_BI_2 + R_CI_5 = E, \\ (R_1)I_1 + (-R_2)I_2 + (R_5)I_5 &= R_DI_1 + R_EI_2 + R_FI_5 = 0, \\ (R_3)I_1 + (-R_4)I_2 + (-R_3 - R_4 - R_5)I_5 &= R_GI_1 + R_HI_2 + R_JI_5 = 0, \end{aligned}$$

where the substitutions of the R's with literal subscripts are obvious. These three equations can be solved simultaneously for I by the use of a determinant of the form

$$I_n = \frac{N_n}{D},$$

where:

N_n = *numerator determinant for current* I_n,
D = *denominator determinant for all currents.*

The determinant D consists of an array of the current coefficients of the simultaneous equations in the same pattern that they have in the equations, when those equations are written with corresponding terms in the same relative positions. The determinant N_n is identical with D, except for the fact that the coefficients of I_n have been replaced by the right-hand members of their respective equations.

In this particular case,

$$D = \begin{vmatrix} R_A & R_B & R_C \\ R_D & R_E & R_F \\ R_G & R_H & R_J \end{vmatrix},$$

and

$$N_5 = \begin{vmatrix} R_A & R_B & E \\ R_D & R_E & 0 \\ R_G & R_H & 0 \end{vmatrix}.$$

These two determinants can be expanded by any of the standard methods of expanding third-order determinants. One general method, which is applicable to the expansion of determinants of any order normally encountered, is outlined below for this example. Symbols are used in order to make the method more obvious, although in an actual problem, numerical values would probably have been substituted by this stage, to speed up the work.

$$\begin{aligned} D &= R_A\begin{vmatrix} R_E & R_F \\ R_H & R_J \end{vmatrix} - R_B\begin{vmatrix} R_D & R_F \\ R_G & R_J \end{vmatrix} + R_C\begin{vmatrix} R_D & R_E \\ R_G & R_H \end{vmatrix} \\ &= R_A(R_ER_J - R_FR_H) - R_B(R_DR_J - R_FR_G) + R_C(R_DR_H - R_ER_G). \end{aligned}$$

In similar manner

$$\begin{aligned} N_5 &= R_A\begin{vmatrix} R_E & 0 \\ R_H & 0 \end{vmatrix} - R_B\begin{vmatrix} R_D & 0 \\ R_G & 0 \end{vmatrix} + E\begin{vmatrix} R_D & R_E \\ R_G & R_H \end{vmatrix} \\ &= R_A(R_E \times 0 - 0 \times R_H) - R_B(R_D \times 0 - 0 \times R_G) + E(R_DR_H - R_ER_G) \\ &= E(R_DR_H - R_ER_G). \end{aligned}$$

We can now evaluate these determinants numerically and solve for I_5 by taking their ratio:

$$\begin{aligned} &R_A = (R_0 + R_1 + R_3) = (1 + 2 + 4) = 7; \qquad R_B = R_0 = 1; \\ &R_C = (-R_3) = -4; \qquad R_D = R_1 = 2; \qquad R_E = (-R_2) = -3; \\ &R_F = R_5 = 6; \qquad R_G = R_3 = 4; \qquad R_H = -R_4 = -5; \\ &R_J = (-R_3 - R_4 - R_5) = -4 - 5 - 6 = -15. \end{aligned}$$

Then $D = 7[(-3)(-15) - (6)(-5)] - 1[(2)(-15) - (6)(4)] + (-4)[(2)(-5) - (-3)(4)]$

$$\begin{aligned} &= 7(45 + 30) - 1(-30 - 24) - 4(-10 + 12) \\ &= 7 \times 75 + 54 - 4 \times 2 = 525 + 54 - 8 \end{aligned}$$

or

$$D = 571.$$

Also, $N_5 = 10[(2)(-5) - (-3)(4)] = 10[-10 + 12] = 10 \times 2 = 20.$

Then, by substitution, we obtain

$$I_5 = \frac{N_5}{D} = \frac{20}{571} = 0.035 \text{ ampere.}$$

Since I_5 was assumed to be directed from b to c, and since the answer turned out to be positive, we conclude that the direction assumed for this current was consistent with the other assumptions employed in the solution (see Article 3.04).

EXAMPLE 11.04B. The following information is available concerning the circuit of Fig. 11.04B:

$$\begin{aligned} e_1 &= 141.4 \sin \omega t \text{ volts},\\ e_2 &= 169.6 \sin (\omega t + 30°) \text{ volts},\\ \mathbf{Z}_1 &= 3 + j4 \text{ complex ohms},\\ \mathbf{Z}_2 &= 8.66 + j5 \text{ complex ohms},\\ \mathbf{Z}_3 &= 10 - j10 \text{ complex ohms}. \end{aligned}$$

Find: (a) The three complex currents, (b) the potential across $\mathbf{Z}_3$, (c) the active power of $\mathbf{Z}_3$, (d) the reactive power of $\mathbf{Z}_3$.

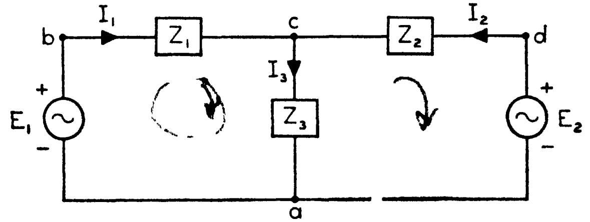


Fig. 11.04B. Circuit of Example 11.04B.

(a) We shall first *assume* the positive directions of current indicated on the figure by the arrows. Inasmuch as there are two major nodes, we may write (2 − 1 = 1) independent equation based on Kirchhoff's current law. There are three major branches, a fact which permits (3 − 1 = 2) independent equations based on Kirchhoff's potential law. This totals three equations—the requisite number for finding the three unknown currents. Three satisfactory equations are, calling potential drops positive,

At node c: $\mathbf{I}_1 + \mathbf{I}_2 - \mathbf{I}_3 = 0,$

Path abca: $-\mathbf{E}_1 + \mathbf{I}_1\mathbf{Z}_1 + \mathbf{I}_3\mathbf{Z}_3 = 0,$

Path adca: $-\mathbf{E}_2 + \mathbf{I}_2\mathbf{Z}_2 + \mathbf{I}_3\mathbf{Z}_3 = 0.$

These three equations may now be solved simultaneously by any one of several algebraic methods. Although there is no particular advantage here, we shall again employ determinants in order to illustrate again this important method. Incorporating the current equation in the two potential equations by substituting $(\mathbf{I}_1 + \mathbf{I}_2)$ for $\mathbf{I}_3$, we obtain

$$-\mathbf{E}_1 + \mathbf{I}_1\mathbf{Z}_1 + (\mathbf{I}_1 + \mathbf{I}_2)\mathbf{Z}_3 = 0,$$

and

$$-\mathbf{E}_2 + \mathbf{I}_2\mathbf{Z}_2 + (\mathbf{I}_1 + \mathbf{I}_2)\mathbf{Z}_3 = 0.$$

Rearranging produces

$$(\mathbf{Z}_1 + \mathbf{Z}_3)\mathbf{I}_1 + (\mathbf{Z}_3)\mathbf{I}_2 = \mathbf{E}_1,$$

and

$$(\mathbf{Z}_3)\mathbf{I}_1 + (\mathbf{Z}_2 + \mathbf{Z}_3)\mathbf{I}_2 = \mathbf{E}_2.$$

Solving for $\mathbf{I}_1$ and $\mathbf{I}_2$ by the method of determinants, we may write

$$\mathbf{I}_1 = \frac{\begin{vmatrix} \mathbf{E}_1 & \mathbf{Z}_3 \\ \mathbf{E}_2 & (\mathbf{Z}_2 + \mathbf{Z}_3) \end{vmatrix}}{\begin{vmatrix} (\mathbf{Z}_1 + \mathbf{Z}_3) & \mathbf{Z}_3 \\ \mathbf{Z}_3 & (\mathbf{Z}_2 + \mathbf{Z}_3) \end{vmatrix}} = \frac{(\mathbf{E}_1)(\mathbf{Z}_2 + \mathbf{Z}_3) - (\mathbf{E}_2)(\mathbf{Z}_3)}{(\mathbf{Z}_1 + \mathbf{Z}_3)(\mathbf{Z}_2 + \mathbf{Z}_3) - (\mathbf{Z}_3)^2},$$

and

$$\mathbf{I}_2 = \frac{\begin{vmatrix} (\mathbf{Z}_1 + \mathbf{Z}_3) & \mathbf{E}_1 \\ \mathbf{Z}_3 & \mathbf{E}_2 \end{vmatrix}}{\begin{vmatrix} (\mathbf{Z}_1 + \mathbf{Z}_3) & \mathbf{Z}_3 \\ \mathbf{Z}_3 & (\mathbf{Z}_2 + \mathbf{Z}_3) \end{vmatrix}} = \frac{(\mathbf{E}_2)(\mathbf{Z}_1 + \mathbf{Z}_3) - (\mathbf{E}_1)(\mathbf{Z}_3)}{(\mathbf{Z}_1 + \mathbf{Z}_3)(\mathbf{Z}_2 + \mathbf{Z}_3) - (\mathbf{Z}_3)^2}.$$

Several terms appear in common in these two equations. Let us evaluate these terms first:

$$\begin{aligned} (\mathbf{Z}_1 + \mathbf{Z}_3) &= (3 + j4) + (10 - j10) = 13 - j6, \\ (\mathbf{Z}_2 + \mathbf{Z}_3) &= (8.66 + j5) + (10 - j10) = 18.66 - j5. \end{aligned}$$

With $\mathbf{E}_1$ employed as horizontal reference, the complex emf's are

$$\begin{aligned} \mathbf{E}_1 &= \frac{141.4}{\sqrt{2}} (\cos 0^\circ + j \sin 0^\circ) \\ &= 100 + j0 = 100\underline{/0^\circ} \text{ complex volts,} \end{aligned}$$

and

$$\begin{aligned} \mathbf{E}_2 &= \frac{169.6}{\sqrt{2}} (\cos 30^\circ + j \sin 30^\circ) \\ &= 120(0.866 + j0.5) = 103.9 + j60 \\ &= 120\underline{/30^\circ} \text{ complex volts.} \end{aligned}$$

The denominator of both equations is

$$\begin{aligned} \mathbf{D} &= (13 - j6)(18.66 - j5) - (10 - j10)^2 \\ &= 212.6 + j23.04 = 213.5\underline{/6.18^\circ}. \end{aligned}$$

The numerator of the first equation is

$$\begin{aligned} \mathbf{N}_1 &= (100 + j0)(18.66 - j5) - (103.9 + j60)(10 - j10) \\ &= 227 - j61 = 235\underline{/-15^\circ}; \end{aligned}$$

and the numerator of the second equation is

$$\begin{aligned} \mathbf{N}_2 &= (103.9 + j60)(13 - j6) - (100 + j0)(10 - j10) \\ &= 710 + j1156.6 = 1356\underline{/58.42^\circ}. \end{aligned}$$

We may now solve for the current values:

$$\mathbf{I}_1 = \frac{\mathbf{N}_1}{\mathbf{D}} = \frac{235\underline{/-15^\circ}}{213.5\underline{/6.18^\circ}} = 1.10\underline{/-21.18^\circ} = 1.03 - j0.398 \text{ complex amperes;}$$

$$\mathbf{I}_2 = \frac{\mathbf{N}_2}{\mathbf{D}} = \frac{1356\underline{/58.42^\circ}}{213.5\underline{/6.18^\circ}} = 6.35\underline{/52.24^\circ} = 3.89 + j5.02 \text{ complex amperes;}$$

$$\mathbf{I}_3 = \mathbf{I}_1 + \mathbf{I}_2 = 4.92 + j4.62 = 6.74\underline{/43.3^\circ} \text{ complex amperes.}$$

(b) The potential drop across $\mathbf{Z}_3$ from *c* to *a* is

$$\begin{aligned} \mathbf{V}_3 = \mathbf{I}_3\mathbf{Z}_3 &= (4.92 + j4.62)(10 - j10) \\ &= 95.4 - j3.00 = 95.4\underline{/-1.73^\circ} \text{ complex volts.} \end{aligned}$$

(c) By the method of conjugates, the complex vector power is

$$\begin{aligned} \mathbf{U}_3' = P + jQ = \mathbf{V}_3\hat{\mathbf{I}}_3 &= (95.4 - j3.00)(4.91 - j4.62) \\ &= 455 - j456 \text{ complex volt-amperes.} \end{aligned}$$

Then $P = 455$ watts, and $Q = -456$ vars (capacitive).

The reactive power is known to be leading because we employed the conjugate of current where potential drops are considered positive. As explained in Article 9.24, this method will agree with convention, producing negative capacitive vars and positive inductive vars.

The accompanying figure shows all vector quantities.

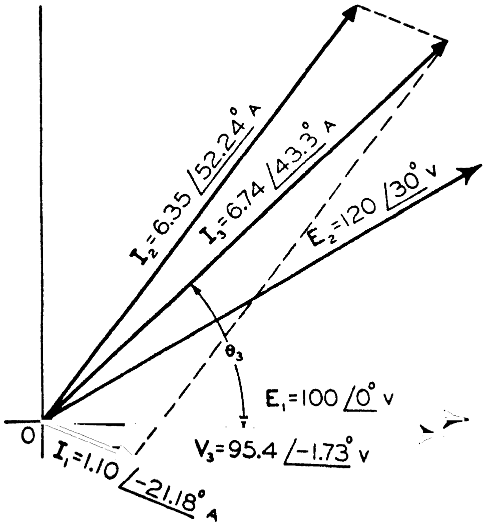


Fig. 11.04C. Vector diagram of Example 11.04B.

The next step is to check the values obtained, by writing the potential equation around another closed path and inserting the values already obtained. Thus:

CHECK

Along path *bacefdb:*

$$-\mathbf{E}_1 + \mathbf{I}_1\mathbf{Z}_1 - \mathbf{I}_2\mathbf{Z}_2 + \mathbf{E}_2 = 0,$$

$$(-100 - j0) + (1.03 - j0.398)(3 + j4) - (3.89 + j5.02)(8.66 + j5) + (103.9 + j60) = 0,$$

$$(-100 - j0) + (4.68 + j2.92) = (8.5 + j62.83) - (103.9 + j60)$$

$$-95.32 + j2.92 = -95.4 + j2.83.$$

This check is observed to hold within slide-rule precision.

11.05. Network Solutions Involving Higher-order Determinants

The preceding article showed, for two specific cases, the use of determinants in solving simultaneous network equations. The first of these solutions, for a three-mesh network, was found to involve ultimately three simultaneous equations and corresponding third-order determinants (three rows and three columns). The second of these solutions, for a two-mesh network, was found to involve ultimately two simultaneous equations and corresponding second-order determinants. *In general, an n-mesh network solution requires a minimum of n simultaneous equations and corresponding nth-order determinants.* Although many specialized methods of handling determinants of various types and orders have been evolved to expedite solution, they properly form the subject matter of textbooks on algebra and will not be discussed here. The remainder of this article will be devoted to discussion of a general method of solution which may be applied to determinants of any order normally encountered in this work. In many cases this method is not so short as other more specialized methods may be, but it has the advantage of being applicable generally. It is the method employed in both of the examples of the preceding article.

Let us consider the following *fourth-order* determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}. \tag{11-12}$$

This group may be expanded into a group of *four* terms, each of which contains a *third-order* determinant:

$$a_1\begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{vmatrix} - a_4\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}. \tag{11-13}$$

Now we can see the rule for reduction of the fourth-order determinant to equation (11-13) containing only third-order determinants. We multiply, in turn, the first term of each column by a determinant containing, with unchanged sequence, all of the terms which are in neither the same row nor the same column as this term. These products are given alternating signs, beginning with plus, and are added. There must, of course, be as many of these products as there were columns in the original determinant—four in this case.

Each of these new determinants of third order is then expanded according to the same rule. Thus, the first term of (11-13) expands into

$$a_1\left[b_2\begin{vmatrix}c_3 & c_4\\ d_3 & d_4\end{vmatrix} - b_3\begin{vmatrix}c_2 & c_4\\ d_2 & d_4\end{vmatrix} + b_4\begin{vmatrix}c_2 & c_3\\ d_2 & d_3\end{vmatrix}\right]. \qquad (11\text{-}14)$$

Each of the terms of (11-13) expands into three terms, giving 12 terms in all. Another expansion is necessary to reduce the second-order determinants of the type given by (11-14). Thus, (11-14) reduces to

$$a_1[b_2(c_3d_4 - c_4d_3) - b_3(c_2d_4 - c_4d_2) + b_4(c_2d_3 - c_3d_2)]. \qquad (11\text{-}15)$$

Each of the terms of (11-14) has been replaced by two terms in (11-15). The total number of terms resulting from the total expansion of the fourth-order determinant of equation (11-12) is seen to be

$$4 \times 3 \times 2 = 4! = 24 \text{ terms.}$$

A review of the examples of the preceding article reveals that the number of terms resulting from expansion of a third-order determinant is

$$3 \times 2 = 3! = 6 \text{ terms;}$$

and the number resulting from expansion of a second-order determinant is

$$2 = 2! = 2 \text{ terms.}$$

This result leads us to the conclusion that this method of expansion of an nth-order determinant leads to a solution involving $n!$ terms.

In practice, judicious rearranging of the terms of the original determinant according to the rules governing such operations may often shorten the work considerably by causing certain terms of zero value to drop out at an early stage of the expansion. The reader is urged to consult an algebra text to refresh his mind on these rules and methods.

11.06. Network Solution by the Loop Method

For very simple networks in which the number of branches is small, solution by direct application of Kirchhoff's laws (known also as the *branch method*) is sometimes the most direct way of obtaining the desired branch currents or potentials. As Examples 11.04A and 11.04B reveal, however, even for relatively simple cases this general method of solution produces a rather formidable array of equations. Where the number of major branches is greater than three or four, it becomes desirable to reduce the number of simultaneous equations to be solved. This reduction may be accomplished by systematizing the writing of the equations through the use of either the *loop method* or the *node method.* The loop method forms the content of this article, and the node method is discussed in Article 11.08.

The loop method of solving networks is based, of course, on Kirchhoff's two laws. Essentially, it consists of assigning a circulating current to each independent loop and deriving the branch currents therefrom. In general, the shortest paths—the meshes—are chosen for convenience, a fact which leads some authors to call this the *method of mesh currents.* In each of these loops a current is assumed to be positive in an arbitrarily designated direction. Because such a loop current leaves every node that it enters, the otherwise necessary independent Kirchhoff current equations are automatically satisfied. Therefore only the independent Kirchhoff potential equations are left to be written and solved for the values of the loop currents, which now constitute the unknowns. The branch currents are readily obtained by algebraic (or vector) addition of these component loop currents.

We shall utilize this loop method to re-solve the circuit of Fig. 11.04B. This circuit is redrawn in Fig. 11.06A, with the assumed positive directions of loop currents $\mathbf{I}_1'$ and $\mathbf{I}_2'$ indicated as following, respectively, the two meshes in a clockwise direction. To systematize application of the loop method, it is well to assume the same positive direction of circulation, either clockwise or counterclockwise, for all the loop currents of a given problem. It is also advisable to utilize the meshes as the paths for the loop currents. The resultant positive current in any branch then becomes the algebraic (or vector) sum of the loop currents through that branch. For example, the current from *c* to *a* of the present problem is $(\mathbf{I}_1' - \mathbf{I}_2')$, and the current from *d* to *a* is $\mathbf{I}_2'$.

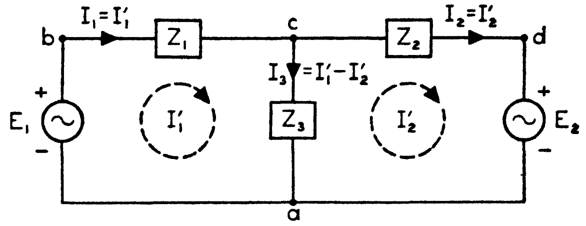


Fig. 11.06A. Circuit for solution of Example 11.04B by the loop method.

Kirchhoff's potential law can now be applied to write the loop equations about the two meshes. Denoting potential drops as positive, we may write

$$-\mathbf{E}_1 + \mathbf{I}_1'\mathbf{Z}_1 + (\mathbf{I}_1' - \mathbf{I}_2')\mathbf{Z}_3 = 0, \tag{11-16a}$$

and

$$\mathbf{E}_2 + \mathbf{I}_2'\mathbf{Z}_2 + (\mathbf{I}_2' - \mathbf{I}_1')\mathbf{Z}_3 = 0. \tag{11-17a}$$

Rearranging terms, we have

$$-\mathbf{E}_1 + \mathbf{I}_1'(\mathbf{Z}_1 + \mathbf{Z}_3) - \mathbf{I}_2'\mathbf{Z}_3 = 0, \tag{11-16b}$$

and

$$\mathbf{E}_2 - \mathbf{I}_1'\mathbf{Z}_3 + \mathbf{I}_2'(\mathbf{Z}_2 + \mathbf{Z}_3) = 0. \tag{11-17b}$$

It is convenient to define the *self impedance* of a loop as the sum of the impedances encountered in a traverse of that loop. The self impedances of the two meshes of Fig. 11.06A are

Self impedance of loop 1,

▶ $$\mathbf{Z}_{11} = \mathbf{Z}_1 + \mathbf{Z}_3, \tag{11-18}$$

and

Self impedance of loop 2,

▶ $$\mathbf{Z}_{22} = \mathbf{Z}_2 + \mathbf{Z}_3. \tag{11-19}$$

Similarly, it is convenient to define the following *mutual impedance: Mutual impedance of loops* 1 *and* 2,

▶ $$\mathbf{Z}_{12} = \mathbf{Z}_{21} = -\mathbf{Z}_3. \tag{11-20}$$

Equations (11-16) and (11-17) now may be written

$$\mathbf{I}_1'\mathbf{Z}_{11} + \mathbf{I}_2'\mathbf{Z}_{12} = \mathbf{E}_1, \tag{11-21}$$

$$\mathbf{I}_1'\mathbf{Z}_{21} + \mathbf{I}_2'\mathbf{Z}_{22} = -\mathbf{E}_2. \tag{11-22}$$

The sole reason for defining the mutual impedance as a negative quantity is to systematize the writing of equations so as to have all terms preceded by (+) signs in the symbolic expressions. Had there been more meshes, there would have been more equations, one for each mesh, and each equation would have contained as many mutual impedance terms as there were adjoining meshes.

By the method of determinants, then,

$$\mathbf{I}_1' = \frac{\begin{vmatrix} \mathbf{E}_1 & \mathbf{Z}_{12} \\ -\mathbf{E}_2 & \mathbf{Z}_{22} \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{vmatrix}} = \frac{\mathbf{E}_1\mathbf{Z}_{22} - (-\mathbf{E}_2)\mathbf{Z}_{12}}{\mathbf{Z}_{11}\mathbf{Z}_{22} - (\mathbf{Z}_{12})^2}, \tag{11-23a}$$

$$\mathbf{I}_2' = \frac{\begin{vmatrix} \mathbf{Z}_{11} & \mathbf{E}_1 \\ \mathbf{Z}_{21} & -\mathbf{E}_2 \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{vmatrix}} = \frac{(-\mathbf{E}_2)\mathbf{Z}_{11} - \mathbf{E}_1\mathbf{Z}_{21}}{\mathbf{Z}_{11}\mathbf{Z}_{22} - (\mathbf{Z}_{21})^2}. \tag{11-24a}$$

Substituting (11-18), (11-19), and (11-20) gives us

$$\mathbf{I}_1' = \frac{\mathbf{E}_1(\mathbf{Z}_2 + \mathbf{Z}_3) - (-\mathbf{E}_2)(-\mathbf{Z}_3)}{(\mathbf{Z}_1 + \mathbf{Z}_3)(\mathbf{Z}_2 + \mathbf{Z}_3) - (-\mathbf{Z}_3)^2} = \frac{\mathbf{E}_1\mathbf{Z}_2 + \mathbf{E}_1\mathbf{Z}_3 - \mathbf{E}_2\mathbf{Z}_3}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_1\mathbf{Z}_3}, \tag{11-23b}$$

and

$$\mathbf{I}_2' = \frac{(-\mathbf{E}_2)(\mathbf{Z}_1 + \mathbf{Z}_3) - \mathbf{E}_1(-\mathbf{Z}_3)}{(\mathbf{Z}_1 + \mathbf{Z}_3)(\mathbf{Z}_2 + \mathbf{Z}_3) - (-\mathbf{Z}_3)^2} = \frac{-\mathbf{E}_2\mathbf{Z}_1 - \mathbf{E}_2\mathbf{Z}_3 + \mathbf{E}_1\mathbf{Z}_3}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_1\mathbf{Z}_3}. \tag{11-24b}$$

If we wish to know the current from c to a, we may write

$$\mathbf{I}_{ca} = \mathbf{I}_1' - \mathbf{I}_2' = \frac{\mathbf{E}_1\mathbf{Z}_2 + \mathbf{E}_2\mathbf{Z}_1}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_1\mathbf{Z}_3}. \tag{11-25}$$

The potential drop from c to a is

$$\mathbf{V}_{ca} = \mathbf{I}_{ca}\mathbf{Z}_3 = \frac{\mathbf{E}_1\mathbf{Z}_2 + \mathbf{E}_2\mathbf{Z}_1}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_1\mathbf{Z}_3}\mathbf{Z}_3. \tag{11-26}$$

In a simple two-mesh network such as this, the advantage of the systematized loop method is slight, if it exists at all. In more complicated networks, however, this method possesses the distinct advantage that it quickly and unerringly provides in minimum possible number and in simplest form the independent simultaneous equations needed to obtain a solution by the potential law.

Example 11.06A. Solve Example 11.04A by the loop methoa.

Figure 11.06B shows the circuit, with the assumed positive directions of loop currents indicated. Designating potential drops as positive, we may write the loop equations as follows:

$$\begin{aligned} I_1'R_{11} + I_2'R_{12} + I_3'R_{13} &= E \\ I_1'R_{12} + I_2'R_{22} + I_3'R_{23} &= 0 \\ I_1'R_{13} + I_2'R_{23} + I_3'R_{33} &= 0, \end{aligned}$$

where:

$$\begin{aligned} R_{11} &= (R_0 + R_1 + R_3) = 1 + 2 + 4 = 7, \\ R_{22} &= (R_1 + R_2 + R_5) = 2 + 3 + 6 = 11, \\ R_{33} &= (R_3 + R_4 + R_5) = 4 + 5 + 6 = 15, \\ R_{12} &= (-R_1) = -2, \\ R_{23} &= (-R_5) = -6, \\ R_{13} &= (-R_3) = -4 \text{ ohms.} \end{aligned}$$

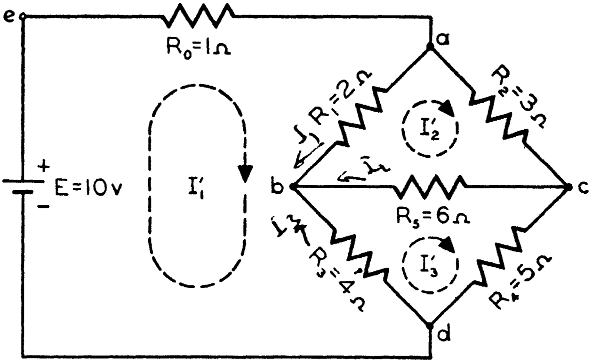


Fig. 11.06B. Circuit for solution of Example 11.04A by the loop method.

Solving the three simultaneous equations for I_2' and I_3', we write

$$I_2' = \frac{\begin{vmatrix} 7 & 10 & -4 \\ -2 & 0 & -6 \\ -4 & 0 & 15 \end{vmatrix}}{\begin{vmatrix} 7 & -2 & -4 \\ -2 & 11 & -6 \\ -4 & -6 & 15 \end{vmatrix}} = \frac{(7)\begin{vmatrix} 0 & -6 \\ 0 & 15 \end{vmatrix} - (10)\begin{vmatrix} -2 & -6 \\ -4 & 15 \end{vmatrix} + (-4)\begin{vmatrix} -2 & 0 \\ -4 & 0 \end{vmatrix}}{(7)\begin{vmatrix} 11 & -6 \\ -6 & 15 \end{vmatrix} - (-2)\begin{vmatrix} -2 & -6 \\ -4 & 15 \end{vmatrix} + (-4)\begin{vmatrix} -2 & 11 \\ -4 & -6 \end{vmatrix}}$$

$$= \frac{0 - 10(-30 - 24) + 0}{7(165 - 36) + 2(-30 - 24) - 4(12 + 44)} = 0.945 \text{ ampere,}$$

and

$$I_3' = \frac{\begin{vmatrix} 7 & -2 & 10 \\ -2 & 11 & 0 \\ -4 & -6 & 0 \end{vmatrix}}{\begin{vmatrix} 7 & -2 & -4 \\ -2 & 11 & -6 \\ -4 & -6 & 15 \end{vmatrix}} = \frac{(7)\begin{vmatrix} 11 & 0 \\ -6 & 0 \end{vmatrix} - (-2)\begin{vmatrix} -2 & 0 \\ -4 & 0 \end{vmatrix} + (10)\begin{vmatrix} -2 & 11 \\ -4 & -6 \end{vmatrix}}{(7)\begin{vmatrix} 11 & -6 \\ -6 & 15 \end{vmatrix} - (-2)\begin{vmatrix} -2 & -6 \\ -4 & 15 \end{vmatrix} + (-4)\begin{vmatrix} -2 & 11 \\ -4 & -6 \end{vmatrix}}$$

$$= \frac{0 + 0 + 10(12 + 44)}{7(165 - 36) + 2(-30 - 24) - 4(12 + 44)} = 0.980 \text{ ampere.}$$

Then $I_5 = I_3' - I_2' = 0.980 - 0.945 = 0.035$ ampere, which checks exactly the answer to Example 11.04A. Comparison of the two solutions reveals that the loop method is considerably more convenient than the branch method for solving this three-mesh network problem.

Example 11.06B. Solve Example 11.04B by the loop method.

Figure 11.06A shows the circuit, with the assumed positive directions of loop currents indicated. The text material of this article develops this solution symbolically. All we need do is to substitute the proper numerical values in equations (11-25) and (11-26) to obtain the answers:

$$\mathbf{I}_{ca} = \frac{(100\underline{/0^\circ})(10\underline{/30^\circ}) + (120\underline{/30^\circ})(5\underline{/53.1^\circ})}{(5\underline{/53.1^\circ})(10\underline{/30^\circ}) + (10\underline{/30^\circ})(14.14\underline{/-45^\circ}) + (5\underline{/53.1^\circ})(14.14\underline{/-45^\circ})}$$

$$= 6.74\underline{/43.2^\circ} \text{ complex amperes, and}$$

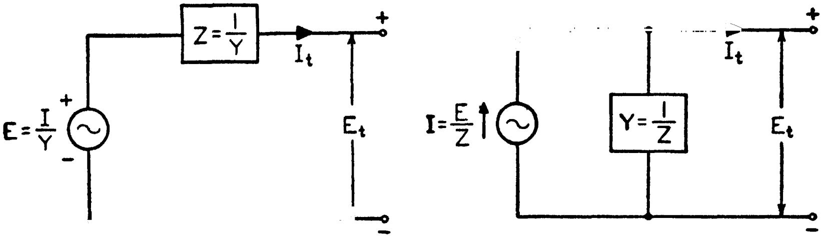
$$\mathbf{V}_{ca} = \mathbf{I}_{ca}\mathbf{Z}_3 = (6.74\underline{/43.2^\circ})(14.14\underline{/-45^\circ}) = 95.1\underline{/-1.8^\circ} \text{ complex volts.}$$

11.07. Potential and Current Sources

The foregoing development of the loop method of solving networks indicates that it is desirable to know the potential of the source in applying this method. The next article will show that it is often desirable to know the current of the source in applying the node method. The present article therefore devotes itself to a consideration of *equivalent potential and current sources.*

A practical source of electric energy, such as a generator, always contains not only electromotive force but also resistance, inductance, and capacitance. As a result, the potential difference between the terminals of a practical source is a function of the current which it supplies. From a circuit viewpoint, it is frequently convenient to simulate a practical source by a pure emf which is independent of terminal potential and current, in series with an impedance. Such a simulation of an actual source is called a *potential source.* Solution by the loop method is readily performed with a potential source by including the internal series impedance with the loop impedance. Part (a) of Fig. 11.07 shows such a source diagrammatically.

Part (b) of Fig. 11.07 shows a *current source* which is equivalent to the potential source of part (a) in the sense that it possesses the same potential-current characteristics at its terminals. It differs internally, however, in that it consists of a pure current source in parallel with a shunt admittance. Part (b) of Fig. 11.07 shows this type of source diagrammatically.



(a) Potential source. (b) Current source.

Fig. 11.07. Equivalent potential and current sources.

The following equations apply, respectively, to the potential source and to the equivalent current source.

Potential source:

▶
$$\mathbf{E}_t = \mathbf{E} - \mathbf{I}_t\mathbf{Z}, \tag{11-27}$$

and

▶
$$\mathbf{I}_t = \frac{\mathbf{E} - \mathbf{E}_t}{\mathbf{Z}} = \frac{\mathbf{E}}{\mathbf{Z}} - \mathbf{E}_t\frac{1}{\mathbf{Z}}; \tag{11-28}$$

Current source:

$$\mathbf{I}_t = \mathbf{I} - \mathbf{E}_t\mathbf{Y}, \tag{11-29}$$

and

$$\mathbf{E}_t = \frac{\mathbf{I} - \mathbf{I}_t}{\mathbf{Y}} = \frac{\mathbf{I}}{\mathbf{Y}} - \mathbf{I}_t\frac{1}{\mathbf{Y}}, \tag{11-30}$$

where:

$\mathbf{E}_t$ = *terminal potential of source* (a function of $\mathbf{I}_t$),
$\mathbf{I}_t$ = *terminal current of source* (a function of $\mathbf{E}_t$),
$\mathbf{E}$ = *emf of potential source* (independent of $\mathbf{E}_t$ and $\mathbf{I}_t$),

$\mathbf{I}$ = *internal current of current source* (independent of $\mathbf{E}_t$ and $\mathbf{I}_t$),
$\mathbf{Z}$ = *series impedance of potential source,*
$\mathbf{Y}$ = *shunt admittance of current source.*

Inasmuch as either one of these sources can replace the other without in any way affecting conditions in the connected circuit, we can equate corresponding quantities, as follows:

$$\mathbf{E}_t = \mathbf{E} - \mathbf{I}_t\mathbf{Z} = \frac{\mathbf{I}}{\mathbf{Y}} - \mathbf{I}_t\frac{1}{\mathbf{Y}}, \tag{11-31}$$

and

$$\mathbf{I}_t = \mathbf{I} - \mathbf{E}_t\mathbf{Y} = \frac{\mathbf{E}}{\mathbf{Z}} - \mathbf{E}_t\frac{1}{\mathbf{Z}}. \tag{11-32}$$

From these equations we obtain the following conversion relations:

▶ $$\mathbf{E} = \frac{\mathbf{I}}{\mathbf{Y}}, \tag{11-33}$$

▶ $$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}}, \tag{11-34}$$

▶ $$\mathbf{Z} = \frac{1}{\mathbf{Y}}. \tag{11-35}$$

It is well to emphasize that although these three relations make a potential source and a current source equivalent in the sense that terminal potentials and currents are matched, they do not, in general, result in sources of equal efficiency.

It is instructive to observe that *a potential source is idle when the external circuit is open* (an infinite impedance, or zero admittance, connects its terminals), whereas *a current source is idle when its terminals are short-circuited* (a zero impedance, or infinite admittance, connects its terminals). The terminal potential of a potential source is a function of the $\mathbf{I}_t\mathbf{Z}$ drop in the internal series impedance; the terminal current of a current source is a function of the $\mathbf{E}_t\mathbf{Y}$ current loss through the internal shunt admittance. Short-circuiting the terminals of a potential source of low impedance (high efficiency) results in excessively high current; opening the circuit connected to the terminals of a current source of low admittance (high efficiency) results in excessively high terminal potential. However, by equation (11-35), a high-admittance current source is equivalent to a low-impedance potential source. Therefore, no more disturbance is caused by open-circuiting a given current source than is caused by short-circuiting the potential source to which it is equivalent.

Example 11.07. A certain potential source consists of an emf, $\mathbf{E} = 100\underline{/0^\circ}$ volts, in series with an impedance, $\mathbf{Z} = 10\underline{/70^\circ}$ ohms. What is the equivalent current source?

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}} = \frac{100\underline{/0^\circ}}{10\underline{/70^\circ}} = 10\underline{/-70^\circ} \text{ amperes,}$$

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{1}{10\underline{/70^\circ}} = 0.1\underline{/-70^\circ}$$

$$= (0.0342 - j0.09396) \text{ mho.}$$

11.08. Network Solution by the Node Method

The *node method* of solving networks, like the loop method, is based on Kirchhoff's two laws. Essentially, it consists of assigning a potential to each node with respect to some reference node and deriving the branch potential differences therefrom. More specifically, one node to which numerous branches converge is arbitrarily designated the *reference node.* The other nodes are then assigned symbolic potentials, each of which represents the amount by which the potential of a particular node is assumed to differ from that of the reference node. We shall express these potentials as *drops* from the reference node to the node in question. The potential difference across a given branch can be expressed as the difference between the node potentials at the two ends of that branch. Inasmuch as the potential of any node is independent of the path by which it is approached, the necessary Kirchhoff potential equations are automatically satisfied when we write, in terms of node potentials, Kirchhoff current equations at $(n - 1)$ of the n nodes. Thus, to solve a network by the node method, we need merely to write the independent Kirchhoff current equations and solve them for the values of the node potentials, which now constitute the unknowns. If a current, rather than a potential, source is employed, the node equations are identical in mathematical form to the loop equations, and may be solved by the same mathematical procedures. It should be noted, however, that conversion of a potential source to an equivalent current source is not an essential preliminary to the use of the node method. The reader is also cautioned to observe that the alternative loop and node methods of formulating network equations have nothing to do with duality.

For purposes of illustration, we shall now utilize the node method to solve the circuit of Fig. 11.04A. This circuit is redrawn in Fig. 11.08A, with a current source substituted for the potential source which constitutes major branch *dea* of the original diagram.

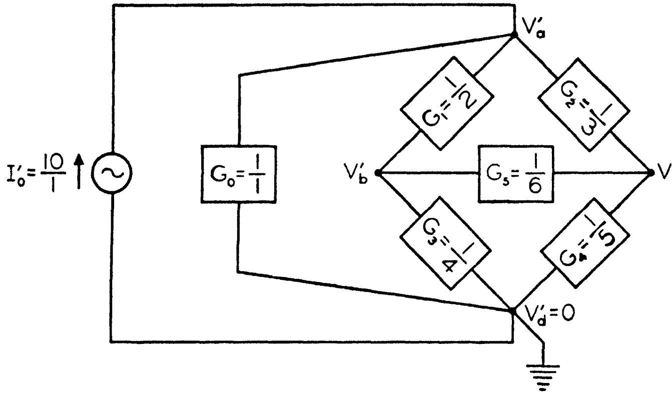


Fig. 11.08A. Circuit for solution of Example 11.04A by the node method.

The internal current and shunt admittance of the equivalent current source are, respectively,

$$I'_0 = \frac{E}{R_0} \quad \text{and} \quad G_0 = \frac{1}{R_0}. \tag{11-36}$$

Employing node *d* as reference and calling *approaching* currents positive, we may write the following equations by Kirchhoff's current law:

Node a: $V'_aG_0 + (V'_a - V'_b)G_1 + (V'_a - V'_c)G_2 + I'_0 = 0,$ (11-37a)
Node b: $V'_bG_3 + (V'_b - V'_c)G_5 + (V'_b - V'_a)G_1 = 0,$ (11-38a)
Node c: $V'_cG_4 + (V'_c - V'_b)G_5 + (V'_c - V'_a)G_2 = 0.$ (11-39a)

The terms of these equations may be rearranged to provide a cyclical order, as follows:

$$V'_a(G_0 + G_1 + G_2) + V'_b(-G_1) + V'_c(-G_2) = -I'_0, \tag{11-37b}$$
$$V'_a(-G_1) + V'_b(G_1 + G_3 + G_5) + V'_c(-G_5) = 0, \tag{11-38b}$$
$$V'_a(-G_2) + V'_b(-G_5) + V'_c(G_2 + G_4 + G_5) = 0. \tag{11-39b}$$

Let us now define *self conductance* (or *self admittance*, in general) of a given node as the sum of the conductances (or admittances) terminating at that node. The self conductances of the nodes a, b, and c are, respectively,

$$\blacktriangleright \qquad G_{aa} = (G_0 + G_1 + G_2), \tag{11-40}$$
$$\blacktriangleright \qquad G_{bb} = (G_1 + G_3 + G_5), \tag{11-41}$$
$$\blacktriangleright \qquad G_{cc} = (G_2 + G_4 + G_5). \tag{11-42}$$

Let us also define the *mutual conductances* (or *mutual admittances*, in general) as the *negative* conductances (or admittances) between all pairs of nodes excluding the reference node. The mutual conductances of this circuit are

$$\blacktriangleright \qquad G_{ab} = G_{ba} = (-G_1), \tag{11-43}$$
$$\blacktriangleright \qquad G_{bc} = G_{cb} = (-G_5), \tag{11-44}$$
$$\blacktriangleright \qquad G_{ca} = G_{ac} = (-G_2). \tag{11-45}$$

Equations (11-37) through (11-39) may be written as follows:

$$\blacktriangleright \quad \textit{Node a:} \qquad V'_a G_{aa} + V'_b G_{ab} + V'_c G_{ac} = -I'_0, \tag{11-46}$$
$$\blacktriangleright \quad \textit{Node b:} \qquad V'_a G_{ba} + V'_b G_{bb} + V'_c G_{bc} = 0, \tag{11-47}$$
$$\blacktriangleright \quad \textit{Node c:} \qquad V'_a G_{ca} + V'_b G_{cb} + V'_c G_{cc} = 0. \tag{11-48}$$

The mutual conductance is defined as a negative quantity purely to systematize the writing of node equations in such a way as to place (+) signs in front of all terms in the symbolic expressions. A careful study of equations (11-46) through (11-48) in the light of Fig. 11.08A should reveal the underlying principle for writing such equations directly and systematically.

We note that equations (11-46) through (11-48) have the same mathematical pattern as do loop equations, and can be solved for the unknowns by identical mathematical methods. Through these methods the values of the node potentials can be found. The branch potential drops may then be found as the differences between node potentials, and the corresponding branch currents may be determined by applications of Ohm's law.

EXAMPLE 11.08A. Solve Example 11.04A by the node method, and compare the result with that obtained by the branch method (Example 11.04A) and with that obtained by the loop method (Example 11.06A).

Referring to Fig. 11.08A and designating *approaching* currents as positive, we may immediately write the node equations, (11-46), (11-47), and (11-48). Evaluating the self and mutual conductances gives

$$
\begin{aligned}
G_{aa} &= (G_0 + G_1 + G_2) = (\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3}) = (1 + 0.5 + 0.333) = 1.833,\\
G_{bb} &= (G_1 + G_3 + G_5) = (\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6}) = (0.5 + 0.25 + 0.167) = 0.917,\\
G_{cc} &= (G_2 + G_4 + G_5) = (\tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{6}) = (0.333 + 0.2 + 0.167) = 0.700,\\
G_{ab} &= (-G_1) = (-\tfrac{1}{2}) = -0.5,\\
G_{bc} &= (-G_5) = (-\tfrac{1}{6}) = -0.167,\\
G_{ca} &= (-G_2) = (-\tfrac{1}{3}) = -0.333 \text{ mho.}
\end{aligned}
$$

Also, $$I_0' = \frac{E}{R_0} = \frac{10}{1} = 10 \text{ amperes.}$$

Substitution of these values in the node equations gives

$$
\begin{aligned}
V_a'(1.833) + V_b'(-0.5) + V_c'(-0.333) &= -10,\\
V_a'(-0.5) + V_b'(0.917) + V_c'(-0.167) &= 0,\\
V_a'(-0.333) + V_b'(-0.167) + V_c'(0.700) &= 0.
\end{aligned}
$$

Solving these equations simultaneously for V_b' and V_c' by the method of determinants,

$$
V_b' = \frac{\begin{vmatrix} 1.833 & -10 & -0.333 \\ -0.50 & 0 & -0.167 \\ -0.333 & 0 & +0.700 \end{vmatrix}}{\begin{vmatrix} 1.833 & -0.5 & -0.333 \\ -0.50 & 0.917 & -0.167 \\ -0.333 & -0.167 & 0.700 \end{vmatrix}}
$$

$$
= \frac{0 - (-10)(-0.350 - 0.056) + 0}{1.833(0.642 - 0.028) + 0.5(-0.350 - 0.056) - 0.333(0.084 + 0.306)}
$$

$$= -5.11 \text{ volts,}$$

and

$$
V_c' = \frac{\begin{vmatrix} 1.833 & -0.5 & -10 \\ -0.5 & 0.917 & 0 \\ -0.333 & -0.167 & -\ 0 \end{vmatrix}}{\begin{vmatrix} 1.833 & -0.5 & -\ 0.333 \\ -0.5 & 0.917 & -\ 0.167 \\ -0.333 & -0.167 & 0.700 \end{vmatrix}}
$$

$$
= \frac{0 - 0 + (-10)(0.084 + 0.306)}{1.833(0.642 - 0.028) + 0.5(-0.350 - 0.056) - 0.333(0.084 + 0.306)}
$$

$$= -4.91 \text{ volts.}$$

Then $$V_{bc} = V_c' - V_b' = -4.91 - (-5.11) = +0.20 \text{ volt}$$

and $$I_{bc} = \frac{V_{bc}}{R_5} = \frac{0.20}{6} = 0.033 \text{ ampere.}$$

Within slide-rule precision, this value compares favorably with the value, 0.035, obtained by both the branch and the loop methods.

Example 11.08B. Solve Example 11.04B by the node method, and compare the result with that obtained by the branch method (Example 11.04B) and with that obtained by the loop method (Example 11.06B).

Figure 11.08B shows the circuit redrawn so as to substitute an equivalent current source for each of the outside major branches and so as to make node a the reference node.

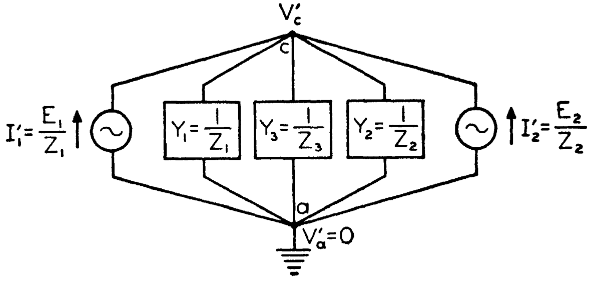


Fig. 11.08B. Circuit for solution of Example 11.04B by the node method.

Designating *approaching* currents as positive, we may write the following node equation:

$$\mathbf{V}_c'\mathbf{Y}_{cc} + \mathbf{I}_1' + \mathbf{I}_2' = 0,$$

where:

$$\begin{aligned}
\mathbf{Y}_{cc} &= \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} \\
&= \frac{1}{3 + j4} + \frac{1}{8.66 + j5} + \frac{1}{10 - j10} \\
&= \left(\frac{3}{25} + j\frac{-4}{25}\right) + \left(\frac{8.66}{100} + j\frac{-5}{100}\right) + \left(\frac{10}{200} + j\frac{+10}{200}\right) \\
&= 0.12 - j0.16 + 0.0866 - j0.05 + 0.05 + j0.05 \\
&= 0.257 - j0.16 \text{ complex mho,}
\end{aligned}$$

$$\mathbf{I}_1' = \frac{\mathbf{E}_1}{\mathbf{Z}_1} = \mathbf{E}_1\mathbf{Y}_1 = (100 + j0)(0.12 - j0.16) = 12 - j16 \text{ amperes,}$$

$$\mathbf{I}_2' = \frac{\mathbf{E}_2}{\mathbf{Z}_2} = \mathbf{E}_2\mathbf{Y}_2 = (104 + j60)(0.0866 - j0.05) = 12 + j0 \text{ amperes.}$$

Solving the single node equation for the single unknown and substituting numerical values, we obtain

$$\begin{aligned}
\mathbf{V}_c' &= \frac{-(\mathbf{I}_1' + \mathbf{I}_2')}{\mathbf{Y}_{cc}} = \frac{-(24 - j16)}{0.257 - j0.16} = \frac{28.8\underline{/146.3^\circ}}{0.3025\underline{/-31.9^\circ}} \\
&= 95.1\underline{/178.2^\circ} \text{ complex volts.}
\end{aligned}$$

The current from *c* to *a* is easily obtained as follows:

$$\begin{aligned}\mathbf{I}_{ca} &= -\mathbf{I}_{ac} = -\mathbf{V}_c'\mathbf{Y}_3 = -(95.1\underline{/178.2^\circ})(\sqrt{2} \times 0.05\underline{/45^\circ}) \\ &= -6.74\underline{/136.8^\circ} = 6.74\underline{/43.2^\circ} \text{ complex amperes.}\end{aligned}$$

Within slide-rule precision, this result compares favorably with the value $6.74\underline{/43.3^\circ}$ obtained by the branch method and the value $6.74\underline{/43.2^\circ}$ obtained by the loop method.

11.09. Choice between Loop and Node Methods of Solution

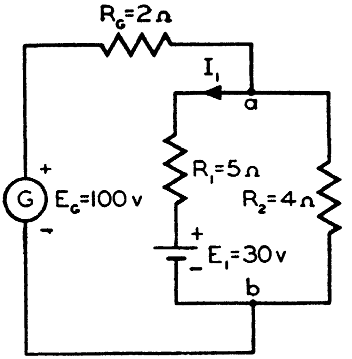
In a given problem, one has a choice between the loop and node methods of solution. As the numerical examples of this chapter illustrate, one of these methods may prove to be considerably shorter than the other. Only one node equation was needed, for example, to solve the two-mesh problem, whereas two simultaneous loop equations were needed to solve the same problem, with a considerable difference between the lengths of manipulative procedures required. In other problems, however, it might well turn out that the loop method would prove the shorter. It is apparent that some way of ascertaining in advance which is the shorter method for a given problem is highly desirable.

In general, it can be stated that the method which requires the smaller number of simultaneous equations is the shorter. Inasmuch as the necessary number of loop equations is equal to the number of meshes and inasmuch as the necessary number of node equations is one less than the number of major nodes, it becomes apparent that one can usually select the shorter method by a preliminary examination of the circuit. If the number of nodes-less-one is less than the number of meshes, the node method appears to be the shorter. If the number of nodes-less-one exceeds the number of meshes, the loop method appears to be the shorter. In actual cases, other factors may dictate a choice of method opposed to that of this rule of thumb.

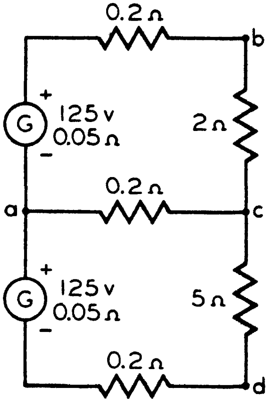
The next chapter is devoted to a discussion of various simplifying procedures which may be employed to make network solutions less difficult than they would be otherwise. Judicious use of one or more of these procedures to reduce the complexity of the circuit before applying either the loop or node methods may shorten the total task of solution materially.

PROBLEMS

11-1. Use the branch method of solution to find the current, I_1, and the potential drop, V_{ab}, in the accompanying circuit.



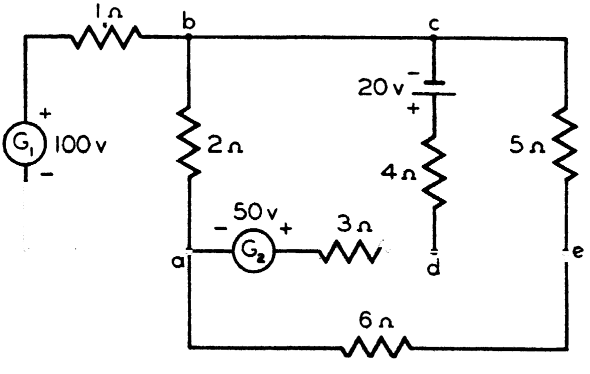
Prob. 11-1.



Prob. 11-2.

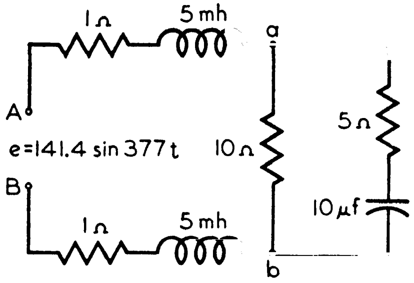
11-2. Use the branch method of solution to find the potential drops V_{bc} and V_{cd} of the accompanying circuit.

11-3. Use the branch method of solution to find the value and positive direction of the current through the 6-ohm resistor of the accompanying circuit.



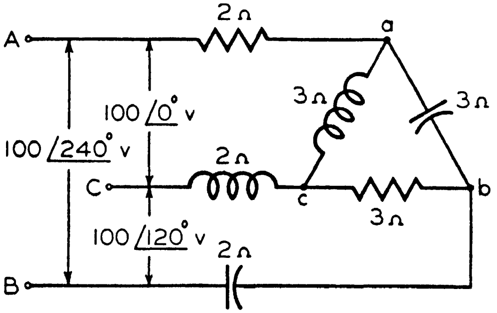
Prob. 11-3.

11-4. Use the branch method of solution to find the equation for the potential drop from *a* to *b*, as a function of ω and t, in the accompanying circuit.



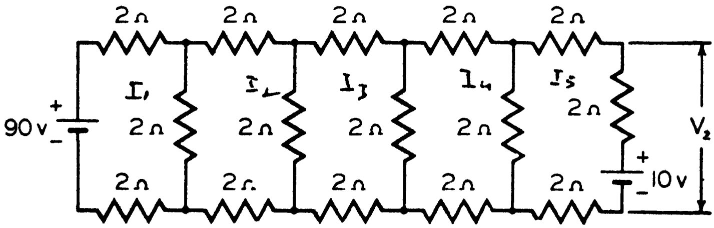
Prob. 11-4.

11-5. Solve the accompanying circuit by the branch method to obtain all values of current.



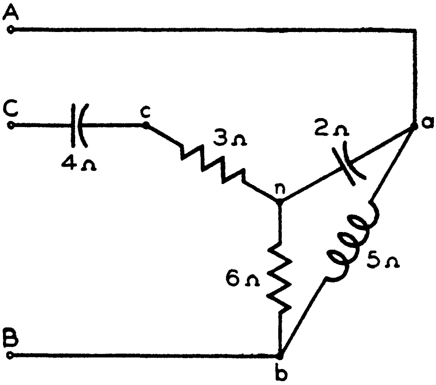
Prob. 11-5.

11-6. Use the branch method of solution to find the value of V_2 in the accompanying circuit.



Prob. 11-6.

11-7. Use the branch method of solution to find the value of $\mathbf{I}_{an}$ and $\mathbf{V}_{an}$ in the accompanying circuit, in which $\mathbf{V}_{AB} = 100\underline{/0^\circ}$, $\mathbf{V}_{BC} = 100\underline{/240^\circ}$, $\mathbf{V}_{CA} = 100\underline{/120^\circ}$ volts.



Prob. 11-7.

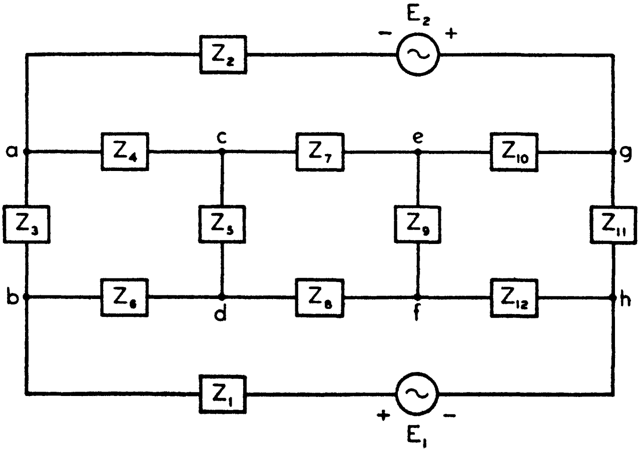
11-8. A certain street railway line is 7 miles long and is supplied with power by power stations at its two ends. The station at end *A* maintains a d-c potential between trolley wire and track of 600 volts, and the station at end *B* maintains a d-c potential between trolley wire and track of 570 volts. The trolley wire and its feeder have a combined resistance of 0.15 ohm per mile. The welded rails of the track present a combined resistance of 0.05 ohm per mile.

(a) How far from station *A* will a car drawing 150 amperes receive a minimum potential, and what will be the value of that potential?

(b) If a second car drawing the same current and running at the same speed leaves station *B* at the same time that the first car leaves station *A*, how far from station *A* will the first car now experience a minimum potential, and what will be that potential? (No need for alarm—when the cars reach the center, they will pass each other on a turnout!)

11-9. Use the method discussed in Article 11.05 to expand completely the determinant of equation 11-12.

11-10. Write the loop equations for the accompanying circuit.



11-11. Solve Prob. 11-1 by the loop method.

11-12. Solve Prob. 11-2 by the loop method.

11-13. Solve Prob. 11-3 by the loop method.

11-14. Solve Prob. 11-4 by the loop method.

11-15. Solve Prob. 11-5 by the loop method.

11-16. Solve Prob. 11-6 by the loop method.

11-17. Solve Prob. 11-7 by the loop method.

11-18. (a) Determine the equivalent current source which could replace the generator branch in Prob. 11-1.

(b) Determine the equivalent current source which could replace the battery branch in Prob. 11-1.

11-19. Determine the equivalent current source which could replace the branch *ab* of Prob. 11-2.

11-20. Wherever possible, in the circuit of Prob. 11-3, replace a potential source by an equivalent current source.

11-21. Solve Prob. 11-1 by the node method.

11-22. Solve Prob. 11-2 by the node method.

11-23. Solve Prob. 11-3 by the node method.

11-24. Solve Prob. 11-4 by the node method.

11-25. Solve Prob. 11-5 by the node method.

11-26. Solve Prob. 11-6 by the node method.

11-27. Solve Prob. 11-7 by the node method.

11-28. Solve Prob. 11-8 by the node method.

11-29. Write the node equations for the circuit of Prob. 11-10.

CHAPTER 12

SIMPLIFICATION OF STEADY-STATE NETWORK ANALYSIS

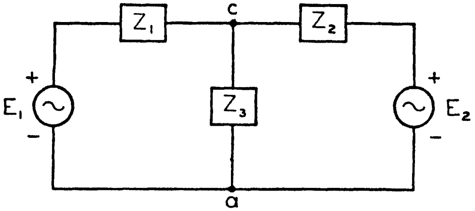
12.01. Introduction

The preceding chapter dealt with three general methods of network solution which stem more or less directly from Kirchhoff's laws: the branch method, the loop method, and the node method. Even the simple examples given in that chapter were sufficient to show that, in many cases, all of these methods may prove exceedingly lengthy and time-consuming. In the search for shorter methods of solution, electrical engineers have evolved a multitude of interesting and useful theorems which have contributed greatly to the technique of network analysis and solution. Some of these theorems are quite general and may be applied to any network, whereas others are highly specialized and apply only to particular types of networks. It is the purpose of this chapter to present a selection of some of the more useful of these theorems and to illustrate their application. All these theorems constitute most useful analytical tools. A preliminary application of one or more of them may so reduce the complexity of an otherwise difficult problem that final solution by the branch, loop, or node method becomes a simple matter.

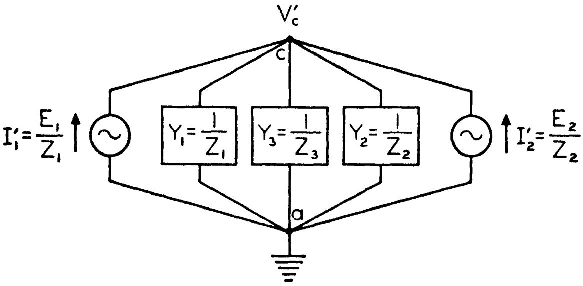
12.02. Principle of Superposition

When several sources are present in a linear bilateral network, it is frequently advantageous to find the currents or potentials in the network resulting from the presence of one source at a time and then to superpose currents or potentials for each branch, loop, or node to obtain the actual solution. This is an electric-circuit application of a basic principle of nature observed to operate under appropriate circumstances in all branches of physical science. It is known as the *principle of superposition.*

In applying the superposition principle to either the branch or loop methods of solution, one must replace the omitted emf's of the potential sources by connections of zero impedance (short circuits). The internal series impedances of all sources must not be disturbed but must be left in the circuit. Example 12.02A illustrates the application of this principle to the loop method of solution for the circuit of Example 11.06B. This circuit is shown again in part (a) of Fig. 12.02.



(a) Circuit to be solved by superposition.



(b) Equivalent circuit to be solved by superposition.

Fig. 12.02. Circuits of Examples 12.02A and 12.02B.

In applying the superposition principle to the node method of solution, one must replace the omitted internal current sources by connections of zero admittance (open circuits). The internal shunt admittances of all sources must not be disturbed but must be left in the circuit. Example 12.02B illustrates the application of this principle to the node method of solution for the same circuit. Fig. 11.08B, which shows this circuit modified by the substitution of current sources for potential sources, is shown again in part (b) of Fig. 12.02.

The principle of superposition generally proves superior to the basic loop or node methods only when the combination of impedances or admittances of the network by the rules for series and parallel combinations or by the delta-wye or wye-delta[1] transformations is easier than the solution of simultaneous equations.

EXAMPLE 12.02A. Use the principle of superposition to solve the problem of Example 11.06B.

Part (a) of Fig. 12.02 shows the circuit which is to be solved. First, assuming only $\mathbf{E}_1$ to be present, we may combine $\mathbf{Z}_2$ and $\mathbf{Z}_3$ in parallel and add in series to $\mathbf{Z}_1$, to obtain the equivalent impedance

$$\begin{aligned}\mathbf{Z}_{T_{E_1}} &= \mathbf{Z}_1 + \frac{1}{\dfrac{1}{\mathbf{Z}_2} + \dfrac{1}{\mathbf{Z}_3}} = \mathbf{Z}_1 + \frac{\mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_2 + \mathbf{Z}_3} \\ &= \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_2 + \mathbf{Z}_3}.\end{aligned}$$

The current through $\mathbf{Z}_1$ due to $\mathbf{E}_1$ alone is, then,

$$\mathbf{I}_{1_{E_1}} = \frac{\mathbf{E}_1}{\mathbf{Z}_{T_{E_1}}} = \frac{\mathbf{Z}_2 + \mathbf{Z}_3}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}\mathbf{E}_1.$$

The potential drop from c to a due to $\mathbf{E}_1$ alone is

$$\begin{aligned}\mathbf{V}_{ca_{E_1}} &= \mathbf{I}_{1_{E_1}}\frac{\mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_2 + \mathbf{Z}_3} = \frac{\mathbf{Z}_2\mathbf{Z}_3(\mathbf{Z}_2 + \mathbf{Z}_3)}{(\mathbf{Z}_2 + \mathbf{Z}_3)(\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1)}\mathbf{E}_1 \\ &= \frac{\mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}\mathbf{E}_1.\end{aligned}$$

Second, assuming only $\mathbf{E}_2$ to be present, we may combine $\mathbf{Z}_1$ and $\mathbf{Z}_3$ in parallel and add in series with $\mathbf{Z}_2$ to obtain the equivalent impedance

$$\begin{aligned}\mathbf{Z}_{T_{E_2}} &= \mathbf{Z}_2 + \frac{\mathbf{Z}_1\mathbf{Z}_3}{\mathbf{Z}_1 + \mathbf{Z}_3} \\ &= \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_3}.\end{aligned}$$

The current through $\mathbf{Z}_2$ due to $\mathbf{E}_2$ alone is, then,

$$\mathbf{I}_{2_{E_2}} = \frac{\mathbf{E}_2}{\mathbf{Z}_{T_{E_2}}} = \frac{\mathbf{Z}_1 + \mathbf{Z}_3}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}\mathbf{E}_2.$$

The potential drop from c to a due to $\mathbf{E}_2$ alone is

---

[1] See Article 12.03.

$$\mathbf{V}_{ca_{E_2}} = \mathbf{I}_{2_{E_2}} \frac{\mathbf{Z}_1\mathbf{Z}_3}{\mathbf{Z}_1 + \mathbf{Z}_3} = \frac{\mathbf{Z}_1\mathbf{Z}_3(\mathbf{Z}_1 + \mathbf{Z}_3)}{(\mathbf{Z}_1 + \mathbf{Z}_3)(\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1)} \mathbf{E}_2$$
$$= \frac{\mathbf{Z}_1\mathbf{Z}_3}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1} \mathbf{E}_2.$$

The resultant positive potential drop from *c* to *a* arising from the presence of *both* $\mathbf{E}_1$ and $\mathbf{E}_2$ is the sum of the two components:

$$\mathbf{V}_{ca} = \mathbf{V}_{ca_{E_1}} + \mathbf{V}_{ca_{E_2}} = \frac{\mathbf{Z}_2\mathbf{Z}_3\mathbf{E}_1 + \mathbf{Z}_3\mathbf{Z}_1\mathbf{E}_2}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}.$$

Substitution of the numerical values leads to the following answer:

$$\mathbf{V}_{ca} = 95.1\underline{/-1.8^\circ} \text{ complex volts.}$$

The current from *c* to *a* is

$$\mathbf{I}_{ca} = \frac{\mathbf{V}_{ca}}{\mathbf{Z}_3} = 6.74\underline{/43.2^\circ} \text{ complex amperes.}$$

This is observed to agree with the answer obtained by other methods.

Example 12.02B. Use the principle of superposition to solve the problem of Example 11.08B.

Part (b) of Fig. 12.02 shows the circuit which is to be solved. First, assuming only $\mathbf{I}_1'$ to be present, we may combine $\mathbf{Y}_1'$, $\mathbf{Y}_2$, and $\mathbf{Y}_3$ in parallel to obtain the equivalent admittance

$$\mathbf{Y}_{T_{I_1}} = \mathbf{Y}_1' + \mathbf{Y}_2 + \mathbf{Y}_3 = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3.$$

The potential drop from *c* to *a* due to $\mathbf{I}_1'$ alone is

$$\mathbf{V}_{ca_{I_1}} = \frac{\mathbf{I}_1'}{\mathbf{Y}_{T_{I_1}}}.$$

Second, assuming only $\mathbf{I}_2'$ to be present, we may combine $\mathbf{Y}_2'$, $\mathbf{Y}_1$, and $\mathbf{Y}_3$ in parallel to obtain the equivalent admittance

$$\mathbf{Y}_{T_{I_2}} = \mathbf{Y}_1 + \mathbf{Y}_2' + \mathbf{Y}_3 = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3.$$

The potential drop from *c* to *a* due to $\mathbf{I}_2'$ alone is

$$\mathbf{V}_{ca_{I_2}} = \frac{\mathbf{I}_2'}{\mathbf{Y}_{T_{I_2}}}.$$

The resultant positive potential drop from *c* to *a* arising from the presence of both $\mathbf{I}_1'$, and $\mathbf{I}_2'$ is the sum of the two components

$$\mathbf{V}_{ca} = \mathbf{V}_{ca_{I_1}} + \mathbf{V}_{ca_{I_2}} = \frac{\mathbf{I}_1'}{\mathbf{Y}_{T_{I_1}}} + \frac{\mathbf{I}_2'}{\mathbf{Y}_{T_{I_2}}} = \frac{\mathbf{I}_1' + \mathbf{I}_2'}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3}$$

Substitution of the numerical values leads to the following answer:

$$\mathbf{V}_{ca} = 95.1\underline{/-1.8^\circ} \text{ complex volts.}$$

The current from *c* to *a* is

$$\mathbf{I}_{ca} = \mathbf{V}_{ca}\mathbf{Y}_3 = 6.74\underline{/43.2^\circ} \text{ complex amperes.}$$

12.03. Delta-Wye Equivalence

In the reduction of networks by simple methods as a preliminary to applying a general method of solution, such as the loop or node methods or the superposition method, situations are sometimes encountered wherein the net impedance between two points cannot be computed by means of series and parallel combinations. The impedance between opposite nodes of a bridge network, such as that of Fig. 12.03A, is an example. As this figure shows, if we wish to determine the net impedance between terminals *a* and *d*, we must reduce the pair of *three-branch meshes*, or *deltas*, of part (a) to some other configuration. If one of these deltas, such as the one formed by $\mathbf{Z}_N$, $\mathbf{Z}_P$, and $\mathbf{Z}_Q$, can be replaced by a *three-branch star*, or *wye*, as indicated by the impedances $\mathbf{Z}_n$, $\mathbf{Z}_p$, and $\mathbf{Z}_q$ of part (b), the net impedance between points *a* and *d* then can be computed as a series-parallel combination. The terms "delta" and "wye" are often symbolized, respectively, as Δ and Y.

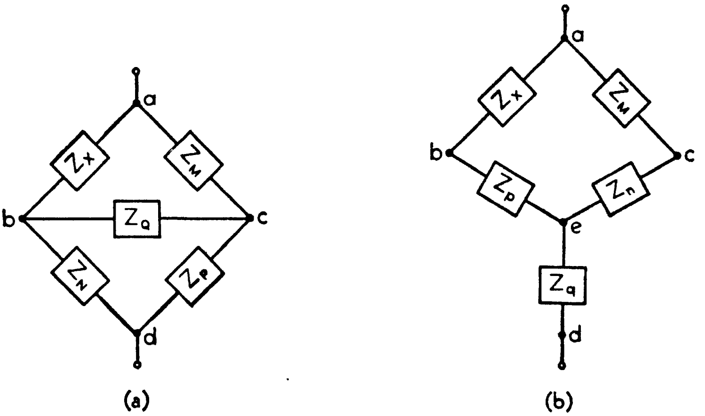


Fig. 12.03A. Network illustrating the utility of the delta-wye transformation.

In order for a wye to be electrically equivalent to a delta, the impedance viewed from any pair of terminals of the wye (excluding the common node) must be the same as the impedance viewed from the corresponding pair of delta terminals. Referring to Fig. 12.03B, we perceive that with no external connections to any of the terminals the following relations must be true:

$$Z_a + Z_b = \frac{1}{\dfrac{1}{Z_C} + \dfrac{1}{Z_B + Z_A}} = \frac{Z_B Z_C}{Z_A + Z_B + Z_C} + \frac{Z_C Z_A}{Z_A + Z_B + Z_C}, \tag{12-1}$$

$$Z_b + Z_c = \frac{1}{\dfrac{1}{Z_A} + \dfrac{1}{Z_C + Z_B}} = \frac{Z_C Z_A}{Z_A + Z_B + Z_C} + \frac{Z_A Z_B}{Z_A + Z_B + Z_C}, \tag{12-2}$$

$$Z_c + Z_a = \frac{1}{\dfrac{1}{Z_B} + \dfrac{1}{Z_A + Z_C}} = \frac{Z_A Z_B}{Z_A + Z_B + Z_C} + \frac{Z_B Z_C}{Z_A + Z_B + Z_C}. \tag{12-3}$$

Simultaneous solution of these relations (by inspection!) yields the following delta-wye transformation relations:

▶ $$Z_a = \frac{Z_B Z_C}{Z_A + Z_B + Z_C}, \tag{12-4}$$

▶ $$Z_b = \frac{Z_C Z_A}{Z_A + Z_B + Z_C}, \tag{12-5}$$

▶ $$Z_c = \frac{Z_A Z_B}{Z_A + Z_B + Z_C}. \tag{12-6}$$

A convenient rule for recalling these relations from memory may be stated as follows: *In a delta-wye transformation, the impedance of an equivalent wye branch equals the product of the corresponding pair of original delta branch impedances divided by the sum of all three original delta-branch impedances.*

If the delta and the wye are actually electrically equivalent so far as measurements at their terminals can determine, any external connections whatever can be made at their terminals without affect-

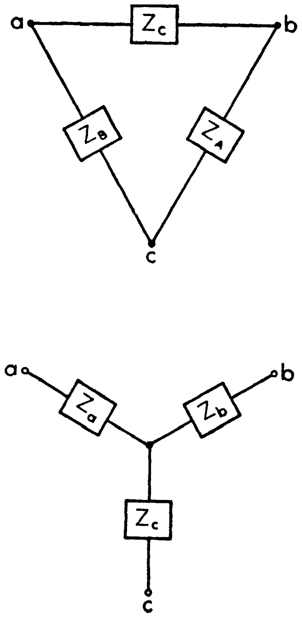


Fig. 12.03B. Delta-wye equivalence.

ing this equivalence. Referring again to Fig. 12.03B, with one pair of terminals joined at a time, we may write the following relations:

Terminals a and b joined:

$$\mathbf{Y}_A + \mathbf{Y}_B = \frac{1}{1 \qquad\qquad 1}, \tag{12-7}$$

Terminals b and c joined:

$$\mathbf{Y}_B + \mathbf{Y}_C = \frac{1}{\dfrac{1}{\mathbf{Y}_a} + \dfrac{1}{\mathbf{Y}_c + \mathbf{Y}_b}}, \tag{12-8}$$

Terminals c and a joined:

$$\mathbf{Y}_C + \mathbf{Y}_A = \frac{1}{\dfrac{1}{\mathbf{Y}_b} + \dfrac{1}{\mathbf{Y}_a + \mathbf{Y}_c}}. \tag{12-9}$$

Expressing these relations in terms of impedances and manipulating the results mathematically leads to the following wye-delta transformation relations:

▶
$$\mathbf{Z}_A = \frac{\mathbf{Z}_a\mathbf{Z}_b + \mathbf{Z}_b\mathbf{Z}_c + \mathbf{Z}_c\mathbf{Z}_a}{\mathbf{Z}_a}, \tag{12-10}$$

▶
$$\mathbf{Z}_B = \frac{\mathbf{Z}_a\mathbf{Z}_b + \mathbf{Z}_b\mathbf{Z}_c + \mathbf{Z}_c\mathbf{Z}_a}{\mathbf{Z}_b}, \tag{12-11}$$

▶
$$\mathbf{Z}_C = \frac{\mathbf{Z}_a\mathbf{Z}_b + \mathbf{Z}_b\mathbf{Z}_c + \mathbf{Z}_c\mathbf{Z}_a}{\mathbf{Z}_c}. \tag{12-12}$$

A convenient rule for recalling these relations from memory may be stated as follows: *In a wye-delta transformation, the impedance of an equivalent delta branch equals the sum of all possible products of original wye-branch impedances divided by the diametrically opposite original wye-branch impedance.*

An important special case of delta-wye equivalence is that in which the three delta (or wye) impedances have the same values. The *balanced delta-wye transformation* then becomes, from equation (12-4),

▶
$$\mathbf{Z}_Y = \frac{\mathbf{Z}_\Delta\mathbf{Z}_\Delta}{3\mathbf{Z}_\Delta} = \frac{1}{3}\mathbf{Z}_\Delta, \tag{12-13}$$

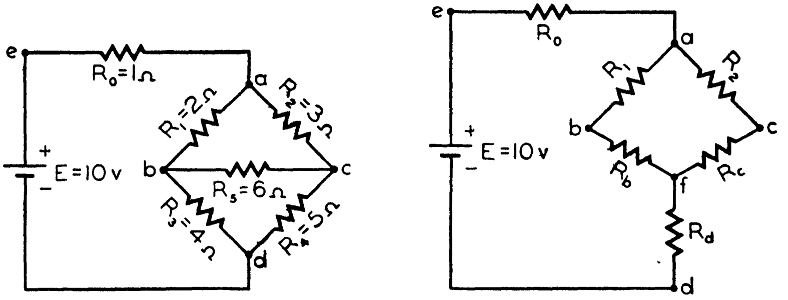
and the *balanced wye-delta transformation* is, correspondingly,

▶
$$\mathbf{Z}_\Delta = 3\mathbf{Z}_Y, \tag{12-14}$$

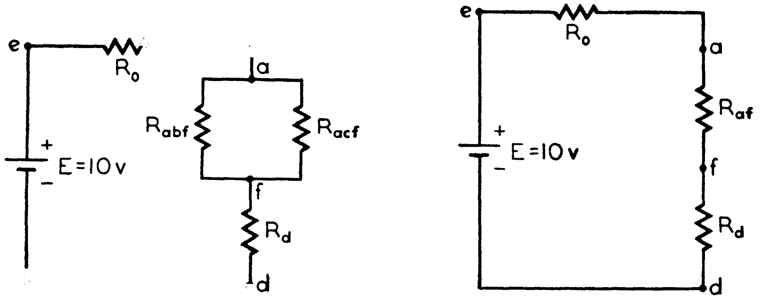
where:

$\mathbf{Z}_\Delta$ = *complex branch impedance of a balanced delta network,* and
$\mathbf{Z}_y$ = *complex branch impedance of the equivalent balanced wye network.*

Example 12.03. Utilize delta-wye transformations to solve the circuit of Fig. 11.04A for the current I_5.



(a) Original circuit. (b) After delta-wye reduction.



(c) Resulting series-parallel circuit. (d) Equivalent series circuit.

Fig. 12.03C. Application of delta-wye transformation to circuit of Example 11.04A.

Substituting an equivalent wye for the delta composed of R_3, R_4, and R_5, as indicated by Fig. 12.03C, we may write the wye-branch resistances as follows:

$$R_b = \frac{4 \times 6}{4 + 5 + 6} = 1.6 \text{ ohms},$$

$$R_c = \frac{5 \times 6}{4 + 5 + 6} = 2.0 \text{ ohms},$$

$$R_d = \frac{4 \times 5}{4 + 5 + 6} = 1.33 \text{ ohms}.$$

The net resistance seen from the emf is, then,

$$R_T = R_0 + R_{ad} = R_0 + \frac{1}{\dfrac{1}{R_1 + R_b} + \dfrac{1}{R_2 + R_c}} + R_d$$

$$= 1 + \frac{1}{\dfrac{1}{2 + 1.6} + \dfrac{1}{3 + 2.0}} + 1.25 = 4.35 \text{ ohms.}$$

The battery current is

$$I_0 = \frac{E}{R_T} = \frac{10}{4.35} = 2.30 \text{ amperes.}$$

The potential drop from a to f is

$$V_{af} = E - I_0(R_0 + R_d) = 4.64 \text{ volts.}$$

The current through branch abf is

$$I_1 = \frac{V_{af}}{R_1 + R_b} = \frac{4.64}{2 + 1.6} = 1.29 \text{ amperes,}$$

and the potential drop from a to b is

$$V_{ab} = I_1R_1 = 1.29 \times 2 = 2.58 \text{ volts.}$$

Similarly, the current through branch acf is

$$I_2 = \frac{V_{af}}{R_2 + R_c} = \frac{4.64}{3 + 2.0} = 0.928 \text{ ampere,}$$

and the potential drop from a to c is

$$V_{ac} = I_2R_2 = 0.928 \times 3 = 2.784 \text{ volts.}$$

The potential drop from b to c is, then,

$$V_{bc} = V_{ac} - V_{ab} = 2.784 - 2.58 = .0204 \text{ volt.}$$

The current is

$$I_5 = \frac{V_{bc}}{R_5} = \frac{0.204}{6} = 0.034 \text{ ampere.}$$

This solution, we note, checks the value obtained by other methods in previous examples.

12.04. Star-to-mesh Transformation[1]

A general n-branch star network of impedances, such as that shown in part (a) of Fig. 12.04 may be replaced by an equivalent mesh arrangement, such as that in part (b) of Fig. 12.04, having branches connecting each node with every other node. This calls for a total of $n(n-1)/2$ branches. Although this transformation is unlikely to be used in practical calculations where n exceeds 3, it is useful as a concept and will form the basis of the development of the next article.

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[1] Rosen, A., "A New Network Theorem," *Institution of Electrical Engineers Journal*, November 1924, Vol. 62, No. 335, page 916.

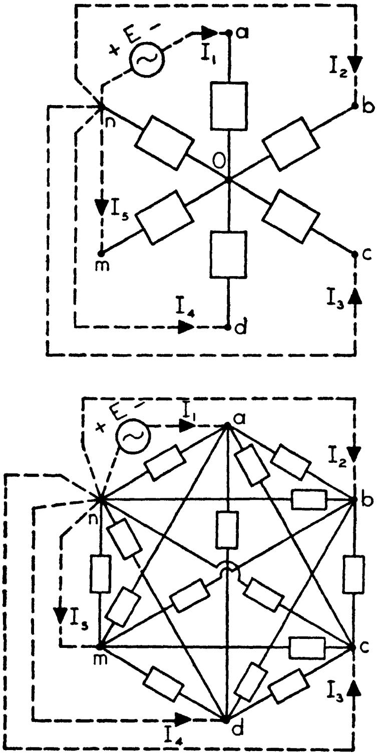


Fig. 12.04. Star-mesh equivalence.

Referring to the figure, let us apply an emf $\mathbf{E}_1$ to the terminals a and n, and connect all other terminals to terminal n. Since the mesh is equivalent to the star, corresponding currents through external connections must be equal. Because of the short circuits, only those mesh branches which connect to terminal a can carry current. We may therefore write the following equations at the various nodes:

$$\mathbf{I}_{oa} = -\mathbf{I}_1, \tag{12-15}$$

$$\mathbf{I}_{ob} = \mathbf{I}_{ab} = \mathbf{E}_1\mathbf{Y}_{ab} = -\mathbf{I}_2, \tag{12-16}$$

$$\mathbf{I}_{oc} = \mathbf{I}_{ac} = \mathbf{E}_1\mathbf{Y}_{ac} = -\mathbf{I}_3, \tag{12-17}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\mathbf{I}_{om} = \mathbf{I}_{am} = \mathbf{E}_1\mathbf{Y}_{am} = -\mathbf{I}_m, \tag{12-18}$$

$$\mathbf{I}_{on} = \mathbf{I}_{an} = \mathbf{E}_1\mathbf{Y}_{an} = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \cdots + \mathbf{I}_m. \tag{12-19}$$

In the star network, the path of current $\mathbf{I}_1$ is from a to o and then back to n through the other radial branches in parallel. We may therefore write

$$\mathbf{I}_1 = -\mathbf{I}_{oa} = \frac{\mathbf{E}_1}{\dfrac{1}{\mathbf{Y}_{oa}} + \dfrac{1}{\mathbf{Y}_{ob} + \mathbf{Y}_{oc} + \cdots + \mathbf{Y}_{om} + \mathbf{Y}_{on}}}. \tag{12-20}$$

This equation reduces to

$$\mathbf{I}_1 = \frac{\mathbf{Y}_{oa}(\Sigma\mathbf{Y}_o - \mathbf{Y}_{oa})}{\Sigma\mathbf{Y}_o}\,\mathbf{E}_1, \tag{12-21}$$

where: $\Sigma\mathbf{Y}_o$ = *the sum of all the star admittances.*

The current $\mathbf{I}_1$, after passing from *a* to *o*, divides among the remaining radial branches in proportion to their admittances. For example,

$$-\mathbf{I}_2 = \mathbf{I}_{ob} = \frac{\mathbf{Y}_{ob}}{\Sigma \mathbf{Y}_o - \mathbf{Y}_{oa}} \mathbf{I}_1. \tag{12-22}$$

Upon substitution of (12-21), this becomes

$$-\mathbf{I}_2 = \mathbf{I}_{ob} = \frac{\mathbf{Y}_{oa}\mathbf{Y}_{ob}}{\Sigma \mathbf{Y}_o} \mathbf{E}_1. \tag{12-23}$$

Upon comparison of this with equation (12-16) we see that we may write

$$\mathbf{Y}_{ab} = \frac{\mathbf{Y}_{oa}\mathbf{Y}_{ob}}{\Sigma \mathbf{Y}_o}. \tag{12-24}$$

By exchanging **E** successively with each of the short-circuit connections and following a similar procedure, we may obtain similar relations for the other admittances of the equivalent mesh. In general, then, the *star-mesh transformation relation* is, for any pair of terminals, *h* and *k*,

▶ $$\mathbf{Y}_{hk} = \frac{\mathbf{Y}_{oh}\mathbf{Y}_{ok}}{\Sigma \mathbf{Y}_o}. \tag{12-25}$$

In terms of impedances, this becomes

▶ $$\mathbf{Z}_{hk} = \mathbf{Z}_{oh}\mathbf{Z}_{ok} \sum \frac{1}{\mathbf{Z}_o}. \tag{12-26}$$

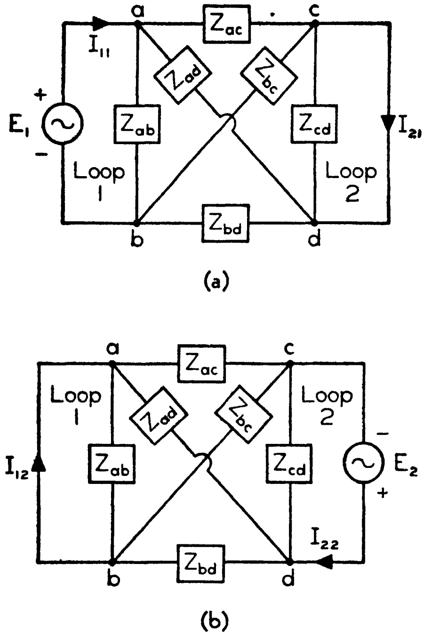
As a special case in which $n = 3$, the wye-delta transformation relations of the preceding article could have been obtained through the use of this relation.

Because the number of conditions for the equivalence of star and mesh networks is exactly equal to the number of branches in the mesh, transformation from star to mesh is always possible. However, except for the three-terminal delta and wye networks considered in the preceding article, transformation from mesh to star is indeterminate or impossible.

12.05. Reciprocity Theorem; Transfer Impedance

Among the interesting and useful network theorems which have been developed to expedite analysis is the *reciprocity theorem.* This theorem may be stated as follows: *The current caused in any branch of a linear network by an emf in any other branch is equal to the current in the other branch which would result from transferring the emf to the first branch.* The reciprocity theorem is not difficult to prove.

It is evident from the preceding article that the star-mesh transformation may be applied successively to any linear impedance network to eliminate as many nodes as desired. The rule of parallel combination may then be employed to reduce to single equivalent branches all groups of parallel branches joining the same two nodes. The net result is an equivalent network having a geometry similar to part (b) of Fig. 12.04. If some of the possible branches do not exist, the missing branches may be treated as though not missing but composed of infinite impedances, or zero admittances.



(a)

(b)

Fig. 12.05A. Application of the reciprocity theorem to a particular circuit.

Let us now suppose that any two meshes of a general network are opened and the free ends provided with terminals, in pairs. By the process outlined above, the network between these two pairs of terminals may be reduced to a lattice of six impedances like that shown in Fig. 12.05A, between the terminal pairs *a-b* and *c-d*.

Let us now place emf E_1, in loop 1 across terminals *a* and *b*, with terminals *c* and *d* short-circuited as indicated in part (a) of the figure. The resulting current in loop 2 is

$$I_{21} = Y_{ac}V_{ac} + Y_{bc}V_{bc} = Y_{ac}V_{ac} - Y_{bc}V_{cb}. \qquad (12\text{-}27)$$

Recalling that *c* and *d* are short-circuited, we see that the net admittance from *a* to *c* is $(Y_{ac} + Y_{ad})$ and the net admittance from

c to b is $(\mathbf{Y}_{bc} + \mathbf{Y}_{bd})$. We also know that if two impedances, $\mathbf{Z}_M$ and $\mathbf{Z}_N$, are in series with a total potential drop, $\mathbf{V}$, across the combination, the component drops across the two impedances are, respectively,

$$\mathbf{V}_M = \frac{\mathbf{I}\mathbf{Z}_M}{\mathbf{I}(\mathbf{Z}_M + \mathbf{Z}_N)}\mathbf{V} = \frac{\mathbf{Y}_N}{\mathbf{Y}_M + \mathbf{Y}_N}\mathbf{V}, \tag{12-28}$$

and

$$\mathbf{V}_N = \frac{\mathbf{I}\mathbf{Z}_N}{\mathbf{I}(\mathbf{Z}_M + \mathbf{Z}_N)}\mathbf{V} = \frac{\mathbf{Y}_M}{\mathbf{Y}_M + \mathbf{Y}_N}\mathbf{V}. \tag{12-29}$$

Applying this principle to the path acb in (a) of Fig. 12.05A, we may write

$$\mathbf{V}_{ac} = \frac{(\mathbf{Y}_{bc} + \mathbf{Y}_{bd})}{(\mathbf{Y}_{ac} + \mathbf{Y}_{ad}) + (\mathbf{Y}_{bc} + \mathbf{Y}_{bd})}\mathbf{E}_1, \tag{12-30}$$

and

$$\mathbf{V}_{cb} = \frac{(\mathbf{Y}_{ac} + \mathbf{Y}_{ad})}{(\mathbf{Y}_{ac} + \mathbf{Y}_{ad}) + (\mathbf{Y}_{bc} + \mathbf{Y}_{bd})}\mathbf{E}_1. \tag{12-31}$$

By substitution, we then obtain

$$\mathbf{I}_{21} = \frac{\mathbf{Y}_{ac}\mathbf{Y}_{bd} - \mathbf{Y}_{bc}\mathbf{Y}_{ad}}{\mathbf{Y}_{ac} + \mathbf{Y}_{ad} + \mathbf{Y}_{bc} + \mathbf{Y}_{bd}}\mathbf{E}_1 = \mathbf{Y}'_{21}\mathbf{E}_1. \tag{12-32}$$

Similarly, when we place emf $\mathbf{E}_2$ in loop 2 across terminals c and d, with terminals a and b short-circuited, as indicated by part (b) of the figure, we find that

$$\mathbf{I}_{12} = \mathbf{Y}_{bd}\mathbf{V}'_{db} + \mathbf{Y}_{bc}\mathbf{V}'_{cb} = \mathbf{Y}_{bd}\mathbf{V}'_{db} - \mathbf{Y}_{bc}\mathbf{V}'_{bc}, \tag{12-33}$$

$$\mathbf{V}'_{db} = \frac{(\mathbf{Y}_{ac} + \mathbf{Y}_{bc})}{(\mathbf{Y}_{ad} + \mathbf{Y}_{bd}) + (\mathbf{Y}_{ac} + \mathbf{Y}_{bc})}\mathbf{E}_2, \tag{12-34}$$

$$\mathbf{V}'_{bc} = \frac{(\mathbf{Y}_{ad} + \mathbf{Y}_{bd})}{(\mathbf{Y}_{ad} + \mathbf{Y}_{bd}) + (\mathbf{Y}_{ac} + \mathbf{Y}_{bc})}\mathbf{E}_2, \tag{12-35}$$

and

$$\mathbf{I}_{12} = \frac{\mathbf{Y}_{ac}\mathbf{Y}_{bd} - \mathbf{Y}_{bc}\mathbf{Y}_{ad}}{\mathbf{Y}_{ac} + \mathbf{Y}_{ad} + \mathbf{Y}_{bc} + \mathbf{Y}_{bd}}\mathbf{E}_2 = \mathbf{Y}'_{12}\mathbf{E}_2. \tag{12-36}$$

Comparison of equations (12-32) and (12-36) shows that

▶
$$\mathbf{Y}'_{12} = \mathbf{Y}'_{21} = \frac{1}{\mathbf{Z}'_{12}} = \frac{1}{\mathbf{Z}'_{21}}, \tag{12-37}$$

where:

$\mathbf{Z}'_{12} = \mathbf{Z}'_{21}$ = *transfer impedance* of the network between the pairs of terminals.

In the words of the AIEE: "*The transfer impedance between any two pairs of terminals of a network is the ratio of a potential difference applied at one pair of terminals to the resultant current at the other pair* *of terminals, all terminals being terminated in any specified manner.*"[2] We may also write

▶ $$\frac{\mathbf{E}_1}{\mathbf{I}_{21}} = \frac{\mathbf{E}_2}{\mathbf{I}_{12}} = \mathbf{Z}'_{21} = \mathbf{Z}'_{12}. \qquad (12\text{-}38)$$

From this relation we see that the reciprocity theorem is merely one way of stating that if a network is composed of linear bilateral impedances, the transfer impedance between any two locations in the network also is linear and bilateral.

EXAMPLE 12.05. Complete solution of the network of Fig. 11.06B by the method of loop currents leads to the following current values: $I'_1 = 2.26$ A, $I'_2 = 0.945$ A, $I'_3 = 0.980$ A, and $I_{bc} = 0.035$ A. What will be the value of the current I'_1 if an emf $E' = 6$ volts is added in the branch bc with such polarity as to oppose the flow of the original current in branch bc? Fig. 12.05B shows the new circuit.

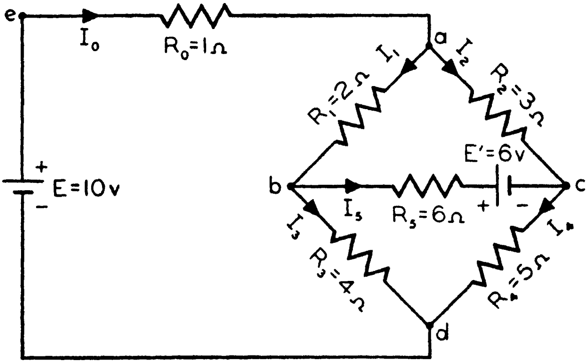


Fig. 12.05B. Circuit of Example 12.05.

Rather than solve this new problem completely by either the loop method or the node method, we shall utilize the principle of superposition and the reciprocity theorem together to obtain the new solution quickly from the old one.

Under the principle of superposition, the old solution becomes one component of the new solution:

$$I'_{1_E} = 2.26 \text{ amperes.}$$

By reciprocity, we note that had the original emf E been transferred to the branch bc it would have given rise to a current in branch ea of the same value as that originally in branch bc, namely, 0.035 ampere. The emf E' in branch *bc* gives rise, by proportion, to a value of current in branch *ea* equal to

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[2] *American Standard Definitions of Electrical Terms*, No. 65.11.515. New York: American Institute of Electrical Engineers, 1942.

$$I'_{1_{E'}} = \frac{-E'}{E} \times 0.035 = \frac{-6}{10} \times 0.035 = -0.021 \text{ ampere.}$$

The resultant value of current in branch *ea* is, then,

$$I'_1 = I'_{1_E} + I'_{1_{E'}} = 2.26 + (-0.021) = 2.24 \text{ amperes.}$$

This solution is considerably shorter, for example, than the simultaneous solution of three new loop equations. This same technique of applying superposition and reciprocity may be employed in many varied ways to shorten the solution of network problems.

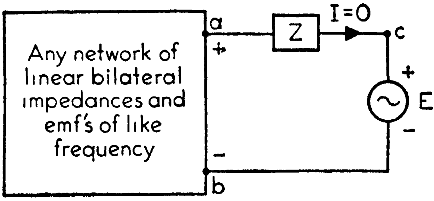
12.06. Thévenin's Theorem

In 1883 M. L. Thévenin first published the highly important network theorem which now bears his name. *Thévenin's theorem* may be stated as follows: *At any given frequency, any linear bilateral network of circuit elements viewed from any two terminals of the network may be replaced by a potential source of emf* $\mathbf{E}'$ *and series impedance* $\mathbf{Z}'$, *where* $\mathbf{E}'$ *is the open-circuit potential of the network measured across the terminals in question and* $\mathbf{Z}'$ *is the impedance of the network viewed from these same terminals with the emf's of all sources replaced by short circuits.* This is easily proved.

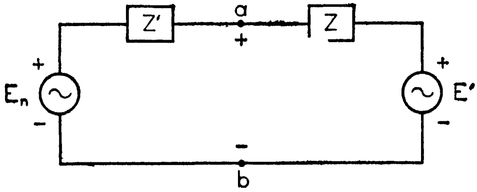
Part (a) of Fig. 12.06A represents a linear bilateral network containing both impedances and emf's of a particular frequency. Terminals *a* and *b* have been brought out. An emf $\mathbf{E}'$ and a load impedance $\mathbf{Z}$ are connected externally in series across the terminals *a* and *b*, and the emf is so adjusted as to be exactly equal in value to the open-circuit potential $\mathbf{V}_{ab}$, with the result that no current leaves or enters the network.

In accordance with the principle of superposition, the zero current in the external loop may be regarded as the sum of the current resulting from $\mathbf{E}'$ acting when all other emf's are short-circuited, plus the current resulting from all other emf's when $\mathbf{E}'$ is short-circuited. It begins to appear that as far as terminal potential and current relations are concerned, the original network of part (a) of the figure may be represented by an internal emf equal to $\mathbf{E}'$ in series with an impedance $\mathbf{Z}'$, as indicated in part (b) of the figure. Furthermore, we see that if the terminal currents of parts (a) and (b) are to equal each other even when the internal emf's are short-circuited and only the external **E**′ is acting, the value of **Z**′ must equal the impedance of the network as viewed from the terminals under this condition. This statement completes the line of reasoning necessary to prove Thévenin's theorem.

Of the many ways in which Thévenin's theorem is useful, two of the most important are that it provides a convenient means of computing the current and potential of any branch of a linear network and that it can be used to determine the behavior of any addition to a linear network. It can be used to determine, for example, the currents taken by potential-measuring instruments or the disturbance such an instrument causes to the circuit. If the original network is a complex power or communication system, such use of this theorem may save considerable time and effort. The following example serves to illustrate one application of the theorem.


Any network of linear bilateral impedances and emf's of like frequency
a
+
Z
I=0
c
+
E′
−
−
b

(a) Original circuit.



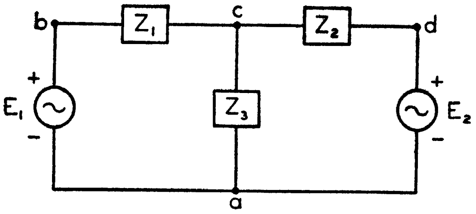
(b) Equivalent circuit.

Fig. 12.06A. Circuit for proof of Thévenin's theorem.

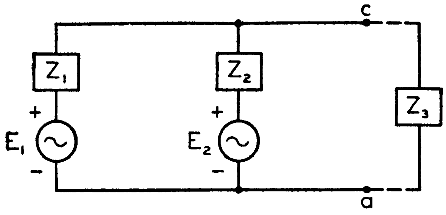
Example 12.06. Re-solve the problem of Example 11.04B by the use of Thévenin's theorem.

Figure 12.06B shows, respectively, the original circuit diagram, a rearranged diagram of the same circuit, and the equivalent circuit diagram derived by Thévenin's theorem.

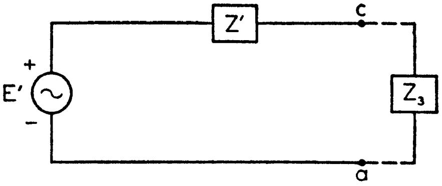
By reference to these diagrams, the solution for $\mathbf{I}_{ca}$ may be carried out in three steps, as follows:



(a) Original circuit.



(b) Rearranged circuit.



(c) Equivalent circuit.

Fig. 12.06B. Circuits for solution of Example 11.04B by the use of Thévenin's theorem.

(a) *Solution for equivalent emf* $\mathbf{E}'$. With terminals *c* and *a* open ($\mathbf{Z}_3$ removed), a circulating current $\mathbf{I}'$ will follow loop *abcd*:

$$\mathbf{I}' = \frac{\mathbf{E}_1 - \mathbf{E}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}.$$

This current causes a drop, $\mathbf{I}'\mathbf{Z}_1$, through the impedance $\mathbf{Z}_1$. The potential between terminals *c* and *a* is the difference between $\mathbf{E}_1$ and this drop. By Thévenin's theorem, then,

$$\mathbf{E}' = \mathbf{E}_1 - \mathbf{I}'\mathbf{Z}_1 = \mathbf{E}_1 - \frac{\mathbf{E}_1 - \mathbf{E}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{Z}_1$$

$$= \frac{\mathbf{E}_1\mathbf{Z}_2 + \mathbf{E}_2\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}.$$

This is the *equivalent emf* of part (c) of Fig. 12.06B.

(b) *Solution for equivalent internal impedance* $\mathbf{Z}'$. With both $\mathbf{E}_1$ and $\mathbf{E}_2$ short-circuited, the net impedance seen by looking into the terminals c and *a* is

$$\mathbf{Z}' = \frac{1}{\dfrac{1}{\mathbf{Z}_1} + \dfrac{1}{\mathbf{Z}_2}} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

This is the *equivalent impedance* of part (c) of Fig. 12.06B.

(c) *Solution for load current* $\mathbf{I}_{ca}$. The series combination of $\mathbf{E}'$ and $\mathbf{Z}'$ can be regarded as an "equivalent generator" possessing *emf* and *internal impedance* of the values $\mathbf{E}'$ and $\mathbf{Z}'$, respectively. The impedance, $\mathbf{Z}_3$, can therefore be regarded as a *load impedance* connected to this generator. The current supplied to $\mathbf{Z}_3$ can then be found as

$$\mathbf{I}_{ca} = \mathbf{I}_3 = \frac{\mathbf{E}'}{\mathbf{Z}' + \mathbf{Z}_3} = \frac{\mathbf{E}_1\mathbf{Z}_2 + \mathbf{E}_2\mathbf{Z}_1}{(\mathbf{Z}_1 + \mathbf{Z}_2)\left(\dfrac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} + \mathbf{Z}_3\right)}$$

$$= \frac{\mathbf{E}_1\mathbf{Z}_2 + \mathbf{E}_2\mathbf{Z}_1}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}.$$

Substituting numerical values from the statement of Example 11.04B, we obtain the same answers that have been found previously by other methods:

$$\mathbf{I}_{ca} = 6.74\underline{/43.2^\circ} \text{ complex amperes.}$$

$$\mathbf{V}_{ca} = \mathbf{I}_{ca}\mathbf{Z}_3 = 95.1\underline{/-1.8^\circ} \text{ complex volts.}$$

12.07. Norton's Theorem

A logical outgrowth of Thévenin's theorem is *Norton's theorem,* which may be stated as follows: *At any given frequency, any linear bilateral network of circuit elements viewed from any two terminals of the network may be replaced by a current source of internal current* $\mathbf{I}'$ *and shunt admittance* $\mathbf{Y}'$, *where* $\mathbf{I}'$ *is the short-circuit current of the two terminals in question and* $\mathbf{Y}'$ *is the admittance of the network viewed from these same terminals with the emf's of all sources replaced by short circuits.*

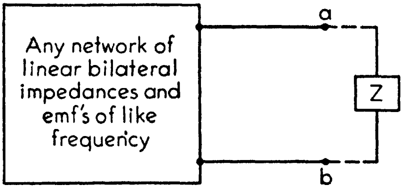
Application of the principles of equivalent current and potential sources to the equivalent network of Fig. 12.07(b) shows that

$$\mathbf{I}' = \frac{\mathbf{E}'}{\mathbf{Z}'}. \tag{12-39}$$

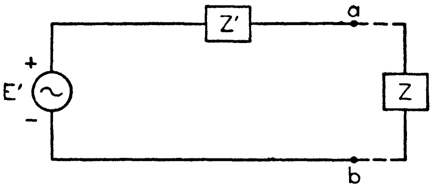
We recognize that this is the current that would be delivered into a short circuit of the terminals *a* and *b* of Fig. 12.07(b), and hence of 12.07(c). This identity proves the first part of the statement of the theorem. Also, under the principles of equivalent current and potential sources, we know that

$$Y' = \frac{1}{Z'}. \tag{12-40}$$

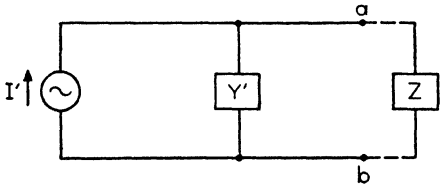
This identity proves the second part of the statement of the theorem.


Any network of linear bilateral impedances and emf's of like frequency
a
Z
b

(a) Original circuit.



(b) Equivalent circuit by Thévenin's theorem.



(c) Equivalent circuit by Norton's theorem.

Fig. 12.07. Circuits for proof of Norton's theorem.

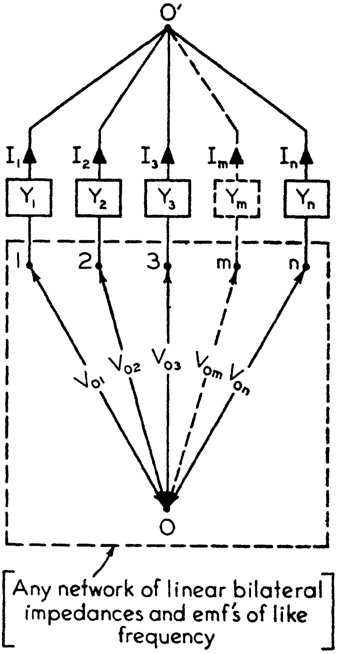


Fig. 12.08A. Circuit for proof of Millman's theorem.

12.08. Millman's Theorem

In 1940 Jacob Millman published a theorem which possesses great utility in solving networks about which a certain type of special information is known.[3] Networks of the form to which this theorem is applicable are encountered frequently both in communication and power applications. *Millman's theorem* may be stated as follows:

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[3] Millman, J., "A Useful Network Theorem," *Proceedings of the Institute of Radio Engineers*, September 1940, Vol. 28, No. 9, page 413.

If any number of linear bilateral admittances join at a common node O′ and the potential drops of any one frequency from any other point O in the network to the other ends of these admittances are known, then the steady-state potential drop from O to O′ is given by the equation

▶ $$V_{OO'} = \frac{V_{O1}Y_1 + V_{O2}Y_2 + V_{O3}Y_3 + \cdots V_{On}Y_n}{Y_1 + Y_2 + Y_3 + \cdots + Y_n}. \qquad (12\text{-}41)$$

Referring to Fig. 12.08A, we may derive this theorem easily as follows. It is evident that

▶ $$\begin{cases} I_1 = V_1Y_1 = (V_{OO}' - V_{O1})Y_1 = V_{OO}'Y_1 - V_{O1}Y_1, & (12\text{-}42a) \\ I_2 = V_2Y_2 = (V_{OO}' - V_{O2})Y_2 = V_{OO}'Y_2 - V_{O2}Y_2, & (12\text{-}42b) \\ I_3 = V_3Y_3 = (V_{OO}' - V_{O3})Y_3 = V_{OO}'Y_3 - V_{O3}Y_3, & (12\text{-}42c) \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \\ I_n = V_nY_n = (V_{OO'} - V_{On})Y_n = V_{OO'}Y_n - V_{On}Y_n. & (12\text{-}42d) \end{cases}$$

The summation of these equations is

$$\begin{aligned} I_T = \sum_{k=1}^{k=n} I_k &= V_{OO'}(Y_1 + Y_2 + Y_3 + \cdots + Y_n) \\ &- (V_{O1}Y_1 + V_{O2}Y_2 + V_{O3}Y_3 + \cdots + V_{On}Y_n) = 0, \qquad (12\text{-}43) \end{aligned}$$

from which equation (12-41) follows directly.

The full implications of the statement and derivation of this theorem are best realized through its actual application to problems. The following example illustrates its application.

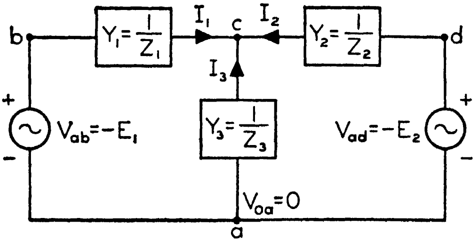

Fig. 12.08B. Circuit for solution of Example 11.04B by means of Millman's theorem.

EXAMPLE 12.08. Re-solve the problem of Example 11.04B by the use of Millman's theorem.

Figure 12.08B shows the circuit diagram redrawn for easy reference. By equation (12-41), calling potential drops positive,

$$\mathbf{V}_{ac} = \frac{\dfrac{-\mathbf{E}_1}{\mathbf{Z}_1} + \dfrac{-\mathbf{E}_2}{\mathbf{Z}_2} + \dfrac{0}{\mathbf{Z}_3}}{\dfrac{1}{\mathbf{Z}_1} + \dfrac{1}{\mathbf{Z}_2} + \dfrac{1}{\mathbf{Z}_3}} = \frac{-\mathbf{Z}_3(\mathbf{E}_1\mathbf{Z}_2 + \mathbf{E}_2\mathbf{Z}_1)}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}.$$

By equation (12-42),

$$\mathbf{I}_{ac} = \frac{-\mathbf{Z}_3(\mathbf{E}_1\mathbf{Z}_2 + \mathbf{E}_2\mathbf{Z}_1)}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1} \cdot \frac{1}{\mathbf{Z}_3} - 0 \cdot \frac{1}{\mathbf{Z}_3},$$

and

$$\mathbf{I}_{ca} = -\mathbf{I}_{ac} = + \frac{\mathbf{E}_1\mathbf{Z}_2 + \mathbf{E}_2\mathbf{Z}_1}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}.$$

This is identical with the final equation derived by the use of Thévenin's theorem, and will, of course, yield the same numerical answer.

12.09. Elementary Bridge Theory

One of the most useful electric circuits ever devised is the *bridge circuit,* first introduced by S. H. Christie in 1833. First practical use of this circuit was made by Charles Wheatstone in 1843 when he applied it to the measurement of d-c resistances. His name has been linked with this application ever since. Rapid evolution since that time has resulted in such a multiplicity of bridge circuits for special purposes that the literature is overwhelming. Several excellent summaries[4] of this information have been prepared, and so only a few basic concepts will be presented here.

"A bridge circuit is a network which is so arranged that, when an electromotive force is present in one branch, the response of a suitable detecting device in another branch may be made zero by a suitable adjustment of the electrical constants of still other branches; and which is characterized by the fact that, if the electromotive force and the detecting device are interchanged after completing an adjustment, the response of the detecting device is still zero."[5]

"A bridge is an instrument which embodies part or all of a bridge circuit, and by means of which one or more of the electrical constants of

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[4] Seeley, W. J., "A Brief Summary of Bridge Networks," *Electrical Engineering,* March 1940, Vol. 59, No. 3, page 108.

[5] *American Standard Definitions of Electrical Terms,* No. 05.45.090. New York: American Institute of Electrical Engineers, 1942.

a bridge circuit may be determined."[6] Figure 12.09 shows the most elementary and the most commonly used configuration for a bridge circuit—the four-arm bridge. We shall restrict our discussion to this type.

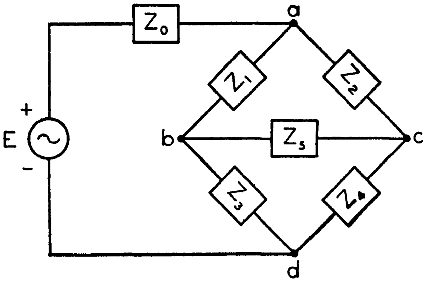

Fig. 12.09. Four-arm bridge circuit.

In the figure, impedance $\mathbf{Z}_5$ is the impedance of a detecting device. By suitable adjustment of the other four impedances, which constitute the arms, a condition of "balance" can be attained, wherein points b and c are at the same potential relative to point a, and no current flows through $\mathbf{Z}_5$. Under this condition of balance,

$$\mathbf{I}_{abd}\mathbf{Z}_1 = \mathbf{I}_{acd}\mathbf{Z}_2 \qquad \text{and} \qquad \mathbf{I}_{abd}\mathbf{Z}_3 = \mathbf{I}_{acd}\mathbf{Z}_4. \tag{12-44}$$

Dividing one of these relations by the other, we obtain

$$\frac{\mathbf{I}_{abd}\mathbf{Z}_1}{\mathbf{I}_{abd}\mathbf{Z}_3} = \frac{\mathbf{I}_{acd}\mathbf{Z}_2}{\mathbf{I}_{acd}\mathbf{Z}_4}, \tag{12-45}$$

or

▶ $$\mathbf{Z}_1\mathbf{Z}_4 = \mathbf{Z}_2\mathbf{Z}_3. \tag{12-46}$$

Evidently, *the condition of balance of a four-arm bridge is that products of impedances of opposite arms are equal.* This statement is true regardless of the value of applied emf, **E**. Interchanging $\mathbf{Z}_5$ and **E** does not disturb the balance, as expressed by (12-46).

In practice, the detector, $\mathbf{Z}_5$, usually takes the form, with alternating current, either of radio earphones, which will provide an audible sound at the frequency of **E** under conditions of unbalance, or of a cathode-ray oscilloscope with which such a signal can be rendered visible. With direct current and pure resistance arms (a *Wheatstone bridge*), a simple galvanometer is employed as the detector.

It is apparent from equation (12-46) that if any three of the four impedances are known in value, the value of the fourth can be found. Thus, bridges have been designed for the measurement of any one of the basic circuit parameters, R, L, or C, as well as for the measurement of impedances consisting of combinations of these parameters. It is apparent, also, from the form of the equation, that *balance is achieved only after both magnitude and phase angle have been balanced.*

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[6] *Ibid.*, No. 30.40.010.

By rewriting (12-46) in polar form, the condition of balance becomes

$$(Z_1\underline{/\theta_1})(Z_4\underline{/\theta_4}) = (Z_2\underline{/\theta_2})(Z_3\underline{/\theta_3}), \qquad (12\text{-}47)$$

or

▶ $$Z_1Z_4\underline{/(\theta_1 + \theta_4)} = Z_2Z_3\underline{/(\theta_2 + \theta_3)}. \qquad (12\text{-}48)$$

Balancing the bridge then consists of achieving the following two conditions simultaneously:

▶ $$Z_1Z_4 = Z_2Z_3 \qquad \text{and} \qquad (\theta_1 + \theta_4) = (\theta_2 + \theta_3). \qquad (12\text{-}49)$$

Rewriting (12-46) in the rectangular form results in another way of viewing the condition of balance, as follows:

$$(R_1 + jX_1)(R_4 + jX_4) = (R_2 + jX_2)(R_3 + jX_3), \qquad (12\text{-}50)$$

and this leads to

$$\begin{aligned}(R_1R_4 - X_1X_4) + j(R_1X_4 + R_4X_1) = (R_2R_3 &- X_2X_3)\\ &+ j(R_2X_3 + R_3X_2), \qquad (12\text{-}51)\end{aligned}$$

from which we obtain the following two conditions of balance:

▶ $$\begin{aligned}(R_1R_4 - X_1X_4) &= (R_2R_3 - X_2X_3)\\ (R_1X_4 + R_4X_1) &= (R_2X_3 + R_3X_2).\end{aligned} \qquad (12\text{-}52)$$

If solution of the bridge under a condition of unbalance is desired, any of the general network theorems can be applied. An important aspect of bridge theory which is beyond the scope of this text is that of *sensitivity* of a given combination of impedance values. The student is referred to any standard text on electrical measurements for a further treatment of this topic, which is practically a science and an art in itself.

12.10. Phase-shifting Networks

In the design of electric apparatus for such special purposes as the control of the firing time of thyratron tubes, it is sometimes desirable to obtain a potential whose phase relative to a given reference emf may be continuously varied while its magnitude remains constant. If no current is drawn from the output terminals, the network of Fig. 12.10 can be used to accomplish this purpose. In operation, both R's of the circuit are kept equal at all times, and both C's are kept equal at all times. It is easy to see that under these conditions the magnitudes of drops $\mathbf{V}_{ab}$ and $\mathbf{V}_{a'b'}$ will be equal,

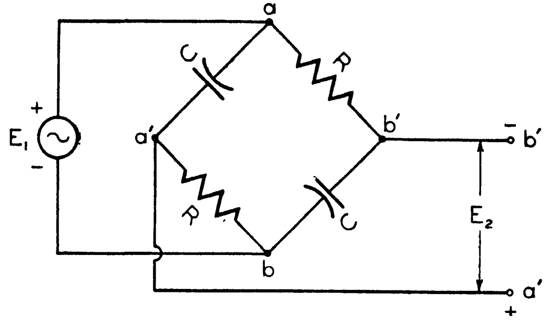

Fig. 12.10. One type of phase-shifting network.

since each is composed of the quadrature (at 90°) combination of a drop equal to IR and a drop equal to IX_C. Thus, the magnitude of $\mathbf{E}_2$ will always equal the magnitude of $\mathbf{E}_1$. This relationship is independent of frequency. Simultaneous variation either of the two R's or of the two C's can be used to vary the phase angle of $\mathbf{E}_2$ relative to $\mathbf{E}_1$, without affecting their magnitudes. The analytical proof of these statements is left as a student exercise.

12.11. External Characteristics of a Two-terminal Network

Thévenin's theorem (Article 12.06) states, in effect, that any two-terminal network consisting of linear bilateral impedances and of sources of emf of any single frequency can be replaced by an equivalent generator consisting of an emf, $\mathbf{E}'$, and internal impedance $\mathbf{Z}'$. One of the most important outgrowths of this theorem is that a general analysis of the behavior of such networks can be conducted as a study of the behavior of a simple generator. This article makes that study for the case of a generator developing a constant emf, $\mathbf{E}_g$, and possessing a constant internal impedance, $\mathbf{Z}_1 = R_1 + jX_1$, as indicated by Fig. 12.11A. A variable load impedance, $\mathbf{Z}_2 = R_2 + jX_2$, is connected across the output terminals, *a* and b. It will be profitable to study some of the *external characteristics* of the network, as characteristics observable at the terminals are called.

A problem which occurs frequently, particularly in communications work, is to determine the net load impedance which will permit maximum power to be transferred from the source to the sink. We know that, in general,

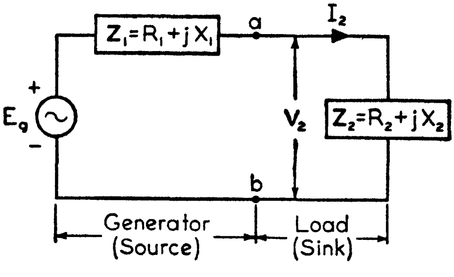

Fig. 12.11A. Equivalent two-terminal supply network with load.

$$I = \frac{E_g}{Z_T} = \frac{E_g}{\sqrt{(R_1 + R_2)^2 + (X_1 + X_2)^2}}. \tag{12-53}$$

It then follows that

$$P_2 = I^2R_2 = \frac{E_g^2R_2}{(R_1 + R_2)^2 + (X_1 + X_2)^2}. \tag{12-54}$$

We realize that the value of R_2 affects both the *magnitude* of $\mathbf{Z}_2$ (and therefore the magnitude of $\mathbf{I}$) and the *angle* of $\mathbf{Z}_2$ (and therefore the angle of $\mathbf{I}$). Since power depends on both magnitude of current relative to potential and angle of current relative to potential, we must express both Z_2 and θ_2 in terms of R_2 and allow both to vary if we seek to find the maximum value of P_2. Equation (12-54) makes use only of Z_2. We can introduce θ_2 by denoting

$$A = \tan\theta_2 = \frac{X_2}{R_2}, \tag{12-55}$$

and then rewriting (12-54) as

$$P_2 = \frac{E_g^2R_2}{(R_1 + R_2)^2 + (X_1 + AR_2)^2}. \tag{12-56}$$

We must now take the derivative of P_2 with respect to R_2 and equate to zero to find the condition for maximum power transfer. Let us do this in two steps, first treating A as a constant, and then allowing A to vary as we differentiate again. This procedure is equivalent to "adjusting" in successive steps both the magnitude and the angle of Z_2.

Treating A as constant and setting $dP_2/dR_2 = 0$, results in

$$R_2 = \frac{\sqrt{R_1^2 + X_1^2}}{\sqrt{1 + A^2}} = \frac{Z_1}{\sqrt{1 + A^2}}. \tag{12-57}$$

Now we must ascertain what effect variation of A (and therefore of θ_2) has on the determination of maximum power.

Substituting (12-57) in (12-56) and rearranging, we have

$$P_2 = \frac{E_g^2}{2Z_1\sqrt{1 + A^2} + 2(R_1 + AX_1)}. \tag{12-58}$$

Now, treating A as variable and setting $dP_2/dA = 0$, we find that

$$A = \pm \frac{X_1}{R_1}. \tag{12-59}$$

Substitution of (12-59) in (12-58) gives

$$P_2 = \frac{E_g^2}{4R_1 + \dfrac{2}{R_1}(X_1^2 \pm X_1^2)}. \tag{12-60}$$

As far as the choice of sign in the last term of the denominator can affect it, the maximum value of P_2 occurs when the negative sign is used, or when

$$A = \frac{X_2}{R_2} = \frac{-X_1}{R_1}. \tag{12-61}$$

Then, for this case,

$$P_{2_{\max}} = \frac{E_g^2}{4R_1}. \tag{12-62}$$

From equation (12-57), *as far as load resistance can affect it,* maximum power is delivered when

▶ $$R_2 = \frac{\sqrt{R_1^2 + X_1^2}}{\sqrt{1 + \dfrac{X_1^2}{R_1^2}}} = \frac{R_1\sqrt{R_1^2 + X_1^2}}{\sqrt{R_1^2 + X_1^2}} = R_1. \tag{12-63}$$

Then, from (12-54), *as far as load reactance can affect it,* maximum power is delivered when

▶ $$X_2 = -X_1. \tag{12-64}$$

The latter two equations can be combined to state that, *for maximum power transfer,*

$$\blacktriangleright \qquad \mathbf{Z}_2 = \hat{\mathbf{Z}}_1, \qquad (12\text{-}65a)$$

or

$$\blacktriangleright \qquad Z_2\underline{/\theta_2} = Z_1\underline{/-\theta_1}, \qquad (12\text{-}65b)$$

or

$$\blacktriangleright \qquad R_2 + jX_2 = R_1 - jX_1. \qquad (12\text{-}65c)$$

The "common sense" of the foregoing conclusions is apparent after a moment of reflection. Equations (12-65) reveal that maximum power transfer occurs, first of all, under a condition of *series resonance.* Our previous study of this phenomenon defined series resonance as a condition of unity power factor in which current magnitude for a given emf is controlled by resistance alone, and therefore is at a maximum for given resistance. It does not sound unreasonable that maximum power transfer should be related to maximum current flow for a given total resistance. We realize, further, that at series resonance the full generator capacity can be utilized in handling active power. The capacitance and the inductance meet each other's reactive power requirements. Secondly, we realize that if unity power factor is maintained, maximum power transfer occurs when load resistance and supply resistance are equal. This statement does not seem unreasonable when one considers the graph of Fig. 12.11B. Although this graph is drawn for the d-c case, where resistance alone is of significance, it is similar to one that could be drawn for the a-c case.

As an example of *two-terminal d-c network characteristics,* Fig. 12.11B deserves some study. It is based on equations (12-62) through (12-65), with the reactance terms dropped out because it is a d-c case. If load resistance, R_2, is caused to vary from an infinitely large value to zero, the load current varies from zero to a large value determined by R_1. A careful study of the figure in the light of the equations leads to a number of interesting and important conclusions.

We note, first of all, that when maximum power is transferred, $R_1 = R_2$, and $I^2R_1 = I^2R_2$, causing a 50 per cent loss of power in the supply circuit. *The efficiency under conditions of maximum power transfer is always exactly* 50 *per cent!* Higher efficiencies can be achieved to the left of this point, as at point *a* in the graph. Here the I^2R loss is only about one-sixth as great as the power transferred, and the efficiency is 85.7 per cent. At the origin, *zero* power is delivered with 100 per cent efficiency! Also, with a d-c network, *maximum power occurs when the terminal potential under load is exactly one-half the no-load terminal potential.*

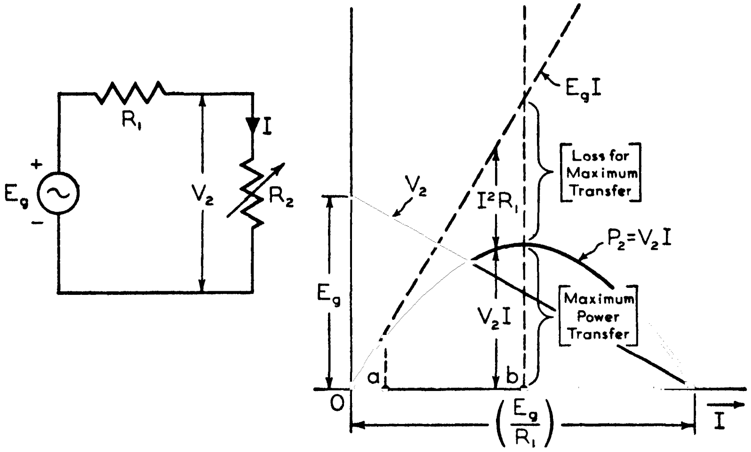

Fig. 12.11B. D-C two-terminal supply network characteristics.

The above statements are of great significance to the application engineer. If the cost of power involved is considerable, as in power systems, an effort must be made to minimize power losses. *Power apparatus is therefore usually designed to operate close to the origin of the graph. In communication systems, on the other hand, where maximum power output is necessary and the cost of power is a secondary factor, operation must be as near as possible to the peak, where power transfer is at a maximum, although efficiency is only* 50 *per cent.*

The foregoing conclusions lead to the highly important concept of *impedance matching* in communications networks. Although this concept has other equally important ramifications which are properly left to a text on communications, maximum power transfer is realized in communications network when impedances viewed both ways from a given pair of junction points are "matched" in accordance with equations (12-65). In some applications, matching of magnitudes alone is employed. If both magnitudes and angles are "matched" in accordance with equations (12-65), maximum power transferred is greater.

12.12. Four-terminal Networks

Figure 12.12A shows a *four-terminal network,* consisting of a combination of linear bilateral impedances from which two pairs of terminals are brought out. Such networks are frequently utilized to connect one part of an electric system at one pair of terminals to another part of the system at another pair of terminals. A four-terminal network so employed is known as a *coupling network.*[7]

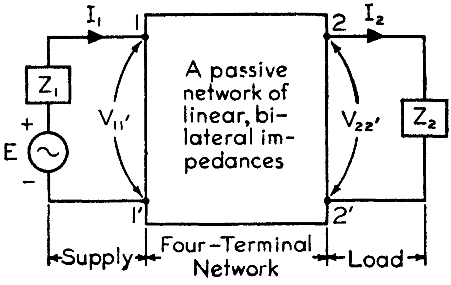

Fig. 12.12A. Four-terminal network with terminations.

So far as currents and potentials at the two pairs of terminals are concerned, any such four-terminal network may be represented at any given frequency by either a simple π or a simple T network. A π *network,* as can be seen from Fig. 12.12B, *is simply a four-terminal delta,* while *a T network is simply a four-terminal wye.* All the statements made concerning the properties of deltas and wyes may be applied to π and T networks.

We can realize from this that only three independent properties of the network (one for each of the three impedances of an equivalent π or T connection) need be expressed to describe the properties of a four-terminal network at any one frequency. Although an unlimited number of ways can be found to define three independent constants, there are three ways most commonly employed in communications practice. These can be found in any good communications textbook or handbook.[8] Since they are somewhat too specialized for this book we shall content ourselves with the expression of the equivalent π or T impedances in terms of open- and short-circuit test measurements at the input and output terminals.

Let us adopt the following notation:

Z_{O1} = *open-circuit impedance of input terminals.* (Impedance at terminals 1-1′ with 2-2′ open.)

Z_{S1} = *short-circuit impedance of input terminals.* (Impedance at terminals 1-1′ with 2-2′ short-circuited.)

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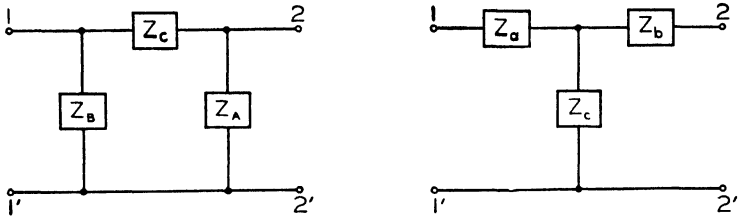
[7] See Chapter 13.

[8] Terman, F. E., *Radio Engineers' Handbook,* Article 3.23. New York: McGraw-Hill Book Company, Inc., 1943.

Z_{O2} = *open-circuit impedance of output terminals.* (Impedance at terminals 2-2′ with 1-1′ open.)

Z_{S2} = *short-circuit impedance of output terminals.* (Impedance at terminals 2-2′ with 1-1′ short-circuited.)



(a) π network.

(b) T network.

Fig. 12.12B. Equivalent π or T networks.

Referring to Fig. 12.12B and assuming input terminals to be at the left, we find, for the T network:

$$Z_{O1} = Z_a + Z_c, \tag{12-66}$$

$$Z_{S1} = Z_a + \frac{Z_bZ_c}{Z_b + Z_c} = \frac{Z_aZ_b + Z_bZ_c + Z_cZ_a}{Z_b + Z_c}, \tag{12-67}$$

$$Z_{O2} = Z_b + Z_c, \tag{12-68}$$

$$Z_{S2} = Z_b + \frac{Z_aZ_c}{Z_a + Z_c} = \frac{Z_aZ_b + Z_bZ_c + Z_cZ_a}{Z_a + Z_c}. \tag{12-69}$$

Since there are but three unknowns, namely, Z_a, Z_b, and Z_c, only three of these equations are independent. We shall use the first three. From (12-66) and (12-68),

$$Z_a = Z_{O1} - Z_c, \tag{12-70}$$

and

$$Z_b = Z_{O2} - Z_c. \tag{12-71}$$

Substituting these values in (12-67), we have

$$Z_{S1} = \frac{(Z_{O1} - Z_c)(Z_{O2} - Z_c) + (Z_{O2} - Z_c)(Z_c) + (Z_c)(Z_{O1} - Z_c)}{(Z_{O2} - Z_c) + Z_c}$$

$$\frac{Z_{O1}Z_{O2} - Z_c^2}{Z_{O2}}. \tag{12-72}$$

From this result, we see that

▶ $$Z_c = \sqrt{Z_{O2}(Z_{O1} - Z_{S1})}. \tag{12-73}$$

Substituting (12-73) in (12-70) and (12-71), we see, further, that

▶ $$Z_a = Z_{O1} - \sqrt{Z_{O2}(Z_{O1} - Z_{S1})}, \tag{12-74}$$

and

▶
$$Z_b = Z_{O2} - \sqrt{Z_{O2}(Z_{O1} - Z_{S1})}. \tag{12-75}$$

These last three equations give, in terms of impedance values which can be obtained by actual measurements at the terminals, the values of the three impedances making up a simple equivalent T network.

By the use of the wye-delta transformation relations, the following expressions can be obtained for the values of the three impedances of a simple π network in terms of values obtainable by measurement at the terminals:

▶
$$Z_A = \frac{Z_{S1}Z_{O2}}{Z_{O1} - \sqrt{Z_{O2}(Z_{O1} - Z_{S1})}}, \tag{12-76}$$

▶
$$Z_B = \frac{Z_{S1}Z_{O2}}{Z_{O2} - \sqrt{Z_{O2}(Z_{O1} - Z_{S1})}}, \tag{12-77}$$

▶
$$Z_C = \frac{Z_{S1}Z_{O2}}{\sqrt{Z_{O2}(Z_{O1} - Z_{S1})}}. \tag{12-78}$$

These last six equations are fundamental in the development of the theory of electric wave filters, as presented in standard communications textbooks.

PROBLEMS

12-1. Solve Prob. 11-1 by the superposition principle.

12-2. Solve Prob. 11-2 by the superposition principle.

12-3. Solve Prob. 11-3 by the superposition principle.

12-4. Solve Prob. 11-5 by the superposition principle.

12-5. Utilize the Δ-Y transformation to simplify the solution of Prob. 11-5.

12-6. Utilize the Δ-Y transformation to reduce the circuit of Prob. 11-6 to a two-mesh network, and solve for V_2.

12-7. Demonstrate the validity of the reciprocity theorem for branches *ab* and *cd* of the circuit of Prob. 11-3 with G_1 and G_2 replaced by short circuits.

12-8. Use superposition and the reciprocity theorem to solve Prob. 11-3.

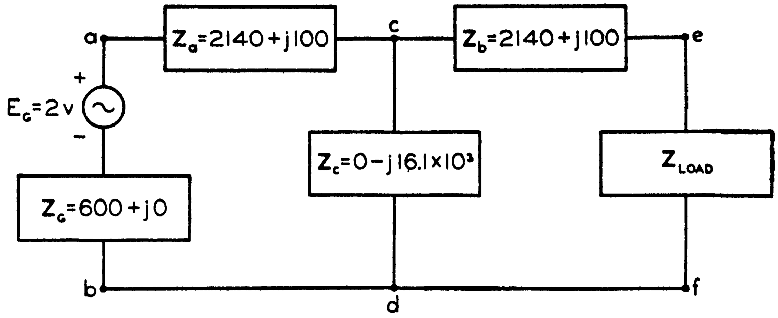
12-9. Solve Prob. 11-1 by the use of Thévenin's theorem.

12-10. Solve Prob. 11-2 by the use of Thévenin's theorem.

12-11. Solve Prob. 11-3 by the use of Thévenin's theorem.

12-12. Solve Prob. 11-4 by the use of Thévenin's theorem.

12-13. The accompanying figure shows a combination of resistive inductors and a capacitor which represent fairly accurately a telephone transmission line at any one frequency. Utilizing Thévenin's theorem, determine the maximum power transfer from the generator to the load.



Prob. 12-13.

12-14. Solve Prob. 11-1 by the use of Millman's theorem.

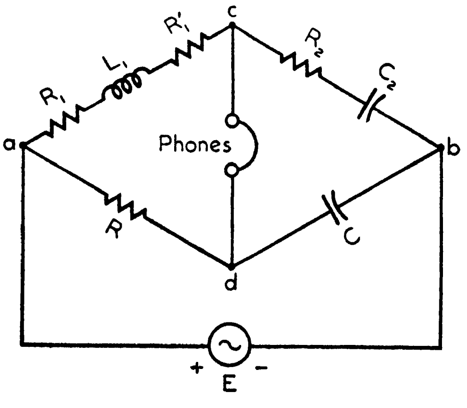
12-15. Solve Prob. 11-2 by the use of Millman's theorem.

12-16. Solve Prob. 11-4 by the use of Millman's theorem.

12-17. Solve Prob. 11-7 by the use of Millman's theorem.

12-18. A very versatile form of bridge, known as the *Owens bridge,* is shown in the accompanying figure. Show that the following two sets of conditions exist when the bridge is balanced:

$$R_1 + R_1' = \frac{RC}{C_2},$$
$$L_1 = CRR_2.$$



Prob. 12-18.

12-19. Prove analytically the statement made in Article 12.10 concerning the phase-shifting network of Fig. 12.10.

12-20. Solve Prob. 9-21 from an "impedance-matching" viewpoint. (*Hint:* imagine an equivalent resistive load in place of the starting motor.)

12-21. Under certain conditions, a given synchronous generator produces a generated emf of 253 volts. If the internal impedance is $Z_G = 3 + j4$ ohms, what is the maximum power that can be delivered by this machine, assuming the generator reactance to remain constant under all load conditions?

12-22. Derive equations (12-76) through (12-78).

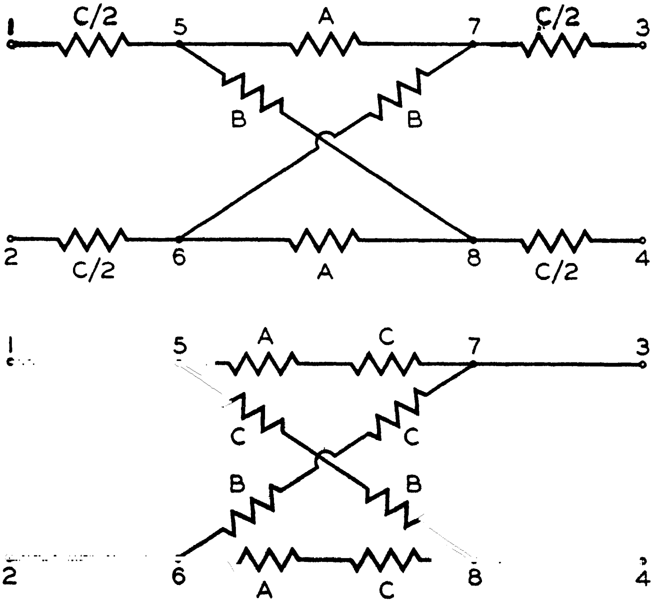
12-23. The following values were obtained by tests on a four-terminal network:

$$\begin{aligned} Z_{O1} &= 30 + j0 \text{ ohms}, \\ Z_{S1} &= 25.5 + j0 \text{ ohms}, \\ Z_{O2} &= 8 + j0 \text{ ohms}. \end{aligned}$$

Find the equivalent T network.

12-24. Repeat Prob. 12-23, finding the equivalent π network.

12-25. Prove that the two four-terminal networks of the accompanying figure are equivalent.



Prob. 12-25.

CHAPTER 13

COUPLED CIRCUITS

13.01. Coupling

"*Coupling is the association of two circuits or systems in such a way that power may be transferred from one to the other.*"[1] In its most general sense, the term "coupling" may refer to any two dynamic systems between which power is transferred. In this sense, a loudspeaker couples an electric circuit to an acoustic system, for example. In the narrower sense in which we shall use the term here, it refers to two electric circuits between which power is transferred. Figure 13.01 shows pictorially the concept underlying this use. Depending on the natures and values of the impedances and the emf's, and on the four-terminal coupling network, a net flow of energy may occur one way or the other between the two circuits.

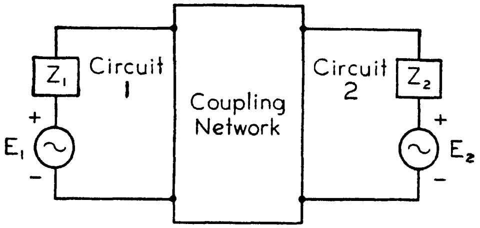


Fig. 13.01. Coupled circuits.

There are three basic types of coupling that can exist between two electric circuits. Each of these makes use of a different type of physical "link" between the two circuits, as follows:

(1) *Resistive coupling* makes use of a common conducting path between the two circuits.

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[1] *American Standard Definitions of Electrical Terms*, No. 65.10.500. New York: American Institute of Electrical Engineers, 1942.

(2) *Capacitive coupling* makes use of a common electric field between the two circuits.

(3) *Inductive coupling* makes use of a common magnetic field between the two circuits.

A given practical case may employ combinations of these types.

From an alternating-current circuit viewpoint, the circuit parameters represented by these couplings may be treated as impedances in suitable combination. As the last article of the preceding chapter showed, any four-terminal network of linear bilateral impedances can be reduced to either an equivalent π or an equivalent T. For this reason, the theory of coupling networks is built around the π and T network relations developed in Chapter 12.

It is important always to remember that, through the mutual element, coupled circuits interact with each other in such a way that the value of current in a given one of the two circuits is governed not only by the emf's and parameters of that circuit itself but also by the parameters of the circuit with which it is coupled. In many applications, coupling may exist between a given circuit and more than one other circuit, or a series of circuits may be coupled successively somewhat like the links of a chain. In such cases, the current of any given circuit is governed, in part, by the emf's and parameters of all other circuits of the coupled group.

13.02. Direct Coupling

"*Direct coupling is the association of two circuits by means of a self inductance, capacitance, resistance, or a combination of them which is common to the circuits.*"[2] If the mutual element is pure resistance, the circuits are said to be *resistance-coupled;* if the mutual element is pure capacitance, the circuit is said to be *capacitance-coupled;* if the mutual element is pure *self* inductance, the circuit is said to be *direct inductance-coupled.* (Mutual inductance as a mode of coupling will be discussed separately.)

Figure 13.02 shows two direct-coupled circuits whose common element is represented as a *mutual impedance.* It is evident that the mutual impedance with direct coupling must, if the mutual parameters are linear and bilateral, have the same numerical value from the viewpoint of either circuit. Following the notation

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[2] *Ibid.*, No. 65.10.515.

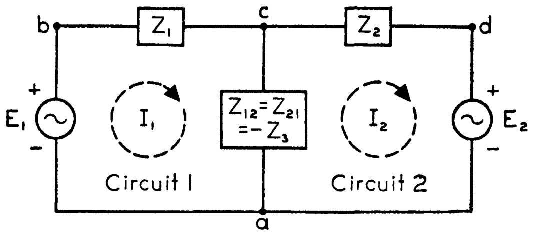

Fig. 13.02. Direct coupling.

adopted in Article 11.06 in connection with the loop method of network solution, let us denote

$$\mathbf{Z}_m = \mathbf{Z}_{12} = \mathbf{Z}_{21} = -\mathbf{Z}_3, \tag{13-1}$$

where:

$\mathbf{Z}_m$ = *mutual impedance of the direct-coupled circuits,*
$\mathbf{Z}_{12}$ = *mutual impedance as it affects Circuit* 1,
$\mathbf{Z}_{21}$ = *mutual impedance as it affects Circuit* 2.

Let us also write

$$\mathbf{Z}_{11} = (\mathbf{Z}_1 - \mathbf{Z}_M) = (\mathbf{Z}_1 + \mathbf{Z}_3), \tag{13-2}$$

$$\mathbf{Z}_{22} = (\mathbf{Z}_2 - \mathbf{Z}_M) = (\mathbf{Z}_2 + \mathbf{Z}_3), \tag{13-3}$$

where:

$\mathbf{Z}_{11}$ = *self impedance of Circuit* 1, and
$\mathbf{Z}_{22}$ = *self impedance of Circuit* 2.

Either the loop or the node method of analysis may be applied to the coupled-circuit problem. By the loop method, referring to the figure and calling potential drops positive, we obtain the following equations:

$$\mathbf{I}_1\mathbf{Z}_{11} + \mathbf{I}_2\mathbf{Z}_{12} = \mathbf{E}_1, \tag{13-4}$$

$$\mathbf{I}_1\mathbf{Z}_{21} + \mathbf{I}_2\mathbf{Z}_{22} = -\mathbf{E}_2. \tag{13-5}$$

These are the same as equations (11-21) and (11-22), for which the following solutions were obtained by the use of determinants:

▶ $$\mathbf{I}_1 = \frac{(\mathbf{E}_1)\mathbf{Z}_{22} - (-\mathbf{E}_2)\mathbf{Z}_{12}}{\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_{12}\mathbf{Z}_{21}} = \frac{(\mathbf{E}_1)\mathbf{Z}_{22} - (-\mathbf{E}_2)\mathbf{Z}_M}{\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_M^2}, \tag{13-6}$$

▶ $$\mathbf{I}_2 = \frac{(-\mathbf{E}_2)\mathbf{Z}_{11} + (-\mathbf{E}_1)\mathbf{Z}_{21}}{\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_{12}\mathbf{Z}_{21}} = \frac{(-\mathbf{E}_2)\mathbf{Z}_{11} + (-\mathbf{E}_1)\mathbf{Z}_M}{\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_M^2}. \tag{13-7}$$

Frequently a source of emf exists in only one of these circuits. If we assume that $\mathbf{E}_2 = 0$ and that $\mathbf{E}_1$ is the driving emf of the network, then Circuit 1 is the *primary circuit* and Circuit 2 is the *secondary circuit*, with energy flowing from Circuit 1 to Circuit 2. Equations (13-6) and (13-7) become, for this case,

$$\blacktriangleright \qquad \mathbf{I}_1 = \frac{\mathbf{E}_1\mathbf{Z}_{22}}{\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_{12}\mathbf{Z}_{21}} = \frac{\mathbf{E}_1\mathbf{Z}_{22}}{\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_M^2}, \qquad (13\text{-}8)$$

$$\blacktriangleright \qquad \mathbf{I}_2 = \frac{-\mathbf{E}_1\mathbf{Z}_{21}}{\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_{12}\mathbf{Z}_{21}} = \frac{-\mathbf{E}_1\mathbf{Z}_M}{\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_M^2}. \qquad (13\text{-}9)$$

A study of the last four equations demonstrates indisputably, for the case of direct coupling, the statement that the current in a given one of two coupled circuits is governed not only by the parameters of that circuit but also by the parameters of the circuit to which it is coupled.

The loop method, as employed above, may be extended to handle a whole "chain" of coupled circuits by writing an appropriate number of simultaneous equations and solving them by determinants of appropriately higher order.

13.03. Driving-point, Transfer, and Mutual Impedances with Direct Coupling

"*The driving-point impedance at any pair of terminals of a network is the ratio of an applied potential difference to the resultant current at these terminals, all terminals being terminated in any special manner.*"[3]

For the direct-coupled circuits of Fig. 13.02, if $\mathbf{E}_2$ is short-circuited, the driving-point impedance as viewed by the source $\mathbf{E}_1$ is, by equation (13-8),

$$\blacktriangleright \qquad \mathbf{Z}_{DP_1} = \frac{\mathbf{E}_1}{\mathbf{I}_1} = \frac{\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_M^2}{\mathbf{Z}_{22}}. \qquad (13\text{-}10)$$

We note that if there is no coupling, then $\mathbf{Z}_M = 0$, and equation (13-10) reduces to

$$\mathbf{Z}_{DP_1} = \frac{\mathbf{E}_1}{\mathbf{I}_1} = \mathbf{Z}_{11} = \mathbf{Z}_1. \qquad (13\text{-}11)$$

This is, of course, the a-c version of Ohm's law for Circuit 1, with $\mathbf{Z}_M$ short-circuited.

Article 12.05 defined transfer impedance between two pairs of terminals of a network as the ratio of the potential difference applied at one pair of terminals to the resultant current at the other pair of terminals, all terminals being terminated in any specified manner. From equation (13-9), we find that the transfer impedance between terminal pairs *ab* and *ca* of the direct-coupled circuit of Fig. 13.02, with terminals *da* short-circuited, is

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[3] *Ibid.*, No. 65.11.510.

$$▶ \qquad \mathbf{Z}_{\text{transfer}} = \frac{\mathbf{E}_1}{\mathbf{I}_2} = \frac{\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_M^2}{-\mathbf{Z}_M}. \tag{13-12}$$

We note that if there is no coupling ($\mathbf{Z}_M$ short-circuited), we may write $\mathbf{Z}_M = 0$, and equation (13-12) becomes

$$\mathbf{Z}_{\text{transfer}} = \frac{\mathbf{E}_1}{\mathbf{I}_2} = \frac{\mathbf{Z}_{11}\mathbf{Z}_{22}}{0} = \frac{\mathbf{E}_1}{0} = \infty. \tag{13-13}$$

No secondary current can flow, in view of this infinite transfer impedance!

"*The mutual impedance between any two pairs of terminals of a network is the ratio of the open-circuit potential difference between either pair of terminals to the current applied at the other pair of terminals, all other terminals being open.*"[4] If this definition is applied to the direct-coupled network of Fig. 13.02 with $\mathbf{E}_2$ replaced by an open circuit, and if potential drops are called positive in the indicated directions of current flow, then

$$▶ \qquad \mathbf{Z}_M = \frac{\mathbf{V}_{acd_0}}{\mathbf{I}_1} = \frac{-\mathbf{V}_{ca}}{\mathbf{I}_1} = \frac{-\mathbf{I}_1\mathbf{Z}_3}{\mathbf{I}_1} = -\mathbf{Z}_3. \tag{13-14}$$

This, we note, agrees with the definition previously stated by equation (13-1).

13.04. Particular Forms of Direct Coupling

A great variety of particular forms of direct coupling can be devised to fit the general definitions of the preceding articles. Figure 13.04 shows a few of the possibilities. The first four diagrams of this figure show the mutual element directly across the line in a single branch. The resulting configuration, including the impedances $\mathbf{Z}_1$ and $\mathbf{Z}_2$ of the coupled circuits, resembles a T. The equations of the previous two articles may be applied directly to these cases. The last four diagrams of this figure show combinations of parameters in π configuration. The equations of the past two articles do not apply directly to this type of coupling circuit. We must therefore develop a relation by means of which the mutual impedance may be evaluated for a π-connected coupling network.

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[4] *Ibid.*, No. 65.11.505.

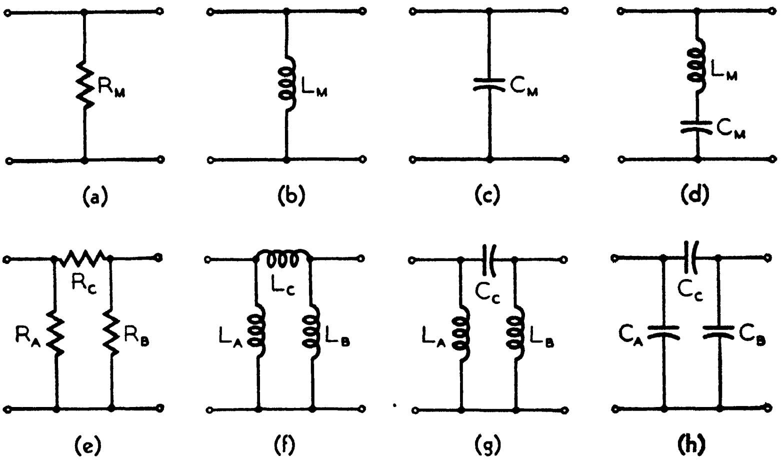


Fig. 13.04. A few of the many forms of direct coupling. (a) Resistive. (b) Inductive. (c) Capacitive. (d) Combined inductive and capacitive. (e) Resistive. (f) Inductive. (g) Capacitive. (h) Combined inductive and capacitive.

13.05. Mutual Impedance of a π-Connected Coupling Network

By the definition of Article 13.04, the mutual impedance between the terminal pairs of a four-terminal network is the ratio of the

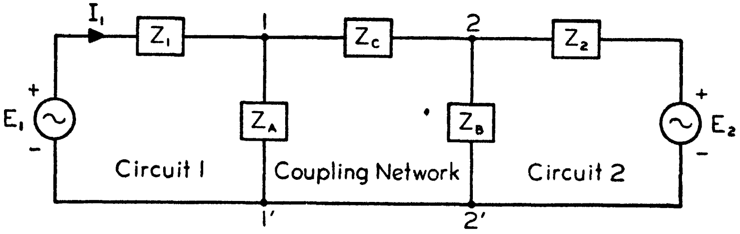


Fig. 13.05A. Circuits coupled by π-connected network.

open-circuit potential difference between either pair of terminals to the current applied at the other pair of terminals. On the basis of this definition, we shall determine the mutual impedance of the π-connected network shown in Fig. 13.05A.

If E_2 is replaced by an open circuit, the mutual impedance of the π-connected coupling network may be found as follows. Considering potential drops to be positive in the indicated direction of current flow, we may write

$$Z_M = \frac{V_{2'2}}{I_1} = \frac{-V_{22'}}{I_1} = \frac{-I_{22'}Z_B}{V_{11'}Y_{ABC}}. \tag{13-15}$$

This may be written

▶ $$Z_M = \frac{-\dfrac{V'_{11}}{Z_B + Z_C} Z_B}{\dfrac{V'_{11}(Z_A + Z_B + Z_C)}{Z_A(Z_B + Z_C)}} = \frac{-Z_AZ_B}{Z_A + Z_B + Z_C}. \tag{13-16}$$

This will be recognized as the impedance of the vertical leg of an equivalent T-connected coupling network!

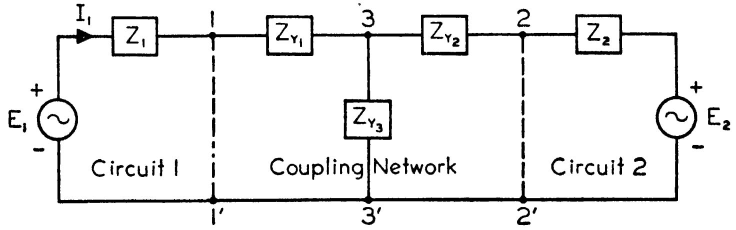


Fig. 13.05B. Circuits coupled by equivalent T-connected network.

The same relation could have been obtained by using the equivalent T-connected coupling network, as indicated by Fig. 13.05B. In this case,

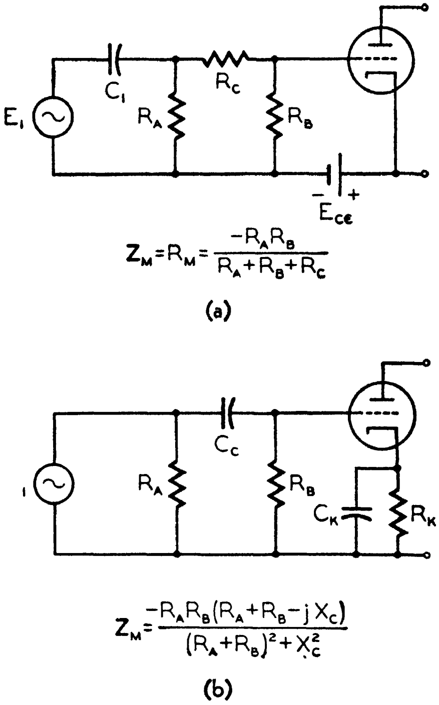
▶ $$Z_M = \frac{V_{2'2}}{I_1} = \frac{V_{3'3}}{I_1} = \frac{-V_{33'}}{I_1} = \frac{-I_1Z_{Y_3}}{I_1} = -Z_{Y_3}$$

$$= \frac{-Z_AZ_B}{Z_A + Z_B + Z_C}. \tag{13-17}$$

As Fig. 13.05B makes apparent, the mutual impedance alone is useful only in determining the *open-circuit* secondary potential in terms of a known primary current. If a solution for the primary currents or for any quantities in the case of a closed secondary circuit is desired, the values of Z_{Y_1} and Z_{Y_2} must be added to Z_1 and Z_2, respectively, before the coupled-circuit relations of Articles 13.02

and 13.03 may be applied to two circuits coupled by a π-connected coupling network.

The approach discussed in this article, it will be realized, reduces what otherwise would be a three-mesh network problem to a two-mesh problem which is especially easy to solve when secondary current is negligible. Figure 13.05C shows some practical applications of π-connected coupling networks in which this condition of negligible secondary current is realized. The derivations of the given mutual impedance expressions are left as student exercises.



$$Z_M = R_M = \frac{-R_A R_B}{R_A + R_B + R_C}$$

(a)

$$Z_M = \frac{-R_A R_B (R_A + R_B - jX_C)}{(R_A + R_B)^2 + X_C^2}$$

(b)

Fig. 13.05C. Some practical applications of π-connected coupling networks.

13.06. Magnetic (Inductive) Coupling

The practical operation of many electrical devices depends on the coupling of one circuit to another by means of a common magnetic field. Such coupling is often called *magnetic*, or *inductive*, *coupling*. "*Inductive coupling is the association of one circuit with another by means of inductance mutual to the circuits . . . . This term, when used without modifying words, is commonly used for coupling by means of mutual inductance, whereas coupling by means of self inductance common to the two circuits is called direct inductive coupling.*"[6]

Figure 13.06 illustrates two circuits which are magnetically (inductively) coupled. For purposes of analysis, the *total flux*, φ_1, which is established by i_1 is divided into two components. One of these, the *leakage flux*, φ_{11}, *of Circuit* 1 is that part which links with

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[6] *Ibid.*, No. 65.10.520.

Circuit 1 but not with Circuit 2. The other of these, the *component of mutual flux established by Circuit* 1, φ_{21}, links not only Circuit 1, but also Circuit 2. The total flux, φ_2, established by i_2 is similarly divided into two components. The *leakage flux*, φ_{22}, *of Circuit* 2 is that part which links with Circuit 2 but not with Circuit 1. The *component of mutual flux established by Circuit* 2, φ_{12}, links not only Circuit 2 but also Circuit 1. The algebraic sum of the two components of mutual flux is the *mutual flux*, φ_M.

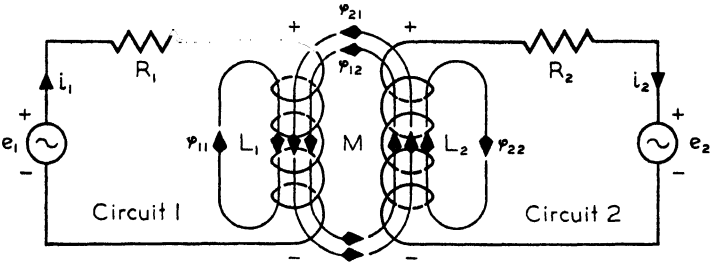

Fig. 13.06. Magnetic coupling between two circuits.

The following mathematical relations come from these definitions:

▶ $$\varphi_1 = \varphi_{11} + \varphi_{21}, \tag{13-18}$$

▶ $$\varphi_2 = \varphi_{22} + \varphi_{12}, \tag{13-19}$$

▶ $$\varphi_M = \varphi_{12} + \varphi_{21}. \tag{13-20}$$

It is helpful, when first becoming acquainted with this notation, to speak of φ_{21} as "the flux linking Coil 2 as a result of Current 1," and so on.

The reader is urged at this point to review the last portion of Article 4.30, with particular reference to equation (4-45). The discussion presented at that point on flux linkages applies equally well to the material of the present chapter.

13.07. Mutual Inductance, M

In Article 4.31 we introduced the definition of *mutual inductance*, *M*. This definition closely parallels that of self inductance, *L*. Dimensionally, it is equivalent, and is measured in **henrys.** Like self inductance, mutual inductance can be defined either in terms of its function of inducing an emf in the secondary of two magnetically coupled circuits when current changes in the primary [see equations

(4.46)], or in terms of change of flux linkages per unit change of current. We shall use the latter approach in this discussion.

Referring to Fig. 13.06, we may write the following equations:

▶ $$L_1 = N_1 \frac{d\varphi_1}{di_1} = \frac{d\lambda_1}{di_1} = \textit{self inductance of Circuit } 1, \qquad (13\text{-}21a)$$

▶ $$L_2 = N_2 \frac{d\varphi_2}{di_2} = \frac{d\lambda_2}{di_2} = \textit{self inductance of Circuit } 2, \qquad (13\text{-}22a)$$

▶ $$M_{12} = N_1 \frac{d\varphi_{12}}{di_2} = \frac{d\lambda_{12}}{di_2} = \textit{mutual inductance of Circuit } 1 \textit{ with respect to Circuit } 2, \text{ and} \qquad (13\text{-}23a)$$

▶ $$M_{21} = N_2 \frac{d\varphi_{21}}{di_1} = \frac{d\lambda_{21}}{di_1} = \textit{mutual inductance of Circuit } 2 \textit{ with respect to Circuit } 1. \qquad (13\text{-}24a)$$

If the magnetic permeabilities of all regions of the magnetic field are constant in value (usually not true of ferromagnetic materials), the self and mutual inductances are constant in value, and the foregoing equations may be rewritten as follows:

$$L_1 = \frac{N_1\varphi_1}{i_1} = \frac{\lambda_1}{i_1}, \qquad (13\text{-}21b)$$

$$L_2 = \frac{N_2\varphi_2}{i_2} = \frac{\lambda_2}{i_2}, \qquad (13\text{-}22b)$$

$$M_{12} = \frac{N_1\varphi_{12}}{i_2} = \frac{\lambda_{12}}{i_2}, \qquad 13\text{-}23b)$$

$$M_{21} = \frac{N_2\varphi_{21}}{i_1} = \frac{\lambda_{21}}{i_1}. \qquad (13\text{-}24b)$$

Under this condition of constant permeability, it is also true that

$$M_{12} = M_{21} = M. \qquad (13\text{-}25)$$

This statement is proved as follows.

Consider two resistanceless coils, 1 and 2, neither of which is carrying current initially. Now increase the current in Coil 1 from zero to I_1. All the energy put into the operation is stored in the field established by the current of Coil 1 and is equal to the integral of the product of current and self-induced emf:

$$W_1 = \int e_{11}i_1\,dt = \int_0^{I_1} L_1 i_1\,di_1 = \tfrac{1}{2}L_1 I_1^2. \qquad (13\text{-}26)$$

Now increase the current in Coil 2 from zero to I_2. While i_2 is increasing, not only is a self-induced emf acting upon the current i_2, but an emf induced into Circuit 1 by the mutual inductance M_{12} acts upon the current I_1. Thus, the energy storage resulting from this operation must contain two terms:

$$\begin{aligned} W_2 &= \int e_{22} i_2 \, dt + \int e_{12} I_1 \, dt \\ &= \int_0^{I_2} L_2 i_2 \, di_2 + I_1 \int_0^{I_2} M_{12} \, di_2 \\ &= \tfrac{1}{2} L_2 I_2^2 + I_1 I_2 M_{12}. \end{aligned} \tag{13-27}$$

The justification for using M_{12} in this manner occurs in the next article. Adding equations (13-26) and (13-27) gives us the final energy storage in the magnetic field:

$$W = W_1 + W_2 = \tfrac{1}{2} L_1 I_1^2 + \tfrac{1}{2} L_2 I_2^2 + I_1 I_2 M_{12}. \tag{13-28}$$

If, now, we start over again and increase first the current of Coil 1 and then that of Coil 2, we obtain

$$W = W_2 + W_1 = \tfrac{1}{2} L_2 I_2^2 + \tfrac{1}{2} L_1 I_1^2 + I_2 I_1 M_{21}. \tag{13-29}$$

From a comparison of (13-28) and (13-29), we conclude that

▶ $$M_{12} = M_{21} = M, \tag{13-30}$$

which is what we set out to prove.

From equations (13-23) and (13-24), it is evident that the mutual inductance of two coils is a function of the geometries of the individual coils, of the relative positions of the two coils, and of the magnetic permeability of the medium.

13.08. Magnetically Coupled Circuit Equations

Figure 13.06 showed a particular case of magnetic coupling in which the relative modes of winding and positive directions of current flow are such as to direct the two positive mutual flux components, φ_{12} and φ_{21}, the same way through the magnetic circuit for positive currents in both windings. This means that if an increase of current i_1 produces in Circuit 1 an induced emf e_1 tending to oppose the increase of current, a simultaneous increase of current i_2 in Circuit 2 will produce in Circuit 1 an induced emf e_{12} of the same polarity as e_1. Both of these emf's which would be thus induced in Circuit 1 would appear as potential drops in the indicated positive direction of current i_1.

By a similar process of reasoning, a simultaneous increase of currents i_1 and i_2 would produce in Circuit 2 induced emf's e_2 and e_{21}, both of the same polarity. In the indicated positive direction of current i_2, these both would appear as potential drops.

Calling potential drops positive in the indicated positive current directions, we may utilize Kirchhoff's potential law to write the following equations applying to the coupled circuits of Fig. 13.06:

$$-e_1 + R_1 i_1 + \left(+N_1 \frac{d\varphi_1}{dt}\right) + \left(+N_1 \frac{d\varphi_{12}}{dt}\right) = 0, \qquad (13\text{-}31a)$$

$$+e_2 + R_2 i_2 + \left(+N_2 \frac{d\varphi_2}{dt}\right) + \left(+N_2 \frac{d\varphi_{21}}{dt}\right) = 0. \qquad (13\text{-}32a)$$

Substituting equations (13-21) through (13-24), we may rewrite these equations as

▶ $$-e_1 + R_1 i_1 + L_1 \frac{di_1}{dt} + M_{12} \frac{di_2}{dt} = 0, \qquad (13\text{-}31b)$$

and

▶ $$+e_2 + R_2 i_2 + L_2 \frac{di_2}{dt} + M_{21} \frac{di_1}{dt} = 0. \qquad (13\text{-}32b)$$

These equations are for the case in which the components of mutual flux due to positive primary current and to positive secondary current are in the same direction and therefore produce a mutually induced emf in a given winding of the same polarity as the self-induced emf of that winding. It will be noticed that if *either* the mode of winding of one of the coils or the positive direction of current flow of one of the circuits is reversed, then the two components of mutual flux will no longer be positive in the same direction but will be positive in opposite directions, and the self-induced emf and the mutually induced emf of a given winding will be of opposite polarities. As a consequence, a reversal of signs would be required in the last terms of equations (13-31) and (13-32), which would then appear as follows:

▶ $$-e_1 + R_1 i_1 + L_1 \frac{di_1}{dt} - M_{12} \frac{di_2}{dt} = 0, \qquad (13\text{-}31c)$$

and

▶ $$+e_2 + R_2 i_2 + L_2 \frac{di_2}{dt} - M_{21} \frac{di_1}{dt} = 0. \qquad (13\text{-}32c)$$

It is clear that the *positive* signs appear with the mutually induced emf terms if the mutual flux components are positive in the same direction, and the *negative* signs appear with the mutually induced emf terms if the mutual flux components are positive in opposite directions.

The relations which have been presented thus far have not depended on any particular manner of variation of potential or current. If the circuit parameters are constant, and if

$$i_1 = I_{m_1} \sin(\omega t + \theta_{i_1}) \qquad \text{and} \qquad i_2 = I_{m_2} \sin(\omega t + \theta_{i_2}), \tag{13-33}$$

we may write

▶ $$\begin{aligned} e_1 = R_1 I_{m_1} \sin(\omega t + \theta_{i_1}) &+ \omega L_1 I_{m_1} \cos(\omega t + \theta_{i_1}) \\ &\pm \omega M_{12} I_{m_2} \cos(\omega t + \theta_{i_2}), \end{aligned} \tag{13-34a}$$

and

▶ $$\begin{aligned} -e_2 = R_2 I_{m_2} \sin(\omega t + \theta_{i_2}) &+ \omega L_2 I_{m_2} \cos(\omega t + \theta_{i_2}) \\ &\pm \omega M_{21} I_{m_1} \cos(\omega t + \theta_{i_1}). \end{aligned} \tag{13-35a}$$

In terms of effective values and complex numbers, these equations become

▶ $$\mathbf{E}_1 = (R_1 + j\omega L_1)\mathbf{I}_1 + (\pm j\omega M_{12})\mathbf{I}_2, \tag{13-34b}$$

and

$$-\mathbf{E}_2 = (\mathbf{R}_2 + j\omega L_2)\mathbf{I}_2 + (\pm j\omega M_{21})\mathbf{I}_1. \tag{13-35b}$$

Let us now denote

▶ $$\mathbf{Z}_{12} = (\pm j\omega M_{12}) = \left(\begin{array}{l} \textit{mutual impedance of Circuit } 1, \\ \textit{with respect to Circuit } 2, \text{ in com-} \\ \text{plex ohms} \end{array}\right), \tag{13-36}$$

▶ $$\mathbf{Z}_{21} = (\pm j\omega M_{21}) = \left(\begin{array}{l} \textit{mutual impedance of Circuit } 2 \\ \textit{with respect to Circuit } 1, \text{ in com-} \\ \text{plex ohms} \end{array}\right). \tag{13-37}$$

Assuming a medium of constant magnetic permeability, these two quantities are equal:

▶ $$\mathbf{Z}_{12} = \mathbf{Z}_{21} = \mathbf{Z}_M = (\pm j\omega M) = \left(\begin{array}{l} \textit{mutual impedance of} \\ \textit{the coupled circuits,} \\ \text{in complex ohms} \end{array}\right). \tag{13-38}$$

Equations (13-34) and (13-35) may, in this case, be written as follows:

$$\blacktriangleright \qquad \mathbf{E}_1 = \mathbf{Z}_1\mathbf{I}_1 + \mathbf{Z}_M\mathbf{I}_2, \tag{13-39}$$

$$\blacktriangleright \qquad -\mathbf{E}_2 = \mathbf{Z}_M\mathbf{I}_1 + \mathbf{Z}_2\mathbf{I}_2. \tag{13-40}$$

These are the general equations for two magnetically coupled electric circuits, with $\mathbf{Z}_M$ defined by equation (13-38) and by the fact that the *positive sign is used when the positive directions of mutual fluxes are the same,* and the *negative sign is used when the positive directions of mutual fluxes are opposites.* The simultaneous solution of these equations by the method of determinants yields

$$\mathbf{I}_1 = \frac{\mathbf{E}_1\mathbf{Z}_2 + \mathbf{E}_2\mathbf{Z}_M}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M^2}, \tag{13-41}$$

and

$$\mathbf{I}_2 = -\frac{\mathbf{E}_2\mathbf{Z}_1 + \mathbf{E}_1\mathbf{Z}_M}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M^2}. \tag{13-42}$$

The form of this solution is seen to be the same as that of the directly coupled case discussed in Article 13.02.

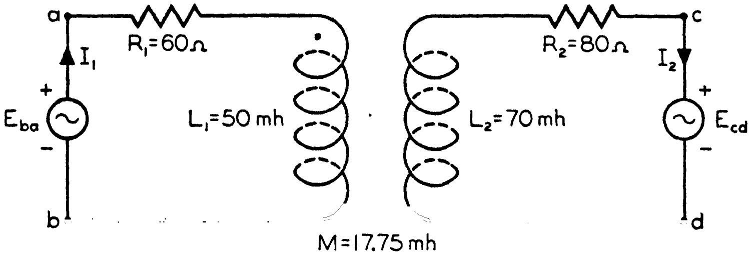

Fig. 13.08A. Circuit for Example 13.08.

Example 13.08. In Fig. 13.08A, $e_{ba} = 169.7 \sin 1885t$, and $e_{cd} = 141.4 \sin (1885t + 45°)$ volts. (a) Find the currents i_1 and i_2. (b) Draw a vector diagram showing all component potential drops and currents for the condition of (a). (c) Find the open-circuit potential of the secondary with the secondary emf removed.

The first step is to determine the relative positive directions of the two components of mutual flux with positive current directions assumed as indicated. By application of the right-hand rule, they are found to be in *opposite* directions. The *negative* sign will then be used in the evaluation of $\mathbf{Z}_M$. We can now write:

$$\begin{aligned}\mathbf{Z}_1 &= R_1 + j\omega L_1 = 60 + j1885 \times 0.050 = 60 + j94.1,\\ \mathbf{Z}_2 &= R_2 + j\omega L_2 = 80 + j1885 \times 0.070 = 80 + j132.0,\\ \mathbf{Z}_M &= 0 \pm j\omega M = 0 - j1885 \times 0.01775 = 0 - j33.4 \text{ ohms,}\end{aligned}$$

$$\mathbf{E}_{ba} = \frac{169.7}{\sqrt{2}} \underline{/0^\circ} = 120\underline{/0^\circ} = 120 + j0 \text{ volts},$$

$$\mathbf{E}_{cd} = \frac{141.4}{\sqrt{2}} \underline{/45^\circ} = 100\underline{/45^\circ} = 70.7 + j70.7 \text{ volts.}$$

Applying Kirchhoff's potential law to each of the circuits:

$$\mathbf{Z}_1\mathbf{I}_1 + \mathbf{Z}_M\mathbf{I}_2 = \mathbf{E}_{ba},$$
$$\mathbf{Z}_M\mathbf{I}_1 + \mathbf{Z}_2\mathbf{I}_2 = \mathbf{E}_{cd},$$

from which

$$\text{(a) } \mathbf{I}_1 = \frac{\mathbf{Z}_2\mathbf{E}_{ba} - \mathbf{Z}_M\mathbf{E}_{cd}}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M^2} = 0.835 - j0.819 = 1.168\underline{/-45.6^\circ} \text{ amperes,}$$

and $$\mathbf{I}_2 = \frac{\mathbf{Z}_1\mathbf{E}_{cd} - \mathbf{Z}_M\mathbf{E}_{ba}}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M^2} = 0.874 - j0.212 = 0.903\underline{/-13.61^\circ} \text{ amperes.}$$

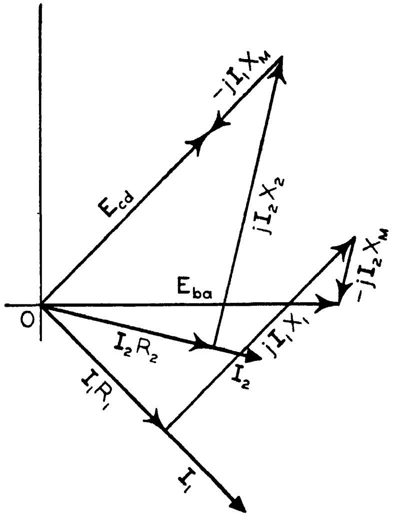

Fig. 13.08B. Vector diagram for Example 13.08.

(b) The vector diagram for this example is obtained by laying out to scale the complex quantities obtained in (a), as indicated by Fig. 13.08B.

(c) If the secondary generator is removed and the circuit left open, no secondary current can circulate, and the open-circuit secondary potential will be equal to the emf induced in the secondary by the primary current. The primary current, it must be noted, differs in value from what it was before. These statements can easily be verified by substituting $\mathbf{I}_2 = 0$ in the original circuit equations:

$$\mathbf{Z}_1\mathbf{I}_1 + 0 = \mathbf{E}_{ba},$$

or $$\mathbf{I}_1 = \frac{\mathbf{E}_{ba}}{\mathbf{Z}_1}.$$

Also, $$\mathbf{Z}_M\mathbf{I}_1 + 0 = -\mathbf{V}_{cd}.$$

Then, by substitution,

$$\mathbf{V}_{dc} = -\mathbf{V}_{cd} = \mathbf{Z}_M\frac{\mathbf{E}_{ba}}{\mathbf{Z}_1} = -30.3 - j19.31 = 35.9\underline{/-147.4^\circ} \text{ volts.}$$

13.09. Leakage and Coupling Coefficients

Figure 13.06 indicates that in general a portion, φ_{11}, of the flux φ_1, established by current i_1 flowing through Circuit 1 fails to link the turns of Circuit 2. This is called the *leakage flux of Circuit* 1 *with respect to Circuit* 2. One can define as follows the *leakage coefficient of Circuit* 1 *with respect to Circuit* 2:

▶
$$k_{11} = \frac{\varphi_{11}}{\varphi_1} = \frac{\varphi_1 - \varphi_{21}}{\varphi_1} = \frac{\varphi_{11}}{\varphi_{11} + \varphi_{21}}. \tag{13-43}$$

Similarly, the *leakage flux of Circuit* 2 *with respect to Circuit* 1 is φ_{22}, and the *leakage coefficient of Circuit* 2 *with respect to Circuit* 1 is

▶
$$k_{22} = \frac{\varphi_{22}}{\varphi_2} = \frac{\varphi_2 - \varphi_{12}}{\varphi_2} = \frac{\varphi_{22}}{\varphi_{22} + \varphi_{12}}. \tag{13-44}$$

In general, the leakage coefficients are not constant and independent of current, and they are not necessarily equal. The leakage coefficients will be equal if the two circuits are of exactly the same shape and size and are symmetrically placed. If there is no ferromagnetic material in the leakage flux paths, the leakage coefficients are constant and independent of current.

Assuming the permeability of the magnetic medium about two coupled circuits to be constant, we know that

$$L_1 = \frac{N_1\varphi_1}{i_1} \quad \text{and} \quad L_2 = \frac{N_2\varphi_2}{i_2}, \tag{13-45}$$

or

$$\frac{L_1}{N_1} = \frac{\varphi_1}{i_1} \quad \text{and} \quad \frac{L_2}{N_2} = \frac{\varphi_2}{i_2}. \tag{13-46}$$

Also,

$$\frac{L_1}{N_1}(1 - k_{11}) = \frac{\varphi_1}{i_1}(1 - k_{11})$$

and

$$\frac{L_2}{N_2}(1 - k_{22}) = \frac{\varphi_2}{i_2}(1 - k_{22}). \tag{13-47}$$

From equations (13-43), (13-44), and (13-45),

$$\frac{\varphi_{21}}{i_1} = \frac{L_1}{N_1}(1 - k_{11}) \quad \text{and} \quad \frac{\varphi_{12}}{i_2} = \frac{L_2}{N_2}(1 - k_{22}). \tag{13-48}$$

We may then write

▶
$$M_{21} = N_2\frac{\varphi_{21}}{i_1} = L_1(1 - k_{11})\frac{N_2}{N_1}, \tag{13-49}$$

and

▶
$$M_{12} = N_1\frac{\varphi_{12}}{i_2} = L_2(1 - k_{22})\frac{N_1}{N_2}. \tag{13-50}$$

Since $M_{12} = M_{21} = M$ if magnetic permeability is constant, it follows, from (13-49) and (13-50), that

▶
$$\begin{aligned} M &= M_{12} = M_{21} = \sqrt{M_{21}M_{12}} \\ &= \sqrt{L_1(1 - k_{11}) \cdot L_2(1 - k_{22})}. \end{aligned} \tag{13-51}$$

Under the condition of no leakage, $k_1 = k_2 = 0$, and

$$M = \sqrt{L_1L_2}. \tag{13-52}$$

In any practical case, there is leakage when two circuits are magnetically coupled. It becomes convenient to define the *coefficient of coupling* between Circuits 1 and 2 as the geometric mean of the ratios of mutual to total flux of the two circuits:

▶
$$\begin{aligned} k_M &= \sqrt{\frac{\varphi_{21}}{\varphi_1} \cdot \frac{\varphi_{12}}{\varphi_2}} = \sqrt{\frac{M_{21}i_1N_2}{L_1i_1N_1} \cdot \frac{M_{12}i_2N_1}{L_2i_2N_2}} \\ &= \sqrt{\frac{M_{21}}{L_1} \cdot \frac{M_{12}}{L_2}}. \end{aligned} \tag{13-53}$$

This becomes, *for the special case of constant permeability,* where $M_{12} = M_{21} = M$,

▶
$$k_M = \sqrt{\frac{M}{L_1} \cdot \frac{M}{L_2}} = \frac{M}{\sqrt{L_1L_2}}. \tag{13-54}$$

Substituting (13-51) in (13-54) enables us to express the coefficient of coupling in terms of the leakage coefficients:

$$\begin{aligned} k_M &= \sqrt{\frac{L_1(1 - k_{11}) \cdot L_2(1 - k_{22})}{L_1L_2}} \\ &= \sqrt{(1 - k_{11})(1 - k_{22})}. \end{aligned} \tag{13-55}$$

It is apparent from (13-53) as well as from (13-55) that the maximum value which the coefficient of coupling could possibly have would be unity, and that this value would occur only under the condition of no leakage. In practical installations, the coefficient of coupling may range in value from 0.01 in certain types of radio circuits to values in excess of 0.99 between iron-core transformer windings.

EXAMPLE 13.09. Find the coefficient of coupling between the circuits of Example 13.08.

Substituting in equation (13-54):

$$k_M = \frac{M}{\sqrt{L_1L_2}} = \frac{0.01775}{\sqrt{00.50 \times 0.070}} = 0.30.$$

13.10. Magnetic Coupling between Series Elements

Frequently it is convenient to connect electrically in series two coils between which there is magnetic coupling. One such arrangement, in which the degree of coupling can be varied by varying the physical orientation of one coil relative to the other, is called a *variometer*, and is used as a *variable inductor*. In this device, the total inductance of the combination will be variable between limits in which the two positive mutual flux components are aiding to maximum degree and in which they are opposing to maximum degree. Occasionally, undesired coupling exists between two parts of a series circuit and must be eliminated by proper placement of the inductive elements of the circuit. Whether desired or undesired, magnetic coupling between parts of the same series circuit may be handled mathematically by proper application of the principles outlined in Article 13.08 for magnetic coupling in general.

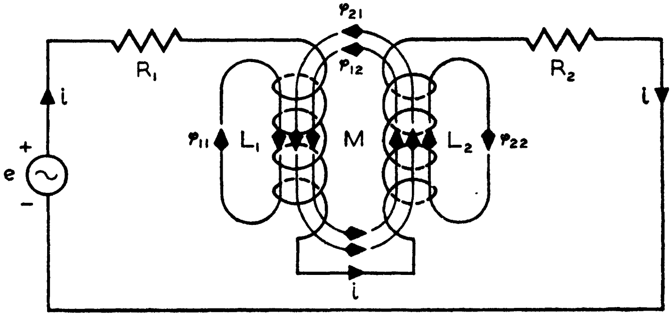

Fig. 13.10. Magnetic coupling between series elements.

Fig. 13.10 shows a particular case in which the mutual flux component, φ_{21}, due to the current i in Coil 1 is positive in the same direction as the mutual flux component, φ_{12}, due to the same current i in Coil 2. Assuming constant permeability of the magnetic medium, $M_{12} = M_{21} = M$, and we may write, by Kirchhoff's potential law, the following summation of potential drops around the circuit:

$$-e + \left(iR_1 + L_1\frac{di}{dt} + M\frac{di}{dt}\right) + \left(iR_2 + L_2\frac{di}{dt} + M\frac{di}{dt}\right) = 0. \quad (13\text{-}56)$$

This may be written

$$e = i(R_1 + R_2) + (L_1 + L_2 + 2M)\frac{di}{dt}. \quad (13\text{-}57)$$

If the mode of winding of Coil 2 were such as to cause φ_{21} to be positive in a direction opposite to the positive direction of φ_{12}, then we would employ the negative sign in front of all mutual inductance terms. The general circuit equation for two magnetically coupled series elements then can be written as

▶ $$e = i(R_1 + R_2) + (L_1 + L_2 \pm 2M)\frac{di}{dt}. \qquad (13\text{-}58)$$

In terms of effective values and complex quantities (assuming a sinusoidal current wave), this becomes

▶ $$\mathbf{E} = \mathbf{I}[(R_1 + R_2) + j\omega(L_1 + L_2 \pm 2M)], \qquad (13\text{-}59)$$

or $$\mathbf{E} = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 \pm 2\mathbf{Z}_M) \qquad (13\text{-}60)$$

It will be observed that equation (13-58) could be rewritten as

▶ $$e = iR' + L'\frac{di}{dt}, \qquad (13\text{-}61)$$

where:

$R' = (R_1 + R_2)$ = *equivalent resistance of the series circuit,*

$L' = (L_1 + L_2 \pm 2M)$ = *equivalent inductance of the series circuit,* and the positive sign is employed when the mutual flux components are positive in the same direction.

13.11. Magnetic Coupling between Parallel Elements

Magnetic coupling between two inductive elements in parallel is sometimes met in practice. The approach used in the preceding article may be employed in analyzing this situation, too.

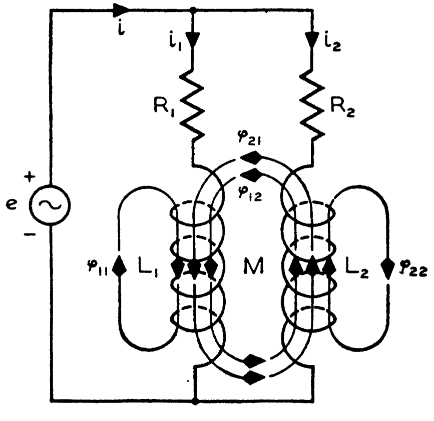

Fig. 13.11. Magnetic coupling between parallel elements.

Figure 13.11 shows the component fluxes for the special case of two magnetically coupled parallel elements in which the mutual flux components are positive in the same direction. From our preceding discussions it is clear that the mutual inductance terms would have the same algebraic sign as the self-inductance terms. If either the mode of winding or the positive direction of current flow (by a reversal of electrical connections) were reversed for *one* of the parallel elements, the two mutual flux components would be positive in opposite directions, and the mutual inductance terms would bear algebraic signs which were the opposite of those borne by the self-inductance terms.

Assuming constant permeability of the magnetic medium, from Kirchhoff's potential law we may write the following general equations for two magnetically coupled parallel elements:

▶ $$e = R_1 i_1 + L_1 \frac{di_1}{dt} \pm M \frac{di_2}{dt}, \qquad (13\text{-}62)$$

▶ $$e = R_2 i_2 + L_2 \frac{di_2}{dt} \pm M \frac{di_1}{dt}. \qquad (13\text{-}63)$$

Assuming sinusoidal currents and constant parameters, we can rewrite these equations as follows, in terms of effective values and complex quantities:

▶ $$\mathbf{E} = (R_1 + j\omega L_1)\mathbf{I}_1 \pm j\omega M \mathbf{I}_2, \qquad (13\text{-}64a)$$

▶ $$\mathbf{E} = (R_2 + j\omega L_2)\mathbf{I}_2 \pm j\omega M \mathbf{I}_1, \qquad (13\text{-}65a)$$

or, where $\mathbf{Z}_M = (\pm j\omega M)$:

▶ $$\mathbf{E} = \mathbf{Z}_1\mathbf{I}_1 + \mathbf{Z}_M\mathbf{I}_2, \qquad (13\text{-}64b)$$

▶ $$\mathbf{E} = \mathbf{Z}_M\mathbf{I}_1 + \mathbf{Z}_2\mathbf{I}_2. \qquad (13\text{-}65b)$$

Then,

$$\mathbf{I}_1 = \frac{\begin{vmatrix} \mathbf{E} & \mathbf{Z}_M \\ \mathbf{E} & \mathbf{Z}_2 \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 & \mathbf{Z}_M \\ \mathbf{Z}_M & \mathbf{Z}_2 \end{vmatrix}} = \frac{\mathbf{E}(\mathbf{Z}_2 - \mathbf{Z}_M)}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M^2}, \qquad (13\text{-}66)$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} \mathbf{Z}_1 & \mathbf{E} \\ \mathbf{Z}_M & \mathbf{E} \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 & \mathbf{Z}_M \\ \mathbf{Z}_M & \mathbf{Z}_2 \end{vmatrix}} = \frac{\mathbf{E}(\mathbf{Z}_1 - \mathbf{Z}_M)}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M^2}. \qquad (13\text{-}67)$$

By Kirchhoff's current law,

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = \frac{\mathbf{E}(\mathbf{Z}_1 + \mathbf{Z}_2 - 2\mathbf{Z}_M)}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M^2}$$
$$= \frac{\mathbf{V}(\mathbf{Z}_1 + \mathbf{Z}_2 - 2\mathbf{Z}_M)}{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M^2}, \qquad (13\text{-}68)$$

since $\mathbf{E} = \mathbf{V}$, where $\mathbf{E}$ is the rise in the i direction and $\mathbf{V}$ is the drop in the i_1 direction. Then the equivalent impedance of two magnetically coupled parallel elements is

$$\blacktriangleright \qquad \mathbf{Z}_p = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\mathbf{Z}_1\mathbf{Z}_2 - \mathbf{Z}_M^2}{\mathbf{Z}_1 + \mathbf{Z}_2 - 2\mathbf{Z}_M}. \tag{13-69}$$

EXAMPLE 13.11. Tests made on a certain variometer whose two coils were series-connected revealed that the obtainable inductance values ranged from 40 millihenrys to 360 millihenrys. Ignoring resistance and assuming the self inductances of the two coils to be equal, find the range of inductance values obtainable if the coils were reconnected so as to be electrically in parallel.

When the coils are in series, we may write

$$L_1 + L_2 + 2M = 2(L + M) = 360 \text{ mh}$$

and

$$L_1 + L_2 - 2M = 2(L - M) = 40 \text{ mh}.$$

These equations contain two unknowns, and may be solved simultaneously to give the following values of self and mutual inductance:

$$L = 100 \text{ mh}; \qquad M = 80 \text{ mh}.$$

By equation (13-69), the equivalent reactance of the two coils when connected in parallel is, at any given frequency,

$$X_p = \frac{X_1X_2 - X_M^2}{X_1 + X_2 - 2X_M} \quad \text{or} \quad \omega L_p = \frac{\omega L_1 \omega L_2 - (\pm\omega M)^2}{\omega L_1 + \omega L_2 - (\pm 2\omega M)}.$$

This result leads to the following expression for equivalent inductance of the parallel combination:

$$L_p = \frac{L^2 - M^2}{2(L \pm M)} = \frac{(100)^2 - (80)^2}{2(100 \pm 80)} = \frac{1800}{100 \pm 80}.$$

The range of inductance values obtainable from the coils in parallel combination is found from this to be from 10 to 90 millihenrys.

13.12. The Air-core Transformer

"*A transformer is an electric device, without continuously moving parts, which by electromagnetic induction transforms electric energy from one or more circuits to one or more other circuits at the same frequency, usually with changed values of voltage and current.*"[6] If the path of the flux of electromagnetic induction is through air, the transformer is an *air-core transformer*. This definition of an air-core transformer describes the principles which we have been studying under the heading "magnetically coupled circuits." Thus, we see that a transformer is, in fact, a device especially designed to accomplish the magnetic coupling of two or more electric circuits, and we can expect an analysis of air-core transformer principles to follow the pattern already developed for coupled circuits.

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[6] *Ibid.*, No. 15.20.010.

Figure 13.12 shows in conventional form the circuit diagram of an air-core transformer. The positive direction of i_2 has been chosen in such a way as to make the two components of mutual flux positive in opposite directions, and the sign of mutual impedance negative.

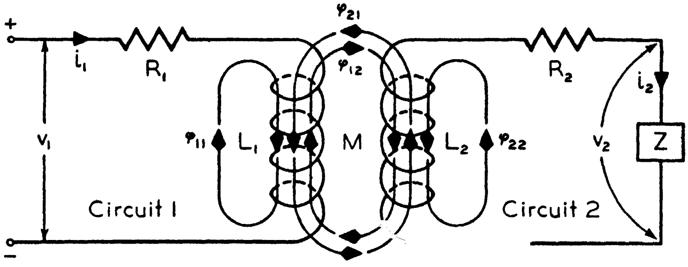


Fig. 13.12. Conventional circuit diagram for the air-core transformer.

Except for the fact that $\mathbf{Z}_M = (-j\omega M)$ and that there is an impedance load instead of a secondary generator, this circuit is exactly like that of Fig. 13.06. The circuit equations, which resemble equations (13-39) and (13-40), will be

$$\mathbf{E}_1 = \mathbf{V}_1 = \mathbf{Z}_1\mathbf{I}_1 + \mathbf{Z}_M\mathbf{I}_2, \tag{13-70}$$

and

$$0 = \mathbf{Z}_M\mathbf{I}_1 + (\mathbf{Z}_2 + \mathbf{Z})\mathbf{I}_2. \tag{13-71}$$

Simultaneous solution of these equations can be performed readily by the method of determinants:

$$\mathbf{I}_1 = \frac{\begin{vmatrix} \mathbf{V}_1 & \mathbf{Z}_M \\ 0 & (\mathbf{Z}_2 + \mathbf{Z}) \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 & \mathbf{Z}_M \\ \mathbf{Z}_M & (\mathbf{Z}_2 + \mathbf{Z}) \end{vmatrix}} = \frac{(\mathbf{Z}_2 + \mathbf{Z})\mathbf{V}_1}{\mathbf{Z}_1(\mathbf{Z}_2 + \mathbf{Z}) - \mathbf{Z}_M^2}, \tag{13-72}$$

$$\mathbf{I}_2 = \frac{\begin{vmatrix} \mathbf{Z}_1 & \mathbf{V}_1 \\ \mathbf{Z}_M & 0 \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 & \mathbf{Z}_M \\ \mathbf{Z}_M & (\mathbf{Z}_2 + \mathbf{Z}) \end{vmatrix}} = \frac{-\mathbf{Z}_M\mathbf{V}_1}{\mathbf{Z}_1(\mathbf{Z}_2 + \mathbf{Z}) - \mathbf{Z}_M^2}. \tag{13-73}$$

We may now write

$$\mathbf{V}_2 = \mathbf{Z}\mathbf{I}_2 = -\mathbf{Z}_M\mathbf{I}_1 - \mathbf{Z}_2\mathbf{I}_2. \tag{13-74}$$

By substitution,

$$\mathbf{V}_2 = \frac{-\mathbf{Z}_M\mathbf{Z}\mathbf{V}_1}{\mathbf{Z}_1(\mathbf{Z}_2 + \mathbf{Z}) - \mathbf{Z}_M^2}. \tag{13-75}$$

Equation (13-74) shows that *the secondary terminal potential of the transformer may be found by treating* $(-\mathbf{Z}_M\mathbf{I}_1)$ *as an induced potential which is supplying energy to the secondary circuit, and* $\mathbf{Z}_2\mathbf{I}_2$ *as an impedance drop which must be deducted in order to obtain* $\mathbf{V}_2$. This is, in fact, the correct physical picture.

Equation (13-71) provides us with the following relation between $\mathbf{I}_1$ and $\mathbf{I}_2$:

▶
$$\frac{\mathbf{I}_1}{\mathbf{I}_2} = \frac{-(\mathbf{Z}_2 + \mathbf{Z})}{\mathbf{Z}_M}. \tag{13-76}$$

From equation (13-75), we obtain the relationship between $\mathbf{V}_1$ and $\mathbf{V}_2$:

▶
$$\frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{\mathbf{Z}_1(\mathbf{Z}_2 + \mathbf{Z}) - \mathbf{Z}_M^2}{-\mathbf{Z}_M\mathbf{Z}}. \tag{13-77}$$

It is of interest—and value—to show that for a given frequency the air-core transformer, as viewed from its primary terminals, can be reduced to an equivalent series circuit having an impedance, $\mathbf{Z}_1'$. From equation (13-72),

$$\mathbf{Z}_1' = \frac{\mathbf{V}_1}{\mathbf{I}_1} = \frac{\mathbf{Z}_1(\mathbf{Z}_2 + \mathbf{Z}) - \mathbf{Z}_M^2}{(\mathbf{Z}_2 + \mathbf{Z})}, \tag{13-78}$$

or

$$\mathbf{Z}_1' = \mathbf{Z}_1 - \frac{\mathbf{Z}_M^2}{\mathbf{Z}_2 + \mathbf{Z}} = \mathbf{Z}_1 + \frac{\omega^2M^2}{\mathbf{Z}_2 + \mathbf{Z}}. \tag{13-79}$$

Let us now denote

$$\mathbf{Z}_s = (\mathbf{Z}_2 + \mathbf{Z}) = (R_s + jX_s). \tag{13-80}$$

Then (13-79) becomes

$$\mathbf{Z}_1' = (R_1 + jX_1) + \frac{\omega^2M^2}{(R_s + jX_s)}, \tag{13-81}$$

or

$$\mathbf{Z}_1' = \left[R_1 + \left(\frac{\omega^2M^2}{Z_s^2}\right)R_s\right] + j\left[X_1 - \left(\frac{\omega^2M^2}{Z_s^2}\right)X_s\right] = R_1' + jX_1'. \tag{13-82}$$

From this, we conclude that the *equivalent resistance referred to the primary* is

▶
$$R_1' = \left[R_1 + \left(\frac{\omega^2M^2}{Z_s^2}\right)R_s\right], \tag{13-83}$$

and that the *equivalent reactance referred to the primary* is

▶
$$X_1' = \left[X_1 - \left(\frac{\omega^2M^2}{Z_s^2}\right)X_s\right]. \tag{13-84}$$

In (13-83) it will be observed that R_s appears at the primary terminals in modified form, and in (13-84) it will be observed that X_s appears at the primary terminals in modified form. Since $R_s = (R_2 + R)$ and $X_s = (X_2 + X)$, it is clear that the impedance of the load, $\mathbf{Z} = (R + jX)$, plus that of the transformer secondary, $\mathbf{Z}_2$, appears at the primary terminals of the transformer in modified form. *This fact suggests the use of transformers in matching the impedance of a load to that of a supply circuit for maximum power transfer.* As we have seen in Article 12.11, this operation is a major consideration in communication circuits. The secondary circuit impedances are said to have been "transferred" from the secondary into the primary.

Under certain conditions the modifying factor of equations (13-83) and (13-84) reduces to approximately $(N_1/N_2)^2$. Under no practical circumstances does it exactly equal the square of the turns ratio. The necessary *simultaneous* conditions under which this factor would be exact are:

(1) $R_s = 0$, making

$$\frac{\omega^2M^2}{R_s^2 + X_s^2} = \frac{M^2}{L_s^2}. \tag{13-85}$$

(2) Unity coupling exists, so that $M = \sqrt{L_1L_2}$, and

$$\frac{M^2}{L_s^2} = \frac{L_1L_2}{L_s^2}. \tag{13-86}$$

(3) All inductance of the secondary circuit is concentrated in the secondary winding, so that $L_s = L_2$, with the result that

$$\left(\frac{L_1L_2}{L_s^2}\right)^2 = \left(\frac{L_1}{L_2}\right)^2 = \left(\frac{\dfrac{N_1\varphi_{11}}{i_1}}{\dfrac{N_2\varphi_{22}}{i_2}}\right)^2 = \left(\frac{N_1}{N_2}\right)^2\left(\frac{\varphi_{11}i_2}{\varphi_{22}i_1}\right)^2. \tag{13-87}$$

Since, for similar one-turn coils, $(\varphi_{11}/i_1) = (\varphi_{22}/i_2)$, it follows that

$$\frac{\omega^2 M^2}{R_s^2 + X_s^2} \approx \left(\frac{N_1}{N_2}\right)^2. \tag{13-88}$$

Inasmuch as all three of these conditions could never be met simultaneously in practice, we conclude that (13-88) at best is only an approximation.

13.13. Primary Resonance with the Air-core Transformer

From equation (13-82), the *total equivalent impedance of an air-core transformer plus its termination, referred to the primary,* can be written as

$$\blacktriangleright \quad \begin{aligned} \mathbf{Z}'_p &= R'_p + jX'_p \\ &= \left[R_p + \left(\frac{\omega^2 M^2}{R_s^2 + X_s^2}\right) R_s\right] + j\left[X_p - \left(\frac{\omega^2 M^2}{R_s^2 + X_s^2}\right) X_s\right], \end{aligned} \tag{13-89}$$

where:

R_p = *total series resistance of the primary circuit,*
X_p = *total series reactance of the primary circuit,*
R_s = *total series resistance of the secondary circuit (including load),*
X_s = *total series reactance of the secondary circuit (including load).*

The *total equivalent reactance, referred to the primary circuit,* is, of course,

$$X'_p = \left[X_p - \left(\frac{\omega^2 M^2}{R_s^2 + X_s^2}\right) X_s\right]. \tag{13-90}$$

In many applications, as has already been pointed out, it is desirable to achieve unity power factor at a given frequency. Under this condition, $X'_p = 0$, or, from (13-90),

$$\blacktriangleright \quad X_p = \left[\left(\frac{\omega^2 M^2}{R_s^2 + X_s^2}\right) X_s\right]. \tag{13-91}$$

It is apparent that, within limits, any one of the quantities X_p, R_s, X_s, or M can be employed as the variable factor to achieve unity power factor in the primary circuit at a given value of ω.

Since the quantities X_p and X_s include, respectively, all of the reactance of the primary circuit and all of the reactance of the secondary circuit, and since X_1 and X_2, the reactances of the primary and secondary windings of the transformer, normally are inductive, it is clear the "adjustment" of X_p or X_s usually involves the insertion of *capacitive* reactance either in the primary or in the secondary circuit. We shall consider three cases of this type, and another in which M is adjusted.

Case 1: *Series Primary Capacitor.* Primary unity power factor can be obtained by introducing a capacitor in series with the primary winding. From equation (13-91),

$$X_p = X_{C_1} + X_1 = \left(\frac{\omega^2 M^2}{R_s^2 + X_s^2}\right) X_s, \tag{13-92}$$

from which

$$X_{C_1} = \left[\left(\frac{\omega^2 M^2}{R_s^2 + X_s^2}\right) X_s - X_1\right], \tag{13-93}$$

and

▶ $$C_1 = \frac{1}{\omega}\left[\frac{Z_s^2}{\omega^2 M^2 X_s - X_1 Z_s^2}\right]. \tag{13-94}$$

Case 2: *Parallel Primary Capacitor.* A capacitor placed in parallel with the primary terminals of the transformer can be used to produce primary unity power factor. This operation involves, of course, making the total susceptance of the combination equal to zero. From equation (13-82),

$$\mathbf{Y}_1' = \frac{1}{\mathbf{Z}_1'} = \left[\frac{R_1'}{(Z_1')^2} + j\left(\frac{-X_1'}{(Z_1')^2}\right)\right]. \tag{13-95}$$

Then, for unity power factor,

$$B_p' = B_1' + B_{C_1} = \left[\left(\frac{-X_1'}{(Z_1')^2}\right) + \left(\frac{-1}{X_{C_1}}\right)\right] = 0. \tag{13-96}$$

From this equation we see that

$$X_{C_1} = \frac{(Z_1')^2}{-X_1'}, \tag{13-97}$$

and

▶ $$C_1 = \frac{1}{\omega}\left(\frac{-X_1'}{(Z_1')^2}\right). \tag{13-98}$$

Case 3: *Secondary Capacitors.* From equation (13-91), it is clear that the insertion of capacitors in the secondary circuit either in series or in parallel with the load impedance can be employed to achieve primary unity power factor. The algebraic expressions which result from a solution in literal form for the values of X_{C_2} and C_2 needed to achieve this are so cumbersome that they will be avoided here. In a given numerical solution, however, these values can be solved for rather easily by a straightforward application of the basic ideas of circuits.

Special Case: Adjustment of M. A study of equation (13-91) reveals that if either X_p or X_s is sufficiently capacitive and the other inductive, an adjustment of M can be employed to obtain primary unity power factor. This adjustment involves changing the degree of coupling between the windings, a feat which is easily accomplished in the case of the air-core transformer by changing the relative locations or orientations of the two windings.

13.14. Partial Resonance

Thus far we have discussed conditions in the primary circuit of an air-core transformer. We shall now transfer our attention to the secondary circuit.

From equation (13-73), we can write the following expression for current in the secondary circuit of an air-core transformer:

$$\mathbf{I}_2 = \frac{-\mathbf{Z}_M\mathbf{V}_1}{\mathbf{Z}_p\mathbf{Z}_s - \mathbf{Z}_M^2}, \qquad (13\text{-}99)$$

where:

$\mathbf{Z}_p$ = *total series impedance of the primary circuit,*
$\mathbf{Z}_s$ = *total series impedance of the secondary circuit,*

and all other quantities are as previously defined. This equation can be expanded as follows:

$$\mathbf{I}_2 = \frac{(-jX_M)\mathbf{V}_1}{(R_p + jX_p)(R_s + jX_s) - (jX_M)^2} \qquad (13\text{-}100)$$

$$= \frac{-X_M\mathbf{V}_1[(X_pR_s + X_sR_p) + j(R_pR_s - X_pX_s + X_M^2)]}{(X_pR_s + X_sR_p)^2 + (R_pR_s - X_pX_s + X_M^2)^2}. \qquad (13\text{-}101)$$

From this result we may obtain the following expression for the magnitude of $\mathbf{I}_2$:

▶ $$I_2 = \frac{V_1X_M}{\sqrt{A}}, \qquad (13\text{-}102)$$

where: A =

$$(X_p^2R_s^2 + X_s^2R_p^2 + R_p^2R_s^2 + 2R_pR_sX_M^2 + X_p^2X_s^2 - 2X_pX_sX_M^2 + X_M^4).$$

It is apparent that variation of any one of the five parameters of equation (13-102) will have an effect on the value of secondary current, I_2. Inasmuch as simple series resonance leads to a maximum value of current with a given applied emf, the variation of any one of these five parameters to give the maximum value of I_2 obtainable by varying that particular parameter leads to a condition akin to series resonance. This condition is termed *partial resonance.* It is clear that partial resonance will be produced by adjusting to zero any parameter which appears in the expression for A in equation (13-102).

In general, the value of any one variable parameter needed to produce partial resonance can be found by differentiating equation (13-102) with respect to that parameter. A great many cases which are of value in the analysis of communications circuits are possible. Study of these cases will be left to texts on communications.

PROBLEMS

13-1. Referring to Fig. 13.04,

(a) Show that the following expression correctly represents the mutual impedance of coupling network (*e*):

$$\mathbf{Z}_M = R_M = \frac{-R_A R_B}{R_A + R_B + R_C}.$$

(b) Write the expression for the mutual impedance of coupling network (*f*) in terms of the reactances.

(c) Write the expression for the "mutual" self inductance, L_M, of coupling network (*f*).

13-2. Referring to Fig. 13.04,

(a) Show that the following expression correctly represents the mutual impedance of coupling network (*g*):

$$\mathbf{Z}_M = jX_M = j\frac{-X_A X_B}{X_A + X_B + X_C}.$$

(b) Write the expression for the "mutual" capacitance, C_M, of coupling network (*g*).

(c) Write the expression for the mutual impedance of coupling network (*f*) in terms of the reactances.

13-3. In the circuit of Fig. 13.02, $\mathbf{Z}_1 = 4 + j3$ ohms, $\mathbf{Z}_2 = 3 - j6$ ohms, and $\mathbf{Z}_3 = 0 - j10$ ohms. If $\mathbf{E}_1 = 100\underline{/0^\circ}$ and $\mathbf{E}_2 = 10\underline{/30^\circ}$ volts, find: (a) The value of current, $\mathbf{I}_{ca}$. (b) The average power supplied by the source of emf $\mathbf{E}_1$. (c) The sum of the I^2R losses in the T network. (d) The average power of the emf $\mathbf{E}_2$.

13-4. (a) Solve Prob. 13-3 for the case where $\mathbf{E}_2 = 0$.

(b) What is the driving-point impedance for this case?

(c) What is the transfer impedance for this case?

(d) If the emf $\mathbf{E}_1$ is replaced by a solid connection in branch 1 and is placed in branch 2 in the position formerly occupied by $\mathbf{E}_2$, what is the new value of $\mathbf{I}_1$?

13-5. Repeat Prob. 13-3 for the following values: $\mathbf{Z}_1 = 4 - j3$ ohms, $\mathbf{Z}_2 = 3 + j6$ ohms, $\mathbf{Z}_3 = 0 + j10$ ohms, $\mathbf{E}_1 = 100\underline{/0^\circ}$ volts, and $\mathbf{E}_2 = 10\underline{/-30^\circ}$ volts.

13-6. Repeat Prob. 13-4 for the values of Prob. 13-5, except to make $\mathbf{E}_2 = 0$.

13-7. In the circuit of Fig. 13.06, the following values apply: $e_1 = 120 \sin 377t$ volts, $e_2 = 100 \sin (377t + 45^\circ)$ volts, $R_1 = 20$ ohms, $L_1 = 100$ millihenrys, $R_2 = 50$ ohms, $L_2 = 50$ millihenrys, and $k = 0.5$.

(a) Find the currents $\mathbf{I}_1$ and $\mathbf{I}_2$.

(b) Write the expressions for i_1 and i_2.

(c) Determine I_1R_1 and I_2R_2.

(d) Draw to scale a vector diagram showing the two applied emf's, the currents, and all potential drops.

(e) What is the secondary open-circuit potential (*one end* of e_2 disconnected)?

13-8. Repeat Prob. 13-7 with the generator e_2 replaced by a solid connection.

13-9. Repeat Prob. 13-7 for $e_1 = 120 \sin 654t$ volts and $e_2 = 100 \sin (654t + 45^\circ)$ volts.

13-10. Repeat Prob. 13-8 for $e_1 = 120 \sin 654t$ volts.

13-11. The circuit of Fig. 13.10 has the following parameters: $R_1 = 60$ ohms, $L_1 = 50$ millihenrys, $R_2 = 80$ ohms, and $L_2 = 70$ millihenrys. The coupling between the two coils can be varied from "boosting" to "bucking," with the maximum coupling coefficient $k = 0.75$. If $e = 169.7 \sin 1885t$ volts,

(a) Find the current and all potential differences for maximum boosting.

(b) Draw to scale a complete vector diagram.

(c) Repeat (a) and (b) for the case where $e' = 141.4 \sin (1885t + 45^\circ)$ volts is added in opposition to e.

13-12. Repeat Prob. 13-11 for the following data: $R_1 = 30$ ohms, $L_1 = 50$ millihenrys, $R_2 = 80$ ohms, and $L_2 = 40$ millihenrys; all other values remain the same.

13-13. The following data apply to the circuit of Fig. 13.11: $R_1 = 10$ ohms, $L_1 = 50$ millihenrys, $R_2 = 20$ ohms, $L_2 = 100$ millihenrys, $k = 0.6$, $e = 169.7 \sin 1885t$ volts.

(a) Find i, i_1, and i_2 as function of ω and t.

(b) Find $\mathbf{I}$, $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_1R_1$, $\mathbf{I}_2R_2$, $\mathbf{I}_1X_{L_1}$, $\mathbf{I}_2X_{L_2}$, $\mathbf{I}_1X_M$, and $\mathbf{I}_2X_M$.

(c) Draw to scale a vector diagram containing all the vectors of (b).

13-14. What value of capacitance placed in the circuit of Prob. 13-11 will produce series resonance at the given frequency, (a) When there is maximum "boosting"? (b) When there is maximum "bucking"?

13-15. Assume the following data to apply to the air-core transformer circuit shown in Fig. 13.12: $R_1 = 4$ ohms; $L_1 = 0.10$ henry; $R_2 = 0.8$ ohm; $L_2 = 0.015$ henry; $M = 0.03$ henry; $Z = 10 + j20$ ohms; $f = 60$ cycles per second; and $\mathbf{V}_1 = 100\underline{/0°}$ volts.

(a) Find $\mathbf{I}_1$, $\mathbf{I}_2$, and all potential drops.
(b) Draw to scale a vector diagram showing the vectors of (a).
(c) Compute the equivalent impedance referred to the primary.
(d) Draw a diagram of the equivalent circuit.
(e) Compute the efficiency of this transformer.

13-16. Re-solve Prob. 13-15 for the following data: $R_1 = 8$ ohms; $L_1 = 0.15$ henry; $R_2 = 0.8$ ohm; $L_2 = 0.015$ henry; $M = 0.04$ henry; $Z = 3 - j4$ ohms; $f = 60$ cycles per second; and $\mathbf{V}_1 = 100\underline{/0°}$ volts.

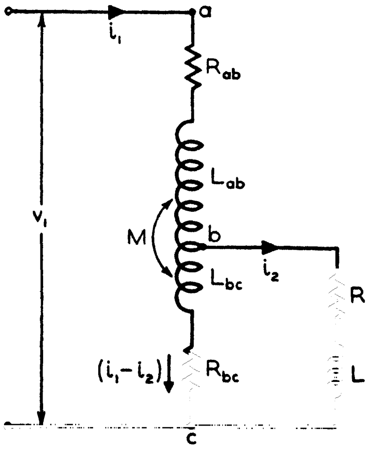
13-17. Find the primary series capacitance required to produce primary unity power factor in the circuit of Prob. 13-15.

13-18. Find the primary series capacitance required to produce primary unity power factor in the circuit of Prob. 13-16.

13-19. Find the primary parallel capacitance required to produce unity power factor at the primary terminals of the circuit of Prob. 13-15.

13-20. Find the primary parallel capacitance required to produce unity power factor at the primary terminals of the circuit of Prob. 13-16.

13-21. The accompanying figure shows an *autotransformer*, in which a portion of the winding which serves one circuit also serves the other.



Prob. 13.21.

Write the general differential equations for potential equilibrium in the two circuits, in terms of R_{ab}, R_{bc}, L_{ab}, L_{bc}, M, R, and L, and i_1 and i_2.

13-22. Assuming v_1, in the circuit of Prob. 13-21, to be a sinusoidal function of t, write the general complex equations for potential equilibrium in terms of the effective values of the currents, and solve the resulting equations for $\mathbf{I}_1$ and $\mathbf{I}_2$.

13-23. Assume the following data to apply to the circuit of Prob. 13-21: $R_{ab} = 1$ ohm, $L_{ab} = 10$ millihenrys, $R_{bc} = 3$ ohms, $L_{bc} = 100$ millihenrys, $M = 30$ millihenrys, $R = 10$ ohms, $L = 10$ millihenrys, $f = 60$ cycles per second, and $V_1 = 100/0°$ volts.

(a) Find $\mathbf{I}_1$, $\mathbf{I}_2$, and $\overline{(\mathbf{I}_1 - \mathbf{I}_2)}$.

(b) Draw to scale a complete vector diagram of all the potentials and currents.

CHAPTER 14

POLYPHASE CIRCUITS

14.01. Introduction

This chapter develops a highly important special application of general alternating-current network theory. This special application utilizes sources of emf which are interrelated in a particular manner, both in magnitude and in phase angle. Because these sources of emf are at *different phase angles*, the circuits which they energize are called *polyphase* circuits. The electrical engineer, regardless of his field of specialization, should be acquainted with the fundamental properties of polyphase circuits, since such circuits are used almost universally in electric-power generation, transmission, and distribution.

14.02. General Polyphase Circuit Definitions

"*A polyphase circuit is a circuit of more than a single phase. This term is ordinarily applied to symmetrical systems.*"[1] A polyphase circuit can be thought of as a combination of several single-phase circuits which are energized by emf's of different initial phase angles. Ordinarily these single-phase component circuits are symmetrically interrelated and interconnected.

The electromotive forces which energize a polyphase circuit comprise a polyphase set of potential differences. "*A polyphase set of potential differences is a group of interrelated alternating potential differences all of which have the same fundamental frequency, but each of which differs from every other in phase and may differ in effective value and wave form.*"[2] In the vast majority of cases, the set of electromotive forces used to energize a polyphase circuit is substantially symmetrical; that is, the vectors used to represent the fundamental frequency components of the individual emf's are uniformly distributed about a circle of 360 electrical degrees measured on the fundamental scale, the individual emf's all have the same wave forms, and they all have the same effective value. Figure 14.02 shows both the waves and the vectors representing a symmetrical set of sinusoidal polyphase potential differences for the case of *six* phases. In more formal terms, the general definition of a symmetrical set of polyphase potential differences may be stated as follows: "*A symmetrical polyphase set of n potential differences is a set in which each potential difference has the same effective value and the same wave form as each other potential difference and in which the members of the set can be arranged in a sequence such that, for the fundamental component, the phase difference between each member and the one following it, and between the first and last members, is* $2\pi/n$ *radians* (360/*n degrees*) *or an integral multiple of this quantity.*"[3] In Fig. 14.02, $n = 6$.

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[1] *American Standard Definitions of Electrical Terms*, No. 35.40.035. New York: American Institute of Electrical Engineers, 1942.

[2] *Ibid.*, No. 05.21.120.

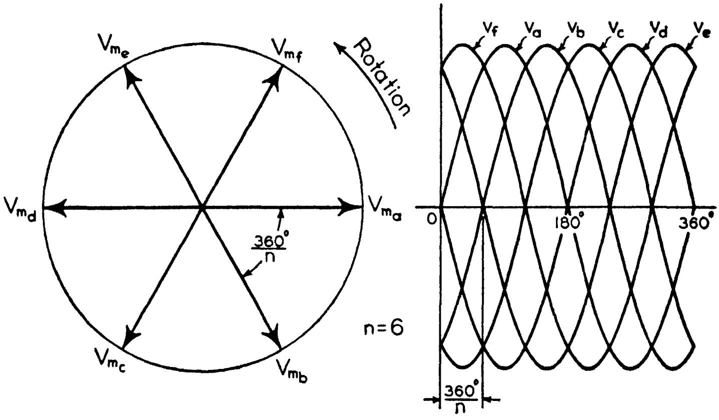

Fig. 14.02. A symmetrical set of six sinusoidal polyphase potential differences.

Substitution of the symbols I and i for V and v in the figure, and of the word "current" for "potential difference" in the definitions, yields the corresponding figures and definitions for *currents* encountered in polyphase circuits.

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[3] *Ibid.*, No. 05.21.125.

Phase sequence is the term applied to the order in which the individual phase potentials (or currents) reach their positive values. In Fig. 14.02, the phase sequence is *a*, *b*, *c*, *d*, *e*, *f*.

14.03. Generation of Polyphase EMF's

Polyphase emf's are generated in the same way as single-phase emf's. The elementary polyphase synchronous generator resembles the single-phase synchronous generator of Fig. 3.15B, except that there is an additional armature coil on the stator for each additional phase. In general, an elementary polyphase synchronous generator of n phases must have n coils per pair of poles, these coils being spaced about the periphery of the armature at intervals of $360/n$ electrical degrees. Figure 14.03 shows an elementary two-pole three-phase alternator. In this case, electrical degrees and mechanical degrees are numerically equal, inasmuch as one complete revolution of the single pair of poles is required to generate a complete cycle of emf in the coil of any given phase. If there were

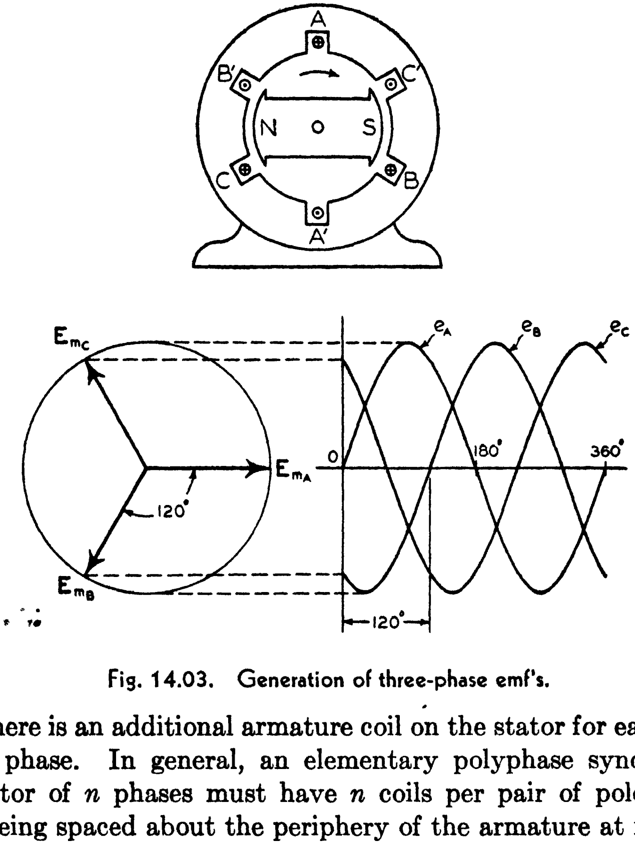


Fig. 14.03. Generation of three-phase emf's.

two pairs of poles on the rotor, each mechanical degree would equal numerically two electrical degrees, inasmuch as one complete revolution of the two pairs of poles would generate two complete cycles of emf in the coil of any given phase. Figure 14.03 shows not only the three emf waves of a three-phase generator but also the three vectors which may be used to represent those waves. In general, reversing the direction of rotation of the rotor of a polyphase synchronous generator reverses the phase sequence of the generated emf's.

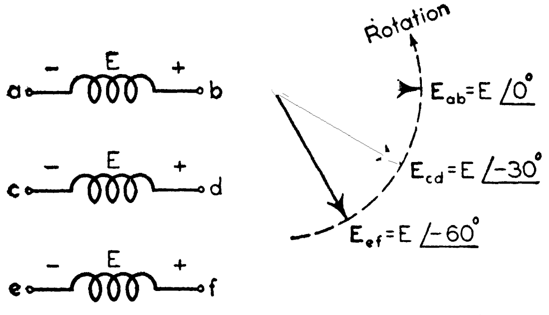
The one exception to the general rule that an elementary n-phase generator has n coils uniformly spaced about the armature at angles of $360/n$ electrical degrees is the *two-phase* generator. If a two-phase generator were designed in accordance with this rule, since $n = 2$, the two coils of such a generator would be separated by an angle of $360/2 = 180$ electrical degrees. The positive emf vectors of two such coils would always point in exactly opposite directions. Inasmuch as the usefulness of polyphase systems depends on emf's whose positive vectors are not in the same straight line, such a generator would have no advantage over a single-phase generator. By special definition, then, an elementary two-phase generator is one whose two coils are 90 electrical degrees apart. The two resulting emf vectors are, therefore, perpendicular to each other. Because of this exception to the rule, general statements made regarding n-phase circuits do not always apply to two-phase circuits.

14.04. Polyphase Vector Diagrams and Notation

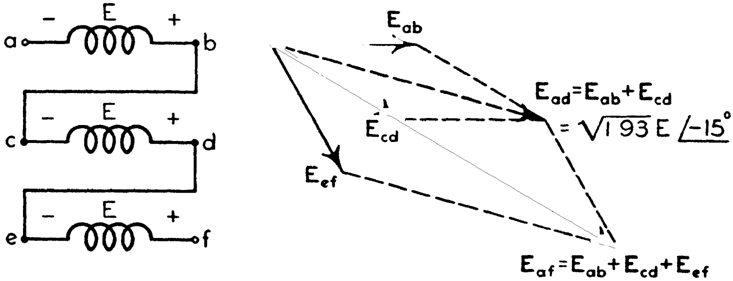
In polyphase circuit work, it is highly important that the assumed positive current directions and the corresponding relative potential polarities of all parts of a circuit be clearly indicated on a circuit diagram. These indicated current directions and potential polarities are those which would prevail when the quantities in question were on the assumed positive portions of their cycles. The corresponding vectors of a vector diagram may then be keyed in, through proper notation, with this circuit diagram. Following is a summary of the procedure which was suggested as far back as Article 3.04 for systematizing the symbolic representation of quantities. This same procedure is recommended for polyphase circuit problems.

1. *Letter or number all nodes of the circuit diagram.*

2. *Use arrows to represent the assumed positive current directions in all branches or loops, and assign a suitable symbol to represent mathematically each such current. If desired, double subscripts may be employed to indicate the sequence in which the current encounters the terminals of a given branch. Thus,* $\mathbf{I}_{ab}$ *signifies "the current from a to b."*



(a) Generated emf's of three generator coils.



(b) Resultant emf's for coils in series.

Fig. 14.04. A particular application of double-subscript notation.

3. *Use pairs of (+) and (−) signs to indicate the assumed relative polarities of pairs of terminals of all branches, and assign a suitable symbol to represent mathematically each significant potential difference. Thus,* $\mathbf{V}_{ab}$ *signifies "the potential drop from a to b."*

Figure 14.04 shows the application of this scheme to a particular case. To forestall any tendency toward confusion in the future, it is pointed out here that, *in general, there need be no geometric simi-*

larity between a polyphase circuit diagram and the associated vector diagram.

14.05. Mesh-connected Polyphase EMF's

If the separate phase windings of an n-phase generator are so interconnected that a continuous closed path is formed and each terminal of each phase winding connects only to that terminal of another phase winding which is nearest to it in effective potential value, the windings of the generator are said to be *mesh-connected.* Figure 14.05A shows the circuit diagram and the associated vector diagram for an n-phase set of mesh-connected emf's.

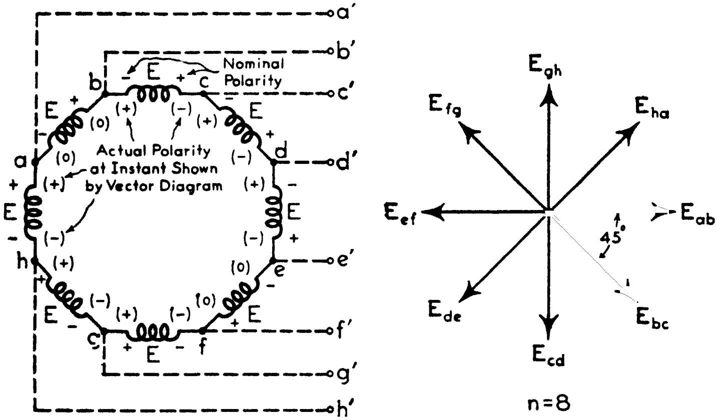


Fig. 14.05A. An eight-phase set of mesh-connected emf's.

It will be noted that, in general, the vector sum of the emf vectors of an n-phase mesh-connected emf source is zero. For this reason, although it might appear that the mesh connection places a short circuit on the generator windings, the circulating current through the closed path around the mesh is zero also. Any unbalance of the generated emf's would, of course, result in circulating current. The physical reason for the fact that the net emf available for circulating currents through the mesh is zero at all times can be understood by a consideration of the diagrams of the figure. It will be noted that although the indicated "positive" polarities of the various windings all provide potential rises around the mesh in a clockwise direction, actual polarities at any given instant are such as to provide rises in clockwise direction through only half of the total number of windings, and drops (negative rises) in that direction through the remaining windings. The physical reason for this phenomenon lies in the fact that at any given instant half of the armature conductors are under north magnetic poles, whereas the other half are under south magnetic poles. In terms of the wave diagram, at any instant the algebraic sum of the ordinates to the various waves is zero. In terms of the vector diagram, the vector sum of all the vectors is always zero.

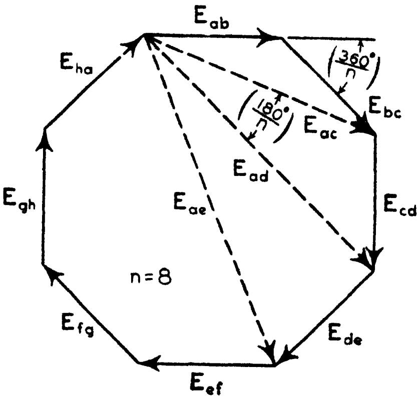


Fig. 14.05B. String diagram for an eight-phase set of mesh-connected emf's.

Figure 14.05A showed vectors emanating from a common origin. This type of vector diagram is known as a *polar vector diagram.* Figure 14.05B shows the same vectors "added end to end." This type of vector diagram is known as a *string*, or *funicular*, *vector diagram.* Either type of diagram may be drawn for any given problem. The string diagrams shown in this figure make it easy to determine the values of potential between any pairs of nodes of the mesh, as the dashed lines indicate.

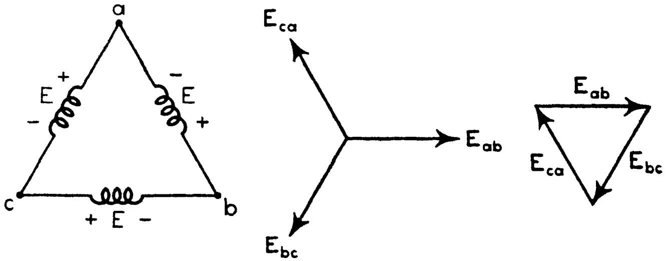


Fig. 14.05C. Three-phase mesh- (delta-) connected emf's.

By definition, the potential between two line wires connected to the line terminals of a symmetrical balanced polyphase circuit is a *line potential,* and that across an individual phase of the circuit is a *phase potential.* From the circuit diagram of Fig. 14.05A it is apparent that, for a given symmetrical balanced *mesh*-connected polyphase source, the magnitude of the line potential equals the magnitude of the phase potential. Mathematically, for this case,

▶ $$E_{\text{line(mesh)}} = E_{\text{phase(mesh)}} \tag{14-1}$$

A *three-phase mesh connection* is known as a *delta connection.* This type of connection is frequently encountered in practice. Figure 14.05C presents the diagrams applying to such a connection.

EXAMPLE 14.05. A voltmeter connected between terminals a and b of a balanced symmetrical three-phase source indicates 100 volts. The phase sequence is known to be e_{ab}, e_{bc}, e_{ca}. (a) Write the equations for the emf's as sinusoidal functions of time, using e_{ab} as reference. (b) Write, in terms of effective values, the complex expressions for the three emf's.

$$\begin{aligned}
\text{(a)}\quad e_{ab} &= E_m \sin(\omega t + 0^\circ) = 100\sqrt{2}\sin(\omega t + 0^\circ) = 141.4 \sin \omega t,\\
e_{bc} &= E_m \sin(\omega t - 120^\circ) = 100\sqrt{2}\sin(\omega t - 120^\circ)\\
&= 141.4 \sin(\omega t - 120^\circ),\\
e_{ca} &= E_m \sin(\omega t - 240^\circ) = 100\sqrt{2}\sin(\omega t - 240^\circ)\\
&= 141.4 \sin(\omega t - 240^\circ) \text{ volts.}\\
\text{(b)}\quad \mathbf{E}_{ab} &= E\underline{/0^\circ} = 100\underline{/0^\circ} = 100 + j0,\\
\mathbf{E}_{bc} &= E\underline{/-120^\circ} = 100\underline{/-120^\circ} = -50 - j86.6,\\
\mathbf{E}_{ca} &= E\underline{/-240^\circ} = 100\underline{/-240^\circ} = -50 + j86.6 \text{ complex volts.}
\end{aligned}$$

14.06. Star-connected Polyphase EMF's

If, when individual phase windings of an n-phase generator are joined at one end to a common junction point, the emf's between the open ends of successive phases are of equal effective value, the generator is said to be *star-connected.* Figure 14.06A shows the

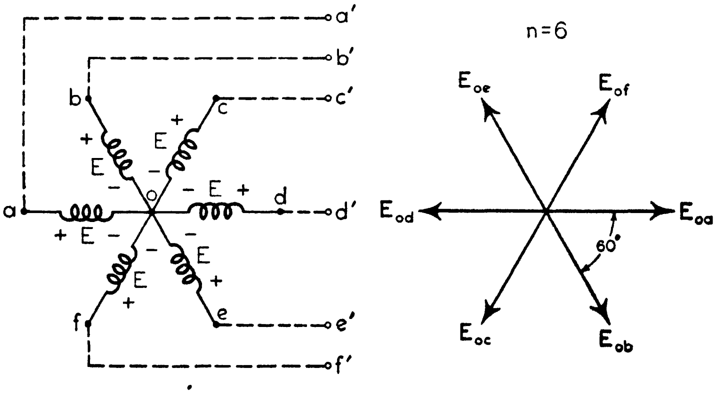


Fig. 14.06A. A six-phase set of star-connected emf's.

wiring diagram and the associated vector diagram for a *six*-phase star-connected generator.

As Fig. 14.06A clearly shows, the line potential and the phase potential need not be equal in the case of a balanced symmetrical star-connected generator. We can explore the relationship between line and phase potentials for this type of connection by reference to Fig. 14.06B. From this figure it is clearly seen that the magnitudes of the line and phase potentials of a balanced symmetrical star connection are related as follows:

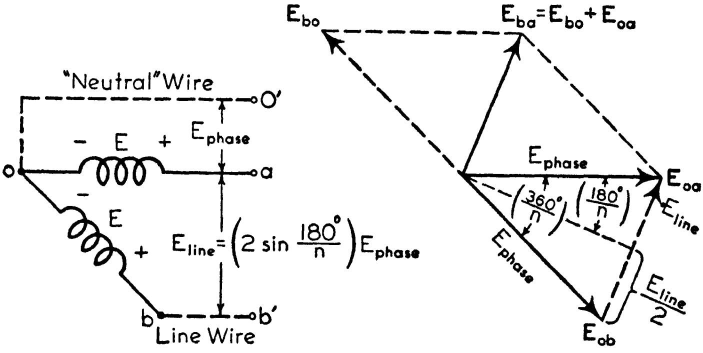


Fig. 14.06B. Determination of relation between line and phase emf's for a star connection.

▶ $$E_{\text{line (star)}} = \left(2 \sin \frac{180^\circ}{n}\right) E_{\text{phase (star)}}. \tag{14-2}$$

A *three-phase star connection* is known as a *wye connection*. This type of connection, too, is frequently encountered in practice. Figure 14.06C presents the diagrams applying to such a connection. From this figure it is apparent that

$$\mathbf{E}_{ab} = \mathbf{E}_{ao} + \mathbf{E}_{ob} = -\mathbf{E}_{oa} + \mathbf{E}_{ob}, \tag{14-3a}$$

$$\mathbf{E}_{bc} = \mathbf{E}_{bo} + \mathbf{E}_{oc} = -\mathbf{E}_{ob} + \mathbf{E}_{oc}, \tag{14-3b}$$

$$\mathbf{E}_{ca} = \mathbf{E}_{co} + \mathbf{E}_{oa} = -\mathbf{E}_{oc} + \mathbf{E}_{oa}. \tag{14-3c}$$

Both this figure and equation (14-2) make it clear that, for a balanced symmetrical wye-connected source,

▶ $$E_{\text{line (wye)}} = \sqrt{3}E_{\text{phase (wye)}}. \tag{14-4}$$

Part (*c*) of the figure shows a string vector diagram based on equations (14-3). This type of diagram will prove especially helpful in subsequent analyses of unbalanced three-wire wye-connected circuits. We note that the delta configuration consists of the line potentials, whereas the enclosed wye configuration consists of the phase potentials. The reader is reminded that angles of these vector diagrams are measured in electrical degrees, not mechanical degrees, and that these diagrams must never be confused with circuit diagrams no matter how similar they may be in appearance.

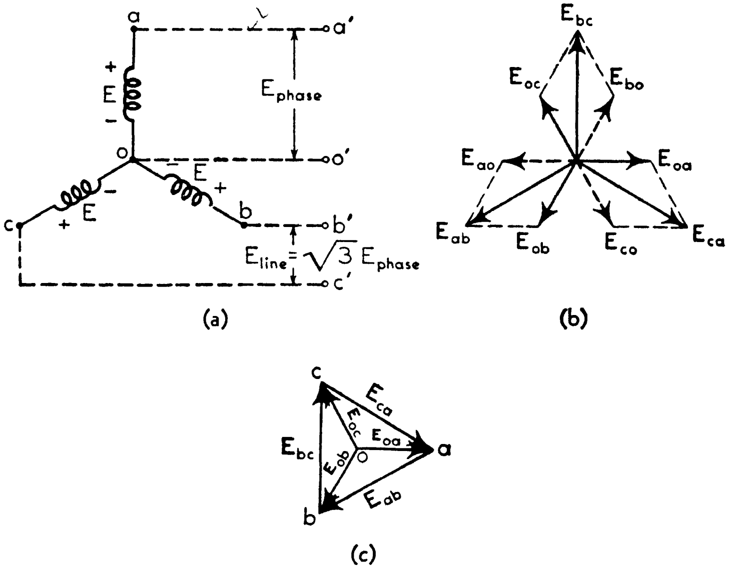


Fig. 14.06C. Line and phase emf's for a three-phase star (wye) connection. (a) Circuit diagram. (b) Polar vector diagram. (c) String vector diagram.

The double subscripts of the emf's of Fig. 14.06C may actually be applied to the ends of the vectors, as shown in part (c), if one is careful to heed the warning issued in the preceding paragraph. This arrangement is sometimes helpful in picturing the physical situation. In this connection, it is often convenient to picture the "center of gravity" of the triangle of line potentials as being at zero electric potential. This "center of gravity" is at the intersection of the bisectors of angles of the triangle. Vectors connecting this center of gravity, or *electric neutral,* with the vertices represent graphically the potentials of the terminals of the circuit relative to the zero of reference. If the system is balanced and symmetrical, as is the case shown here, then the point *o*, whose position represents the relative potential of the common junction point of the wye, falls exactly on top of the center of gravity. Consequently, the common junction (or *neutral*) of a balanced symmetrical system of emf's is also the true electric neutral of the system. The phase potential vectors for such a system represent the absolute potentials of the terminals. In the case shown, if the magnitudes of the vectors were originally expressed in terms of effective values, then the diagram shows in geometric form the relations between the readings to be expected of voltmeters equipped with touch leads which are applied across all possible pairs of terminals of the generator, including the neutral. Similar diagrams may be constructed for star-connected polyphase systems of any number of phases.

Example 14.06. The phase potentials of Fig. 14.06C may be expressed as follows:

$$\begin{aligned} e_{oa} &= 50 \sin (\omega t + 0^\circ), \\ e_{ob} &= 50 \sin (\omega t - 120^\circ), \\ e_{oc} &= 50 \sin (\omega t - 240^\circ) \text{ volts.} \end{aligned}$$

Determine analytically similar expressions for the line potentials e_{ab}, e_{bc}, and e_{ca}.

The phase potential vectors are:

$$\begin{aligned} \mathbf{E}_{m_{oa}} &= 50\underline{/0^\circ} = 50 + j0, \\ \mathbf{E}_{m_{ob}} &= 50\underline{/-120^\circ} = -25 - j43.3, \\ \mathbf{E}_{m_{oc}} &= 50\underline{/-240^\circ} = -25 + j43.3 \text{ complex volts.} \end{aligned}$$

From equation (14-3),

$$\begin{aligned} \mathbf{E}_{m_{ab}} &= -75 - j43.3 = 86.6\underline{/-150^\circ}, \\ \mathbf{E}_{m_{bc}} &= 0 + j86.6 = 86.6\underline{/90^\circ}, \\ \mathbf{E}_{m_{ca}} &= +75 - j43.3 = 86.6\underline{/-30^\circ} \text{ complex volts.} \end{aligned}$$

Then the desired line potential expressions are:

$$\begin{aligned} e_{ab} &= 86.6 \sin (\omega t - 150^\circ), \\ e_{bc} &= 86.6 \sin (\omega t + 90^\circ), \\ e_{ca} &= 86.6 \sin (\omega t - 30^\circ) \text{ volts.} \end{aligned}$$

14.07. Two-phase and Four-phase Systems of EMF's

Figure 14.07A shows the circuit and vector diagram for a two-phase system of emf's. The particular case shown is that of a *three-*

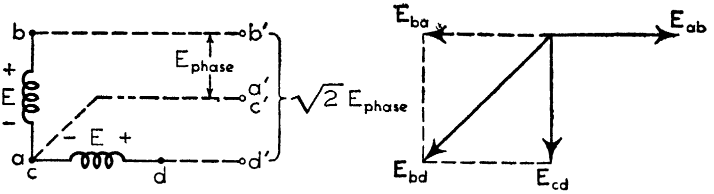


Fig. 14.07A. Emf's for a two-phase connection.

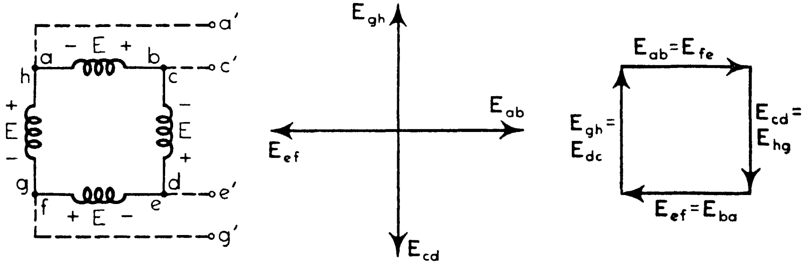


Fig. 14.07B. Line and phase emf's for a four-phase mesh connection.

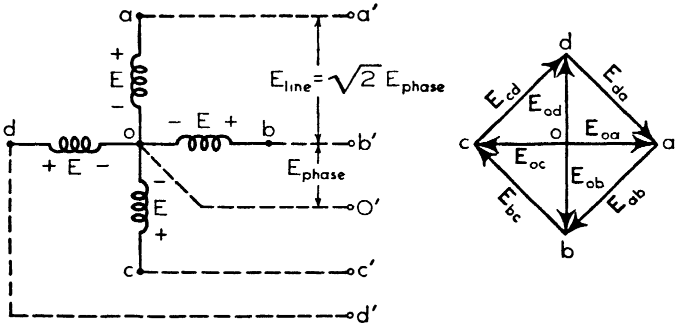


Fig. 14.07C. Line and phase emf's for a four-phase star connection.

wire two-phase system. If winding terminals *a* and *o* were not connected but were provided with their own line wires, the system would be called a *four-wire two-phase system.* Since there would then be no interconnection between the two windings, the vector $\mathbf{E}_{bd}$ of the vector diagram would no longer exist.

Figure 14.07B shows the circuit and vector diagrams of a mesh-connected four-phase system of emf's, and Fig. 14.07C shows the circuit and vector diagrams of a star-connected four-phase system of emf's. If no neutral wire is employed with the star connection, the system is said to be a *four-wire four-phase system,* whereas if there is a neutral wire present, the system is a five-wire four-phase system. It will be noted that with the *balanced* star-connected four-phase system,

▶ $$E_{\text{line}(4\text{-phase star})} = \sqrt{2}\,E_{\text{phase}(4\text{-phase star})}. \qquad (14\text{-}5)$$

14.08. General Approach to the Solution of Polyphase Circuit Problems

Despite the special interrelations existing among the component phases, a polyphase circuit is in the first analysis a network. As a consequence, all the methods of solution applicable to networks in general are applicable to the solution of polyphase circuits. The loop method and the node method are very useful in solving polyphase circuit problems, and one or the other may prove to be the best method of solving certain problems. The delta-wye transformation is frequently useful; and, with unbalanced star-connected loads, Millman's theorem is often advantageous. Under certain conditions, the superposition theorem proves to be a powerful tool, especially in combination with other techniques.

In general, load impedances can be connected in any manner whatsoever to a polyphase emf source. In practice, load impedances are frequently connected in mesh or in star configurations. Circuits of the latter type sometimes are provided with a neutral wire, sometimes not. Because of certain desirable properties that accrue, attempts are made to keep the complex impedances of all phases as nearly equal as possible. A polyphase circuit in which the complex impedances of all phases are exactly equal is said to be *balanced.* In balanced circuits, special relations which effect a real economy of time and effort are available.

The remainder of this chapter is devoted to a study of the properties of polyphase circuits and to the methods of solving them. In general, in writing equations for solving polyphase circuit problems, we shall adhere to the convention of assuming as positive *all line currents flowing toward the load, all clockwise loop currents, all potential drops,* and *all approaching node currents.*

14.09. Three-wire Wye-connected Three-phase Circuits

Figure 14.09A shows a three-wire wye-connected three-phase circuit. As a network, this type of circuit may be solved readily by means of either the loop or node methods, while Millman's theorem presents an especially interesting alternative approach. In the general case, which must allow for either a balanced load or an unbalanced load, the common junction point, or neutral, n, is not necessarily at zero potential relative to the line terminals of the load. This means that, in general, the phase potentials at the load are not necessarily equal in magnitude, even though the line potentials at the load may be both symmetrical and balanced. We shall show several alternative methods of solution for the general case, as well as special methods of handling the balanced case.

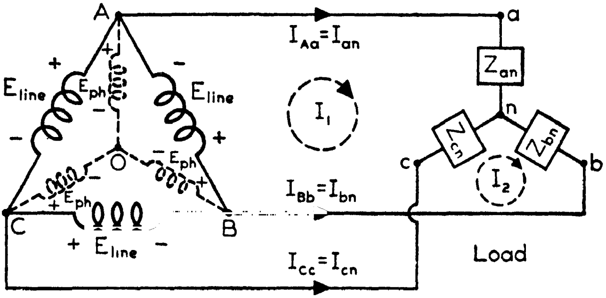


Fig. 14.09A. A three-wire wye-connected three-phase circuit.

1. *Loop Method.* In Fig. 14.09A, we note that the manner of connection of the supply generator is immaterial, so long as we know the line potentials. Knowing these, we may write two simple loop equations which can be solved simultaneously to obtain a solution. The following loop equations apply to this circuit, assuming generator-winding and line impedances to be negligible.

$$\mathbf{I}_1(\mathbf{Z}_{an} + \mathbf{Z}_{cn}) - \mathbf{I}_2\mathbf{Z}_{cn} = \mathbf{E}_{CA}, \tag{14-6}$$

and

$$-\mathbf{I}_1\mathbf{Z}_{cn} + \mathbf{I}_2(\mathbf{Z}_{bn} + \mathbf{Z}_{cn}) = \mathbf{E}_{BC}. \tag{14-7}$$

Simultaneous solution of these equations yields the following results:

$$\mathbf{I}_1 = \frac{\mathbf{E}_{CA}(\mathbf{Z}_{bn} + \mathbf{Z}_{cn}) + \mathbf{E}_{BC}\mathbf{Z}_{cn}}{\mathbf{Z}_{an}\mathbf{Z}_{bn} + \mathbf{Z}_{bn}\mathbf{Z}_{cn} + \mathbf{Z}_{cn}\mathbf{Z}_{an}}, \tag{14-8}$$

and

$$\mathbf{I}_2 = \frac{\mathbf{E}_{CA}\mathbf{Z}_{cn} + \mathbf{E}_{BC}(\mathbf{Z}_{an} + \mathbf{Z}_{cn})}{\mathbf{Z}_{an}\mathbf{Z}_{bn} + \mathbf{Z}_{bn}\mathbf{Z}_{cn} + \mathbf{Z}_{cn}\mathbf{Z}_{an}}. \tag{14-9}$$

The line (and phase) currents are, then,

$$I_{Aa} = I_{an} = I_1 = \frac{E_{BC}Z_{cn} + E_{CA}(Z_{bn} + Z_{cn})}{Z_{an}Z_{bn} + Z_{bn}Z_{cn} + Z_{cn}Z_{an}}, \tag{14-10}$$

$$I_{Bb} = I_{bn} = -I_2 = -\frac{E_{BC}(Z_{an} + Z_{cn}) + E_{CA}Z_{cn}}{Z_{an}Z_{bn} + Z_{bn}Z_{cn} + Z_{cn}Z_{an}}, \tag{14-11}$$

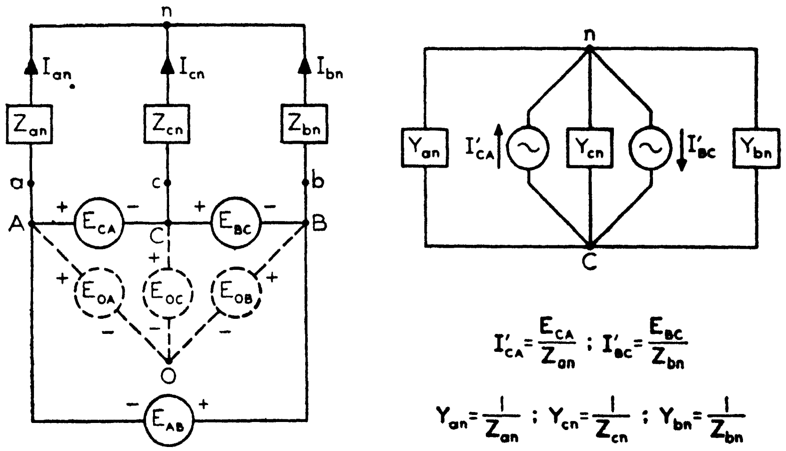
$$I_{Cc} = I_{cn} = I_2 - I_1 = \frac{E_{BC}Z_{an} - E_{CA}Z_{bn}}{Z_{an}Z_{bn} + Z_{bn}Z_{cn} + Z_{cn}Z_{an}}. \tag{14-12}$$

The phase potentials at the load are

$$V_{an} = I_{an}Z_{an} = \frac{E_{BC}Z_{cn} + E_{CA}(Z_{bn} + Z_{cn})}{Z_{an}Z_{bn} + Z_{bn}Z_{cn} + Z_{cn}Z_{an}} Z_{an}, \tag{14-13}$$

$$V_{bn} = I_{bn}Z_{bn} = -\frac{E_{BC}(Z_{an} + Z_{cn}) + E_{CA}Z_{cn}}{Z_{an}Z_{bn} + Z_{bn}Z_{cn} + Z_{cn}Z_{an}} Z_{bn}, \tag{14-14}$$

$$V_{cn} = I_{cn}Z_{cn} = \frac{E_{BC}Z_{an} - E_{CA}Z_{bn}}{Z_{an}Z_{bn} + Z_{bn}Z_{cn} + Z_{cn}Z_{an}} Z_{cn}. \tag{14-15}$$


n
I_an
I_cn
I_bn
Z_an
Z_cn
Z_bn
a
c
b
A
C
B
E_CA
E_BC
E_OA
E_OC
E_OB
O
E_AB
+
−
n
Y_an
I'_CA
Y_cn
I'_BC
Y_bn
C
I'_CA = E_CA / Z_an ; I'_BC = E_BC / Z_bn
Y_an = 1 / Z_an ; Y_cn = 1 / Z_cn ; Y_bn = 1 / Z_bn

(a) (b)

Fig. 14.09B. Circuit of Fig. 14.09A re-drawn for solution by node method. (a) Circuit re-drawn to show potential sources. (b) Equivalent circuit employing current sources.

2. *Node Method.* Part (a) of Fig. 14.09B shows the circuit of Fig. 14.09A redrawn for convenience, and part (b) of the figure shows an equivalent circuit in which the potential sources have been replaced by equivalent current sources. Let us refer to this equivalent circuit in studying this application of the node method to the solution of this problem.

Using terminal C as reference, we may write the following node equation to express the node potential $\mathbf{V}'_n$ (the amount by which the absolute potential of point n is *less* than that of point C):

$$\mathbf{V}'_n(\mathbf{Y}_{an} + \mathbf{Y}_{bn} + \mathbf{Y}_{cn}) + \mathbf{I}'_{CA} + \mathbf{I}'_{CB} = 0, \tag{14-16}$$

or

$$\mathbf{V}'_n = \frac{-\mathbf{I}'_{CA} - \mathbf{I}'_{CB}}{\mathbf{Y}_{an} + \mathbf{Y}_{bn} + \mathbf{Y}_{cn}} = \frac{-\mathbf{E}_{CA}\mathbf{Y}_{an} + \mathbf{E}_{BC}\mathbf{Y}_{bn}}{\mathbf{Y}_{an} + \mathbf{Y}_{bn} + \mathbf{Y}_{cn}} \tag{14-17}$$

The line (and phase) currents are, then,

$$\mathbf{I}_{Aa} = \mathbf{I}_{an} = \mathbf{V}_{an}\mathbf{Y}_{an} = (\mathbf{V}'_n - \mathbf{V}'_a)\mathbf{Y}_{an} = (\mathbf{V}'_n - \mathbf{E}_{CA})\mathbf{Y}_{an}, \tag{14-18}$$

$$\begin{aligned}\mathbf{I}_{Bb} = \mathbf{I}_{bn} = \mathbf{V}_{bn}\mathbf{Y}_{bn} = (\mathbf{V}'_n - \mathbf{V}'_b)\mathbf{Y}_{bn} &= (\mathbf{V}'_n - \mathbf{E}_{CB})\mathbf{Y}_{bn} \\ &= (\mathbf{V}'_n + \mathbf{E}_{BC})\mathbf{Y}_{bn},\end{aligned} \tag{14-19}$$

$$\mathbf{I}_{Cc} = \mathbf{I}_{cn} = \mathbf{V}_{cn}\mathbf{Y}_{cn} = (\mathbf{V}'_n - \mathbf{V}'_c)\mathbf{Y}_{cn} = (\mathbf{V}'_n - 0)\mathbf{Y}_{cn} = \mathbf{V}'_n\mathbf{Y}_{cn}. \tag{14-20}$$

We may demonstrate that the values obtained by the node method will be the same as those obtained by the loop method, by showing, for example, that (14-20) is equivalent to (14-12). Equation (14-20) may be rewritten as follows:

$$\begin{aligned}\mathbf{I}_{Cc} = \mathbf{V}'_n\mathbf{Y}_{cn} &= \frac{-\mathbf{E}_{CA}\mathbf{Y}_{an} + \mathbf{E}_{BC}\mathbf{Y}_{bn}}{\mathbf{Y}_{an} + \mathbf{Y}_{bn} + \mathbf{Y}_{cn}}\mathbf{Y}_{cn} \\ &= \frac{-\mathbf{E}_{CA}/\mathbf{Z}_{an} + \mathbf{E}_{BC}/\mathbf{Z}_{bn}}{\dfrac{1}{\mathbf{Z}_{an}} + \dfrac{1}{\mathbf{Z}_{bn}} + \dfrac{1}{\mathbf{Z}_{cn}}} \cdot \frac{1}{\mathbf{Z}_{cn}} \\ &= \frac{\mathbf{E}_{BC}\mathbf{Z}_{an} - \mathbf{E}_{CA}\mathbf{Z}_{bn}}{\mathbf{Z}_{an}\mathbf{Z}_{bn} + \mathbf{Z}_{bn}\mathbf{Z}_{cn} + \mathbf{Z}_{cn}\mathbf{Z}_{an}}.\end{aligned} \tag{14-21}$$

This is identical with (14-12).

3. *Millman's Theorem.* Millman's theorem provides a very interesting and useful method for solving unbalanced wye-connected, three-wire, three-phase circuits. Part (a) of Fig. 14.09B shows the circuit of Fig. 14.09A in convenient form for applying Millman's theorem. In applying this theorem we may use as reference terminal any one of the generator terminals, A, B, or C, or the generator neutral O (if the generator is wye-connected).

(a) First, let us utilize the terminal C as reference. By Millman's theorem, we may then write the following expression for the potential drop from C to n:

$$\mathbf{V}_{Cn} = \frac{\mathbf{V}_{CA}\mathbf{Y}_{an} + \mathbf{V}_{CB}\mathbf{Y}_{bn} + \mathbf{V}_{Cc}\mathbf{Y}_{cn}}{\mathbf{Y}_{an} + \mathbf{Y}_{bn} + \mathbf{Y}_{cn}} = \frac{-\mathbf{E}_{CA}\mathbf{Y}_{an} + \mathbf{E}_{BC}\mathbf{Y}_{bn} + 0}{\mathbf{Y}_{an} + \mathbf{Y}_{bn} + \mathbf{Y}_{cn}}. \tag{14-22}$$

The line (and phase) currents are, then,

$$\mathbf{I}_{Aa} = \mathbf{I}_{an} = \mathbf{V}_{an}\mathbf{Y}_{an} = (\mathbf{V}_{Cn} - \mathbf{V}_{CA})\mathbf{Y}_{an} = (\mathbf{V}_{Cn} + \mathbf{E}_{CA})\mathbf{Y}_{an}, \tag{14-23}$$

$$\mathbf{I}_{Bb} = \mathbf{I}_{bn} = \mathbf{V}_{bn}\mathbf{Y}_{bn} = (\mathbf{V}_{Cn} - \mathbf{V}_{CB})\mathbf{Y}_{bn} = (\mathbf{V}_{Cn} - \mathbf{E}_{BC})\mathbf{Y}_{bn}, \tag{14-24}$$

$$\mathbf{I}_{Cc} = \mathbf{I}_{cn} = \mathbf{V}_{cn}\mathbf{Y}_{cn} = (\mathbf{V}_{Cn} - \mathbf{V}_{Cc})\mathbf{Y}_{cn} = \mathbf{V}_{Cn}\mathbf{Y}_{cn}. \tag{14-25}$$

We may easily demonstrate that the values obtained by application of Millman's theorem will be the same as those obtained by the other methods by showing, for example, that (14-25) is equivalent to (14-20) and therefore to (14-12). This can be done as follows. Substituting (14-22) in (14-25) and making use of (14-20), we have

$$\mathbf{I}_{Cc} = \left(\frac{-\mathbf{E}_{CA}\mathbf{Y}_{an} + \mathbf{E}_{BC}\mathbf{Y}_{bn}}{\mathbf{Y}_{an} + \mathbf{Y}_{bn} + \mathbf{Y}_{cn}}\right)\mathbf{Y}_{cn} = \mathbf{V}'_{n}\mathbf{Y}_{cn}. \tag{14-26}$$

This is identical with (14-20) which has already been shown to equal (14-12).

(b) It is sometimes convenient to utilize the generator neutral as reference in applying Millman's theorem. If the generator is delta-connected with the result that there is no neutral, then through the delta-wye transformation an equivalent wye-connected generator can be substituted for the actual generator. If the line potentials are symmetrical and balanced, this may readily be done by application of equation (14-4), with reference to Fig. 14.06C. The generator neutral, O, will be the true electrical neutral of the system under these conditions, and we may utilize Millman's theorem to establish for the circuit a vector diagram of the type shown in part (c) of Fig. 14.06C. Figure 14.09C shows such a diagram for a particular case of very bad unbalance. The neutral of the load is seen to be separated in potential from the generator neutral by a potential drop V_{On}. When the potential of point n strays from that of point O, as it does in this figure, the point n is said to be a *roving neutral.* The potential drop V_{On} may easily be determined through the use of Millman's theorem. Employing the generator neutral as reference, we may write

$$\mathbf{V}_{On} = \frac{\mathbf{V}_{Oa}\mathbf{Y}_{an} + \mathbf{V}_{Ob}\mathbf{Y}_{bn} + \mathbf{V}_{Oc}\mathbf{Y}_{cn}}{\mathbf{Y}_{an} + \mathbf{Y}_{bn} + \mathbf{Y}_{cn}} = -\frac{\mathbf{E}_{Oa}\mathbf{Y}_{an} + \mathbf{E}_{Ob}\mathbf{Y}_{bn} + \mathbf{E}_{Oc}\mathbf{Y}_{cn}}{\mathbf{Y}_{an} + \mathbf{Y}_{bn} + \mathbf{Y}_{cn}}. \tag{14-27}$$

The line (and phase) currents are, then,

$$\mathbf{I}_{an} = \mathbf{V}_{an}\mathbf{Y}_{an} = (\mathbf{V}_{On} - \mathbf{V}_{Oa})\mathbf{Y}_{an} = (\mathbf{V}_{On} + \mathbf{E}_{OA})\mathbf{Y}_{an}, \tag{14-28}$$

$$\mathbf{I}_{bn} = \mathbf{V}_{bn}\mathbf{Y}_{bn} = (\mathbf{V}_{On} - \mathbf{V}_{Ob})\mathbf{Y}_{bn} = (\mathbf{V}_{On} + \mathbf{E}_{OB})\mathbf{Y}_{bn}, \tag{14-29}$$

$$\mathbf{I}_{cn} = \mathbf{V}_{cn}\mathbf{Y}_{cn} = (\mathbf{V}_{On} - \mathbf{V}_{Oc})\mathbf{Y}_{cn} = (\mathbf{V}_{On} + \mathbf{E}_{OC})\mathbf{Y}_{cn}. \tag{14-30}$$

The phase potentials of the load are, of course, the quantities in the parentheses of the last three equations.

4. *Balanced Circuit Relations.* A study of Fig. 14.09C leads to the conclusion that if the load is balanced, the point n must coincide with the point O in potential, inasmuch as the load phase potentials, currents, and impedances all must be balanced. The following statements will then be true of the potential vectors of the figure:

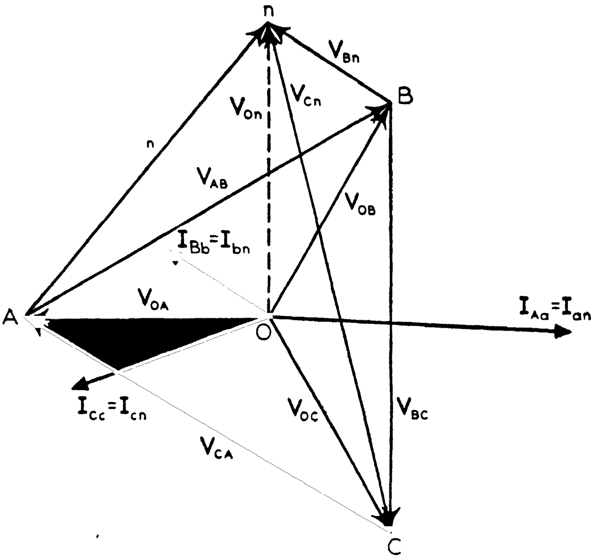


Fig. 14.09C. String diagram for a badly unbalanced three-wire wye-connected circuit with symmetrical three-phase emf's applied.

$$\mathbf{V}_{an} = \mathbf{V}_{AO} = \mathbf{E}_{OA}, \tag{14-31}$$
$$\mathbf{V}_{bn} = \mathbf{V}_{BO} = \mathbf{E}_{OB}, \tag{14-32}$$
$$\mathbf{V}_{cn} = \mathbf{V}_{CO} = \mathbf{E}_{OC}. \tag{14-33}$$

In view of equation (14-4), we conclude that, in a *balanced* three-wire wye-connected three-phase circuit,

▶ $$V_{\text{line(wye)}} = \sqrt{3}\, V_{\text{phase(wye)}}. \tag{14-34}$$

Figure 14.09A makes it clear that we may also write

▶ $$I_{\text{line(wye)}} = I_{\text{phase(wye)}}. \tag{14-35}$$

Upon converting Fig. 14.09C into a polar vector diagram for the balanced case, we also see that the phase sequence is the same for both the phase and line potentials, and for the currents. We also note that there are 30° angles between each line potential and its component phase potentials. Knowledge of these facts helps us to sketch vector diagrams quickly.

The foregoing conclusions lead us to treat the *balanced* three-wire wye-connected circuit exactly as we do the balanced four-wire wye-connected circuit in Article 14.10. This consists of applying the a-c version of Ohm's law to any given phase, exactly as though it were a separate single-phase circuit with the return path for the current through a (nonexistent) neutral wire of zero impedance. The solutions of the other two phases then have the same magnitudes but are displaced in phase position by 120°.

Example 14.09. The following quantities apply to the circuit of Fig. 14.09A:

$$\begin{aligned}
\mathbf{E}_{OA} &= 100 + j0 = 100\underline{/0^\circ},\\
\mathbf{E}_{OB} &= -50 - j86.6 = 100\underline{/-120^\circ},\\
\mathbf{E}_{OC} &= -50 + j86.6 = 100\underline{/120^\circ} \text{ complex volts},\\
\mathbf{Z}_{an} &= 3 + j4 = 5\underline{/53.1^\circ},\\
\mathbf{Z}_{bn} &= 5 + j0 = 5\underline{/0^\circ},\\
\mathbf{Z}_{cn} &= 0 - j10 = 10\underline{/-90^\circ} \text{ complex ohms.}
\end{aligned}$$

Find the phase potentials and the line currents, and construct the string vector diagram of potential drops.

We shall employ Millman's theorem, with point O used as reference.

$$\begin{aligned}
\mathbf{V}_{On} &= \frac{-(100\underline{/0^\circ})(\frac{1}{5}\underline{/-53.1^\circ}) - (100\underline{/-120^\circ})(\frac{1}{5}\underline{/-0^\circ}) - (100\underline{/120^\circ})(\frac{1}{10}\underline{/90^\circ})}{(\frac{1}{5}\underline{/-53.1^\circ}) + (\frac{1}{5}\underline{/-0^\circ}) + (\frac{1}{10}\underline{/90^\circ})}\\
&= \frac{-(12 - j16) - (-10 - j17.32) - (-8.66 - j5)}{0.12 - j0.16 + 0.2 + j0 + 0 + j0.1}\\
&= -1.56 + j120 = 120\underline{/90.75^\circ} \text{ complex volts.}
\end{aligned}$$

The phase potentials are

$$\begin{aligned}
\mathbf{V}_{an} &= (-1.56 + j120) + (100 + j0) = 98.4 + j120 = 154.9\underline{/50.67^\circ},\\
\mathbf{V}_{bn} &= (-1.56 + j120) + (-50 - j86.6) = -51.6 + j33.4 = 61.5\underline{/147.1^\circ},\\
\mathbf{V}_{cn} &= (-1.56 + j120) + (-50 + j86.6) = -51.6 + j206.6\\
&= 212.5\underline{/104^\circ} \text{ complex volts.}
\end{aligned}$$

The line currents are

$$\mathbf{I}_{Aa} = \mathbf{I}_{an} = (154.9\underline{/50.67^\circ})(\tfrac{1}{5}\underline{/-53.1^\circ}) = 31.0\underline{/-2.4^\circ} = 31.0 - j1.31,$$
$$\mathbf{I}_{Bb} = \mathbf{I}_{bn} = (61.5\underline{/147.1^\circ})(\tfrac{1}{5}\underline{/0^\circ}) = 12.3\underline{/147.1^\circ} = -103 + j6.67,$$
$$\mathbf{I}_{Cc} = \mathbf{I}_{cn} = (212.5\underline{/104^\circ})(\tfrac{1}{10}\underline{/90^\circ}) = 21.3\underline{/194^\circ}$$
$$= -20.7 - j5.16 \text{ complex amperes.}$$

Figure 14.09C is the string vector diagram applying to this problem.

Three important facts which are true of three-wire wye-connected three-phase circuits in general can be noted from this solution. They are:

1. The vector sum of the currents approaching the load (in this case, the sum of the line currents) is zero. This statement is, of course, an outgrowth of Kirchhoff's current law as applied to the neutral of the load.
2. The vector sum of the individual phase potentials of a three-wire wye-connected circuit is not necessarily zero.
3. The vector sum of the line potentials is zero. This statement stems directly from Kirchhoff's potential law.

14.10. Four-wire Wye-connected Three-phase Circuits

Figure 14.10A shows a wye-connected load supplied by a wye-connected three-phase generator. Inasmuch as the neutral of the generator is joined to the neutral of the load by a fourth conductor, this circuit is known as a *four-wire three-phase circuit.* In the general case in which the loads may be unbalanced and in which there may be appreciable impedance in the line and neutral wires, one of the general methods of network solution, such as the loop method or the node method, may be the most satisfactory approach to solution. However, in view of the common junction points at *O* and at *n*, Millman's theorem is readily applicable and may prove to be the preferable approach.

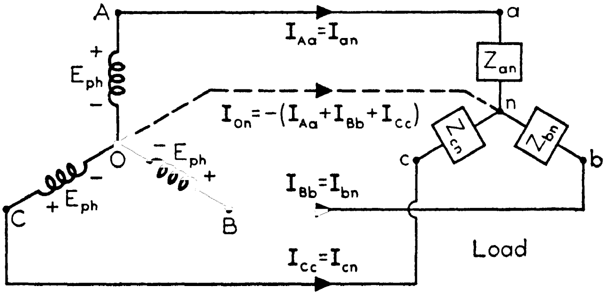


Fig. 14.10A. A four-wire wye-connected three-phase circuit.

By Kirchhoff's current law, it is generally true in a four-wire circuit such as that of Fig. 14.10A that

$$\mathbf{I}_{Aa} + \mathbf{I}_{Bb} + \mathbf{I}_{Cc} + \mathbf{I}_{On} = 0, \tag{14-36}$$

or that

$$\mathbf{I}_{On} = -(\mathbf{I}_{Aa} + \mathbf{I}_{Bb} + \mathbf{I}_{Cc}). \tag{14-37}$$

Several special cases of four-wire three-phase circuits which merit particular mention follow.

1. *Negligible Neutral Impedance.* If the impedance of the neutral wire is negligible, there can be no potential difference between O and n, and solution is considerably facilitated. Each phase may be treated as a separate single-phase circuit. In this case, the line impedances, if not negligible, may be combined in series with their respective load impedances, and the solution may be handled as for the case of negligible line and neutral-wire impedance.

2. *Negligible Line and Neutral Impedances.* If the line, as well as the neutral, impedances are negligible, the phase potentials at the load must have the same values as the corresponding supply phase potentials. Ideally, this condition would make the load phase potentials balanced and symmetrical. If this is the case, an unbalance of load impedances cannot result in an unbalance of load phase potentials but may result in an extreme unbalance of line (and phase) currents. Figure 14.10B represents a particular case of unbalanced load in a four-wire three-phase circuit.

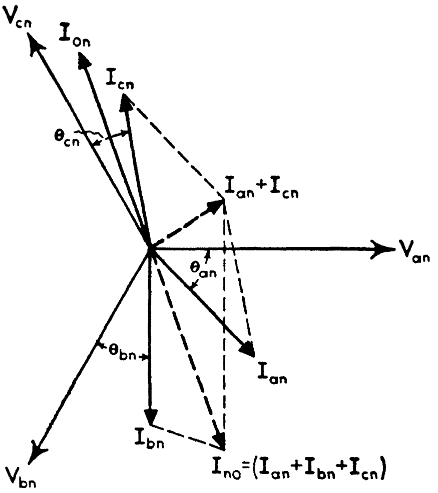


Fig. 14.10B. Polar diagram for a badly unbalanced four-wire wye-connected circuit with symmetrical three-phase emf's applied.

3. *Balanced Load and Line Impedances.* In the special case of a *balanced* four-wire wye-connected circuit in which both the load and the line impedances are balanced, both the potentials and the currents are balanced at the load. As equation (14-37) shows,

there is, then, no current flowing in the neutral wire, and the value of the neutral-wire impedance is immaterial. The neutral wire can even be removed without affecting the situation, so far as the present analysis is concerned. This case may be handled by applying the a-c version of Ohm's law to any given phase, exactly as though it were a separate single-phase circuit. The solutions for the other two phases then have the same magnitudes but are displaced in phase position by 120°.

14.11. Delta-connected Three-phase Circuits

Figure 14.11A shows a delta-connected three-phase load supplied by a system of three-phase emf's. If the impedances of the line wires are appreciable, the loop method often proves to be a highly satisfactory general method of solution. As an alternative method, one may utilize the delta-wye transformation to convert the delta-connected load into an equivalent wye-connected load which then may be solved by any of the methods discussed in the previous article. If this method is followed, the line impedances must be combined in series with the corresponding phase impedances of the equivalent wye before further solution is carried out.

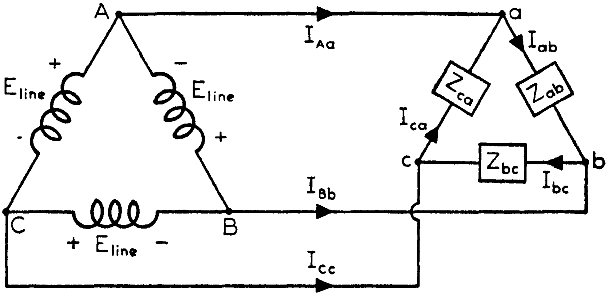


Fig. 14.11A. A delta-connected three-phase circuit.

If the impedances of the line wires are negligible, the emf's of the source appear directly across the individual phases of the load, and the a-c version of Ohm's law can be utilized to determine the individual phase currents. These, in turn, can be combined by Kirchhoff's current law to determine the line currents. Referring to the figure, we may write the following equations for the phase currents, if line impedances are negligible:

$$\mathbf{I}_{ab} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_{ab}} = \frac{-\mathbf{E}_{BA}}{\mathbf{Z}_{ab}} = \mathbf{E}_{AB}\mathbf{Y}_{ab}, \tag{14-38}$$

$$\mathbf{I}_{bc} = \frac{\mathbf{V}_{bc}}{\mathbf{Z}_{bc}} = \frac{-\mathbf{E}_{CB}}{\mathbf{Z}_{bc}} = \mathbf{E}_{BC}\mathbf{Y}_{bc}, \tag{14-39}$$

$$\mathbf{I}_{ca} = \frac{\mathbf{V}_{ca}}{\mathbf{Z}_{ca}} = \frac{-\mathbf{E}_{AC}}{\mathbf{Z}_{ca}} = \mathbf{E}_{CA}\mathbf{Y}_{ca}. \tag{14-40}$$

The line currents are, then,

$$\mathbf{I}_{Aa} = \mathbf{I}_{ab} + \mathbf{I}_{ac} = \mathbf{I}_{ab} - \mathbf{I}_{ca}, \tag{14-41}$$

$$\mathbf{I}_{Bb} = \mathbf{I}_{bc} + \mathbf{I}_{ba} = \mathbf{I}_{bc} - \mathbf{I}_{ab}, \tag{14-42}$$

$$\mathbf{I}_{Cc} = \mathbf{I}_{ca} + \mathbf{I}_{cb} = \mathbf{I}_{ca} - \mathbf{I}_{bc}. \tag{14-43}$$

Example 14.11 illustrates the solution of an unbalanced delta-connected circuit from this approach.

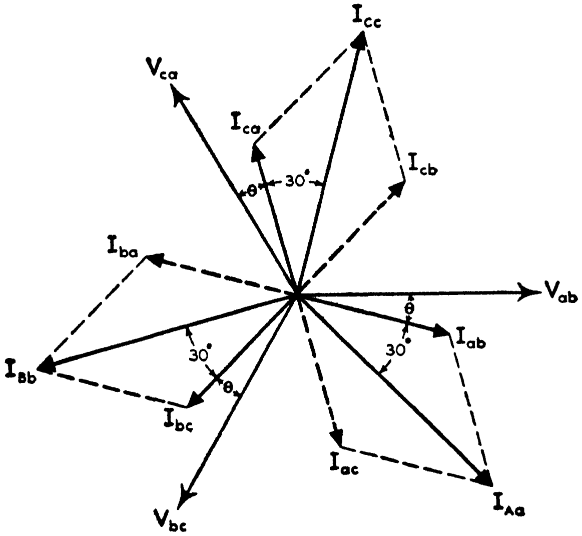


Fig. 14.11B. Polar diagram for a balanced delta-connected circuit.

The solution of this circuit by the node method is left as a student exercise.

In the special case of a *balanced* delta-connected load, very convenient labor-saving relations may be readily deduced. Referring to Fig. 14.11B and recognizing that corresponding quantities in the three individual phases are equal in magnitude, we come to the conclusion that, for a *balanced delta-connected three-phase circuit,*

▶ $$V_{\text{line(delta)}} = V_{\text{phase(delta)}}, \tag{14-44}$$

and

▶ $$I_{\text{line(delta)}} = \sqrt{3}\, I_{\text{phase(delta)}}. \tag{14-45}$$

It is also clear from the figure that the phase sequence of the line currents is the same as that of the phase currents, and that there are 30° phase angles between each line current and its component phase currents.

Example 14.11. The following values apply to the circuit of Fig. 14.11A:

$$\mathbf{E}_{ab} = 100\underline{/0^\circ}, \qquad \mathbf{E}_{bc} = 100\underline{/-120^\circ}, \qquad \mathbf{E}_{ca} = 100\underline{/120^\circ} \text{ complex volts,}$$

and

$$\mathbf{Z}_{ab} = 10\underline{/0^\circ}, \qquad \mathbf{Z}_{bc} = 20\underline{/30^\circ}, \qquad \mathbf{Z}_{ca} = 5\underline{/-60^\circ} \text{ complex ohms.}$$

Using $\mathbf{E}_{ab}$ as reference, (a) find the values of all phase and line currents, and (b) draw to scale a vector diagram showing all these potential and current quantities and indicating the power-factor angles of the individual phases.

(a) We may write

$$\begin{aligned}
\mathbf{V}_{ab} &= \mathbf{E}_{ab} = 100\underline{/0^\circ} = 100 + j0 \text{ complex volts,}\\
\mathbf{V}_{bc} &= \mathbf{E}_{bc} = 100\underline{/-120^\circ} = -50 - j86.6 \text{ complex volts,}\\
\mathbf{V}_{ca} &= \mathbf{E}_{ca} = 100\underline{/120^\circ} = -50 + j86.6 \text{ complex volts.}
\end{aligned}$$

Then

$$\mathbf{I}_{ab} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_{ab}} = \frac{100\underline{/0^\circ}}{10\underline{/0^\circ}} = 10\underline{/0^\circ} = 10 + j0 \text{ complex amperes,}$$

$$\mathbf{I}_{bc} = \frac{\mathbf{V}_{bc}}{\mathbf{Z}_{bc}} = \frac{100\underline{/-120^\circ}}{20\underline{/30^\circ}} = 5\underline{/-150^\circ} = -4.33 - j2.50 \text{ complex amperes,}$$

$$\mathbf{I}_{ca} = \frac{\mathbf{V}_{ca}}{\mathbf{Z}_{ca}} = \frac{100\underline{/120^\circ}}{5\underline{/-60^\circ}} = 20\underline{/180^\circ} = -20 + j0 \text{ complex amperes.}$$

Applying Kirchhoff's current law, we obtain

$$\begin{aligned}
\mathbf{I}_{Aa} &= (\mathbf{I}_{ab} - \mathbf{I}_{ca}) = 10 + j0 + 20 - j0 = 30 + j0\\
&= 30\underline{/0^\circ} \text{ complex amperes,}\\
\mathbf{I}_{Bb} &= (\mathbf{I}_{bc} - \mathbf{I}_{ab}) = -4.33 - j2.50 - 10 - j0 = -14.33 - j2.50\\
&= 14\ 51\underline{/-170.1^\circ} \text{ complex amperes,}\\
\mathbf{I}_{Cc} &= (\mathbf{I}_{ca} - \mathbf{I}_{bc}) = -20 + j0 + 4.33 + j2.50 = -15.67 + j2.50\\
&= 15.89/170.9^\circ \text{ complex amperes.}
\end{aligned}$$

(b) The vector quantities found in (a) may be plotted to scale to yield the vector diagram of Fig. 14.11C.

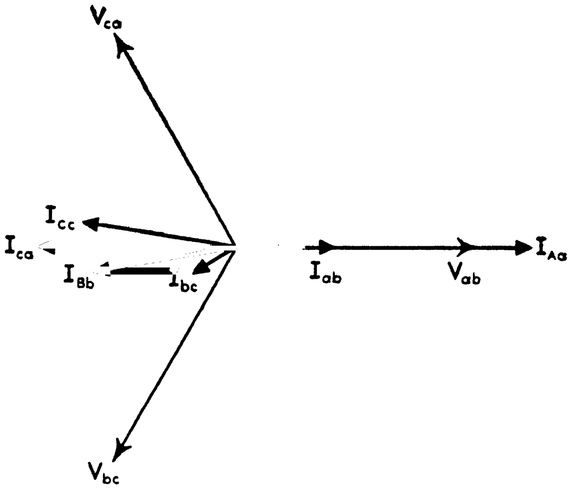


Fig. 14.11C. Vector diagram for Example 14.11.

Several important facts which are true of delta-connected three-phase circuits in general can be noted from this solution. They are:

1. The vector sum of the currents approaching the load (the sum of the line currents) is zero. This follows directly from the fact that it must be true of an equivalent wye-connected load.
2. The vector sum of the individual phase currents of a delta-connected load is not necessarily zero.
3. The vector sum of the line potentials is zero.

14.12. Combined Delta- and Wye-connected Loads

Delta-connected loads are sometimes operated in conjunction with wye-connected loads, as indicated by Fig. 14.12. Either the loop or the node method of solution could be applied to this circuit. However, in view of the large number of simultaneous equations which would result with either of these methods, another approach might prove more desirable.

One alternative procedure is to utilize the wye-delta transformation to convert the enclosed wye into a delta in parallel with the enclosing delta. Corresponding legs of the two deltas may then be combined in parallel to provide a single equivalent delta. The resulting circuit may then be solved by any of the methods outlined in Article 14.11.

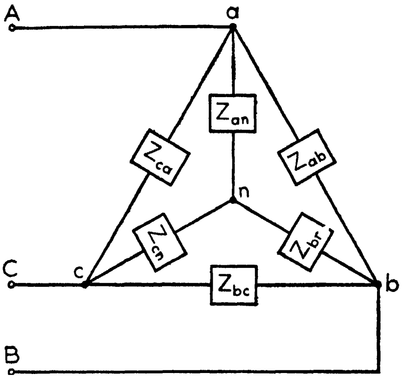


Fig. 14.12. Combined delta- and wye-connected loads.

Another alternative is to ignore the wye and solve the delta by itself, then to ignore the delta and solve the wye by itself, and finally to combine the two sets of results under the superposition principle.

If the impedances of the line wires are negligible, Millman's theorem may conveniently be applied to solve the wye, Ohm's law may conveniently be applied—a phase at a time—to the delta, and the results combined under the superposition principle.

14.13. Instantaneous Polyphase Power

The instantaneous power of a polyphase circuit is the sum of the instantaneous powers of its individual phases. Mathematically,

$$p_{\text{polyphase}} = \Sigma p_{\text{phase}} = \Sigma v_{\text{phase}} i_{\text{phase}}. \tag{14-46}$$

One of the bases for considering polyphase—particularly three-phase (often abbreviated "3ϕ")—power superior to single-phase power can readily be demonstrated by reference to this equation. Inasmuch as three-phase power is much more prevalent in practice than polyphase power of any other number of phases, this discussion will be limited to three-phase power.

Let us assume the following phase potentials and currents associated with a three-phase circuit. It does not matter whether these phases are connected in delta or in wye.

$$v_A = V_{mA} \sin (\omega t + \theta_{vA}), \quad \text{and} \quad i_A = I_{mA} \sin (\omega t + \theta_{iA}); \tag{14-47}$$

$$v_B = V_{mB} \sin (\omega t + \theta_{vB}), \quad \text{and} \quad i_B = I_{mB} \sin (\omega t + \theta_{iB}); \tag{14-48}$$

$$v_C = V_{mC} \sin (\omega t + \theta_{vC}), \quad \text{and} \quad i_C = I_{mC} \sin (\omega t + \theta_{iC}). \tag{14-49}$$

We may now write the following expressions for the individual instantaneous phase powers:

$$p_A = V_{m_A} I_{m_A} \sin (\omega t + \theta_{v_A}) \sin (\omega t + \theta_{i_A}), \tag{14-50a}$$

$$p_B = V_{m_B} I_{m_B} \sin (\omega t + \theta_{v_B}) \sin (\omega t + \theta_{i_B}), \tag{14-51a}$$

$$p_C = V_{m_C} I_{m_C} \sin (\omega t + \theta_{v_C}) \sin (\omega t + \theta_{i_C}). \tag{14-52a}$$

These may be expanded as follows by equation (6-45):

$$\begin{aligned} p_A = \left[\frac{V_{m_A} I_{m_A}}{2} \cos (\theta_{v_A} - \theta_{i_A})\right] &- \left[\frac{V_{m_A} I_{m_A}}{2} \cos (\theta_{v_A} + \theta_{i_A})\right] \cos 2\omega t \\ &+ \left[\frac{V_{m_A} I_{m_A}}{2} \sin (\theta_{v_A} + \theta_{i_A})\right] \sin 2\omega t, \end{aligned} \tag{14-50b}$$

$$\begin{aligned} p_B = \left[\frac{V_{m_B} I_{m_B}}{2} \cos (\theta_{v_B} - \theta_{i_B})\right] &- \left[\frac{V_{m_B} I_{m_B}}{2} \cos (\theta_{v_B} + \theta_{i_B})\right] \cos 2\omega t \\ &+ \left[\frac{V_{m_B} I_{m_B}}{2} \sin (\theta_{v_B} + \theta_{i_B})\right] \sin 2\omega t, \end{aligned} \tag{14-51b}$$

$$\begin{aligned} p_C = \left[\frac{V_{m_C} I_{m_C}}{2} \cos (\theta_{v_C} - \theta_{i_C})\right] &- \left[\frac{V_{m_C} I_{m_C}}{2} \cos (\theta_{v_C} + \theta_{i_C})\right] \cos 2\omega t \\ &+ \left[\frac{V_{m_C} I_{m_C}}{2} \sin (\theta_{v_C} + \theta_{i_C})\right] \sin 2\omega t. \end{aligned} \tag{14-52b}$$

The summation of these three equations is the instantaneous three-phase power:

$$\begin{aligned} p_{3\phi} &= p_A + p_B + p_C \\ &= \left\{\left[\frac{V_{m_A} I_{m_A}}{2} \cos (\theta_{v_A} - \theta_{i_A}) + \frac{V_{m_B} I_{m_B}}{2} \cos (\theta_{v_B} - \theta_{i_B}) \right.\right. \\ &\qquad\qquad \left. + \frac{V_{m_C} I_{m_C}}{2} \cos (\theta_{v_C} - \theta_{i_C})\right] \\ &\qquad - \left[\frac{V_{m_A} I_{m_A}}{2} \cos (\theta_{v_A} + \theta_{i_A}) + \frac{V_{m_B} I_{m_B}}{2} \cos (\theta_{v_B} + \theta_{i_B}) \right. \\ &\qquad\qquad \left.\left. + \frac{V_{m_C} I_{m_C}}{2} \cos (\theta_{v_C} + \theta_{i_C})\right] \cos 2\omega t\right\} \\ &\qquad + \left\{\left[\frac{V_{m_A} I_{m_A}}{2} \sin (\theta_{v_A} + \theta_{i_A}) + \frac{V_{m_B} I_{m_B}}{2} \sin (\theta_{v_B} + \theta_{i_B}) \right.\right. \\ &\qquad\qquad \left.\left. + \frac{V_{m_C} I_{m_C}}{2} \sin (\theta_{v_C} + \theta_{i_C})\right] \sin 2\omega t\right\}. \end{aligned} \tag{14-53}$$

The *instantaneous active three-phase power* is the entire quantity enclosed by the first pair of braces. In terms of effective values, this is

$$\begin{aligned}p_{3\phi(\text{active})} = \{[&V_AI_A \cos(\theta_{iA} - \theta_{iA}) + V_BI_B \cos(\theta_{vB} - \theta_{iB}) \\ &\qquad + V_CI_C \cos(\theta_{vC} - \theta_{iC})] \\ &- [V_AI_A \cos(\theta_{vA} + \theta_{iA}) + V_BI_B \cos(\theta_{vB} + \theta_{iB}) \\ &+ V_CI_C \cos(\theta_{vC} + \theta_{iC})] \cos 2\omega t\}. \qquad (14\text{-}54)\end{aligned}$$

The *instantaneous reactive three-phase power* is the entire quantity enclosed by the second pair of braces. In terms of effective values, this is

$$\begin{aligned}p_{3\phi(\text{reactive})} = \{[&V_AI_A \sin(\theta_{vA} + \theta_{iA}) + V_BI_B \sin(\theta_{vB} + \theta_{iB}) \\ &+ V_CI_C \sin(\theta_{vC} + \theta_{iC})] \sin 2\omega t\}. \qquad (14\text{-}55)\end{aligned}$$

To be of unique advantage, a three-phase circuit must be as nearly balanced and symmetrical as possible. Assuming perfection in these respects, we may write that

$$\left.\begin{aligned}V_A &= V_B = V_C = V, \\ I_A &= I_B = I_C = I, \\ (\theta_{vA} - \theta_{iA}) &= (\theta_{vB} - \theta_{iB}) = (\theta_{vC} - \theta_{iC}) = \theta, \\ \theta_{vB} &= (\theta_{vA} - 120^\circ) \quad \text{and} \quad \theta_{iB} = (\theta_{iA} - 120^\circ), \\ \theta_{vC} &= (\theta_{vA} + 120^\circ) \quad \text{and} \quad \theta_{iC} = (\theta_{iA} + 120^\circ).\end{aligned}\right\} \qquad (14\text{-}56)$$

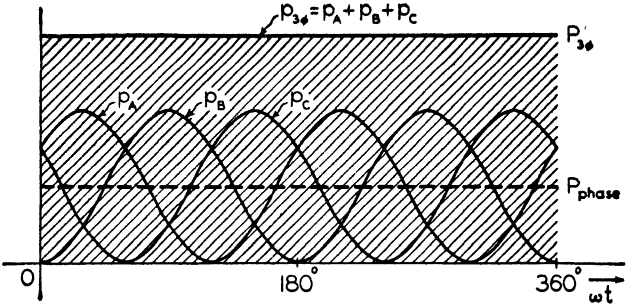
Equation (14-54) becomes, upon substitution from (14-56),

$$\begin{aligned}p_{3\phi(\text{active})} = \{&[3VI \cos\theta] \\ &- VI[\cos(\theta_{vA} + \theta_{iA}) + \cos(\theta_{vA} + \theta_{iA} - 240^\circ) \\ &+ \cos(\theta_{vA} + \theta_{iA} + 240^\circ)] \cos 2\omega t\}. \qquad (14\text{-}57)\end{aligned}$$

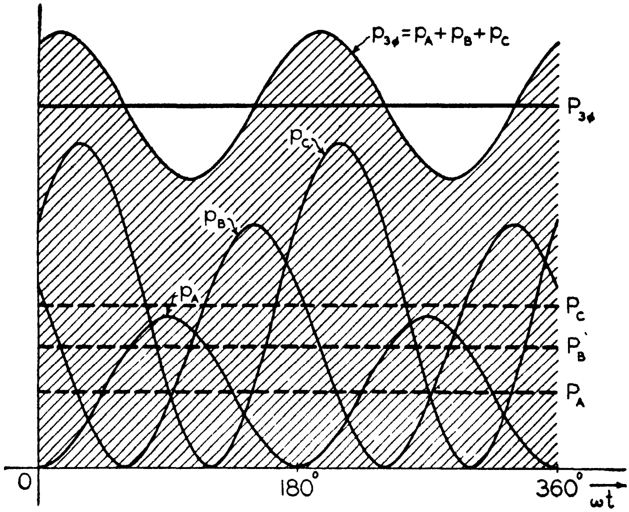
The second bracketed term can be shown, by simple trigonometric substitution, to equal zero. The *instantaneous* **active** *three-phase power for a balanced circuit* is, then,

▶ $$p_{3\phi(\text{active})} = 3VI \cos\theta, \qquad (14\text{-}58)$$

where V, I, and θ are *phase values.* This equation is highly important in that it shows that the *net instantaneous active three-phase power of a balanced circuit is not a function of time.* Although the individual phases receive their active power in double-frequency pulses, the net instantaneous active power flow to the three-phase load is uniform, provided that the circuit is balanced.



(a) Balanced circuit.



(b) Unbalanced circuit.

Fig. 14.13. Three-phase power in balanced and unbalanced circuits.

Upon substitution from (14-56), equation (14-55) becomes

$$p_{3\phi(\text{reactive})} = \{VI\,[\sin(\theta_{v_A} + \theta_{i_A}) + \sin(\theta_{v_A} + \theta_{i_A} - 240°) + \sin(\theta_{v_A} + \theta_{i_A} + 240°)]\sin 2\omega t\}. \quad (14\text{-}59)$$

Simple trigonometric substitution shows the bracketed term to equal zero. The *instantaneous* **reactive** *three-phase power of a balanced circuit* is, then,

▶ $$p_{3\phi(\text{reactive})} = 0. \quad (14\text{-}60)$$

This equation is highly important in that it shows that the *net instantaneous reactive three-phase power of a balanced circuit is zero.* Although the individual phases receive and return reactive power in double-frequency pulses, the net instantaneous reactive power flowing to the total three-phase load is zero at every instant, provided that the load is balanced.

Adding equations (14-58) and (14-60) leads to the following expression for *net instantaneous three-phase power for a balanced circuit:*

▶ $$p_{3\phi} = p_{3\phi(\text{active})} + p_{3\phi(\text{reactive})} = 3VI\cos\theta. \qquad (14\text{-}61)$$

It is apparent that net instantaneous three-phase power is not a function of time if the circuit is balanced.

The net average power taken by a balanced three-phase load is, obviously,

▶ $$P_{3\phi} = 3VI\cos\theta. \qquad (14\text{-}62)$$

This is, of course, the same value as the unvarying instantaneous three-phase power. Figure 14.13(a) shows the power waves for the individual phases as well as the net instantaneous three-phase power for the case of a balanced circuit.

If the three-phase circuit is not balanced, the net instantaneous three-phase power is not constant but contains a "ripple," as Fig. 14.13(b) shows for a particular case.

14.14. Vector Power Quantities in Polyphase Circuits

As in any other type of network, the total active power of a polyphase circuit is the sum of the active powers of the individual circuit elements, and the total reactive power of a polyphase circuit is the algebraic sum of the reactive powers of the individual circuit elements. This relationship leads to the graphical interpretation shown by Fig. 14.14A for the case of a three-phase circuit. The total vector power of a three-phase

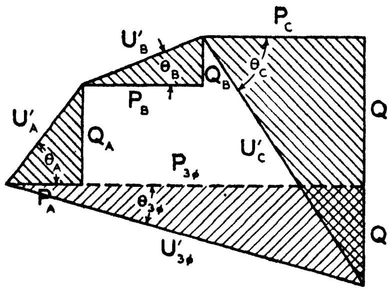


Fig. 14.14A. Power diagram for an unbalanced three-phase circuit.

circuit is, we see, the vector sum of the vector powers of the individual phases. Graphically, this summation consists of "adding end to end" the power triangles of the individual phases.

A study of Fig. 14.14A reveals, for the three-phase case, the validity of the following relations for an n-phase circuit:

▶
$$\begin{aligned} P_{n\phi} &= P_A + P_B + P_C + \cdots + P_n \\ &= I_A^2R_A + I_B^2R_B + I_C^2R_C + \cdots + I_n^2R_n \\ &= V_AI_A\cos\theta_A + V_BI_B\cos\theta_B + V_CI_C\cos\theta_C \\ &\qquad + \cdots + V_nI_n\cos\theta_n, \end{aligned} \tag{14-63}$$

▶
$$\begin{aligned} Q_{n\phi} &= Q_A + Q_B + Q_C + \cdots + Q_n \\ &= I_A^2X_A + I_B^2X_B + I_C^2X_C + \cdots + I_n^2X_n \\ &= V_AI_A\sin\theta_A + V_BI_B\sin\theta_B + V_CI_C\sin\theta_C \\ &\qquad + \cdots + V_nI_n\sin\theta_n, \end{aligned} \tag{14-64}$$

▶
$$\mathbf{U}'_{n\phi} = \mathbf{U}'_A + \mathbf{U}'_B + \mathbf{U}'_C + \cdots + \mathbf{U}'_n, \tag{14-65}$$

▶
$$(\text{P.F.})_{n\phi} = \cos\theta_{n\phi} = \frac{P_{n\phi}}{U'_{n\phi}}, \tag{14-66}$$

▶
$$(\text{R.F.})_{n\phi} = \sin\theta_{n\phi} = \frac{Q_{n\phi}}{U'_{n\phi}}. \tag{14-67}$$

For the special case of a *balanced* n-phase circuit, the power triangles of the individual phases are identical, and the following relations hold:

▶
$$P_{n\phi} = nP_{\text{phase}}, \tag{14-68}$$

▶
$$Q_{n\phi} = nQ_{\text{phase}}, \tag{14-69}$$

▶
$$U'_{n\phi} = nU'_{\text{phase}}, \tag{14-70}$$

▶
$$\theta_{n\phi} = \theta_{\text{phase}}. \tag{14-71}$$

The graphical interpretation of these relations for the case of a balanced three-phase circuit is shown by Fig. 14.14B. All of the foregoing three-phase power relations are applicable to either star- or mesh-connected polyphase circuits.

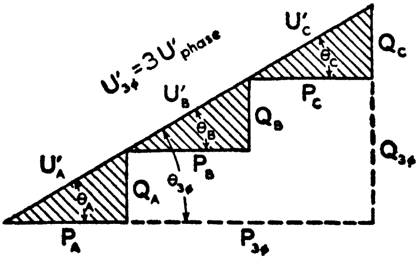


Fig. 14.14B. Power diagram for a balanced three-phase circuit.

Example 14.14A. (a) Find the vector power of a three-phase circuit whose individual phases are loaded as follows: phase A, 10 kva at 0.8 lagging power factor; phase B, 5 kva at unity power factor; phase C, 10 kva at 0.2 leading power factor. (b) What is the three-phase power factor? (c) What is the three-phase reactive factor?

The individual complex vector powers are, calling inductive reactive power positive,

$$\mathbf{U}'_A = 10\underline{/\cos^{-1} 0.8} = 10\underline{/36.9^\circ} = 8 + j6 \text{ complex kilovolt-amperes,}$$
$$\mathbf{U}'_B = 5\underline{/\cos^{-1} 1.0} = 5\underline{/0^\circ} = 5 + j0 \text{ complex kilovolt-amperes,}$$
$$\mathbf{U}'_C = 10\underline{/-\cos^{-1} 0.2} = 10\underline{/-78.7^\circ} = 2 - j9.8 \text{ complex kilovolt-amperes.}$$

(a) The complex vector power of the three-phase circuit is

$$\mathbf{U}'_{3\phi} = \mathbf{U}'_A + \mathbf{U}'_B + \mathbf{U}'_C = 15 - j3.8 = 15.5\underline{/-14.2^\circ} \text{ complex kilovolt-amperes.}$$

(b) The three-phase power factor is

$$(\text{P.F.})_{3\phi} = \frac{P_{3\phi}}{U'_{3\phi}} = \frac{15}{15.5} = 0.968 \text{ leading.}$$

(c) The three-phase reactive factor is

$$(\text{R.F.})_{3\phi} = \frac{Q_{3\phi}}{U'_{3\phi}} = \frac{-3.8}{15.5} = -0.245 \text{ capacitive.}$$

EXAMPLE 14.14B. A balanced three-phase load of 300 kw at 0.9 lagging power factor is supplied by a symmetrical balanced set of emf's at 2300 volts, line to line. (a) Assuming a delta connection, what is the line current? (b) Assuming a wye connection, what is the line current?

With a balanced circuit:

$$(\text{P.F.})_{3\phi} = (\text{P.F.})_{\text{phase}} = \frac{P_{\text{phase}}}{U'_{\text{phase}}} = \frac{P_{\text{phase}}}{(VI)_{\text{phase}}} = \frac{P_{3\phi}/3}{V_{\text{phase}}I_{\text{phase}}}.$$

$$I_{\text{phase}} = \frac{P_{3\phi}}{3V_{\text{phase}}(\text{P.F.})_{3\phi}} = \frac{300{,}000}{3V_{\text{phase}}0.9} = \frac{100{,}000}{0.9V_{\text{phase}}}.$$

(a) *With a delta connection:* $V_{\text{line}} = V_{\text{phase}}$; $I_{\text{line}} = \sqrt{3}\, I_{\text{l}}$

$$I_{\text{line}} = \sqrt{3}\, I_{\text{phase}} = \frac{100{,}000 \times \sqrt{3}}{0.9 \times 2300} = 83.5 \text{ amperes.}$$

(b) *With a wye connection:* $V_{\text{line}}/\sqrt{3} = V_{\text{phase}}$; $I_{\text{line}} = I_{\text{phase}}$.

$$I_{\text{line}} = I_{\text{phase}} = \frac{100{,}000}{0.9 \times 2300/\sqrt{3}} = 83.5 \text{ amperes.}$$

14.15. Measurement of Polyphase Power Quantities

Because electric energy is a primary commodity of the electrical engineer, the measurement of energy and of power (the rate of flow of energy) is one of the most important aspects of power system operation. The *watt-hour meter*, an instrument which integrates power over time, is the energy-measuring instrument. The *wattmeter* is the power-measuring instrument. Inasmuch as the watt-hour meter and the wattmeter are connected into a circuit similarly, we need study only the principles underlying the proper connection of the wattmeter to understand the connection of the watt-hour meter. The study of the design and operating characteristics of these two instruments is a specialized topic which properly forms no part of a basic circuits text and hence will not be pursued at length here.

The measurement of single-phase power by means of a wattmeter is relatively simple, aside from a consideration of effects introduced into the circuit by the instrument itself. The wattmeter, in both principle and operation, is a combination voltmeter and ammeter. It indicates the average of the product of instantaneous potential and current. The wattmeter mechanism consists essentially of a stationary coil through which passes a fixed proportion of the *current* to the load, and a pivoted coil through which passes a current proportional to the *potential* across which it is applied. The current through each of these coils establishes a magnetic field of proportionate strength. The interaction of these two fields tends to produce, in opposition to the torque of a calibrated spring, a rotational deflection of the pivoted coil which is proportional to the product, vi. Rotational inertia of the moving element renders it impossible for the instrument to follow the double-frequency variations of an a-c instantaneous power wave. Consequently, *the instrument deflects to an average position, thereby indicating the average of the product of v and i.* From Article 14.13 we see that the wattmeter will indicate, for a circuit with sinusoidal potential and current waves,

▶ $$P = VI \cos(\theta_v - \theta_i), \tag{14-72}$$

where:

P = *wattmeter indication* = *average* (*active*) *power, in* **watts,**

V = *effective value of potential,* in **volts,**

I = *effective value of current,* in **amperes,**

$(\theta_v - \theta_i)$ = *angular phase difference of potential and current.*

Figure 14.15A shows the circuit diagram of a wattmeter properly connected to indicate the power of a single-phase circuit.

Sometimes it is desirable to be able to measure the *reactive power* taken by a load. Such measurement is accomplished by means of a *varmeter*. The varmeter is essentially a wattmeter with the proper combination of electric circuit parameters inserted in series with its potential coil to shift the phase of the current through that coil by an angle of 90° without affecting the current magnitude. One of the simplest means is, of course, to substitute a capacitor of suitable size for the series resistor of the potential coil circuit. With proper design, a varmeter connected to a single-phase circuit in the same manner as the wattmeter of Fig. 14.15A will indicate, for a circuit with sinusoidal current and potential waves,

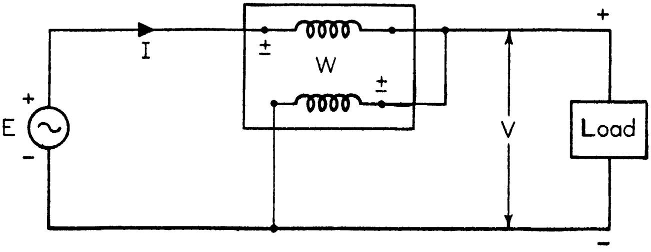


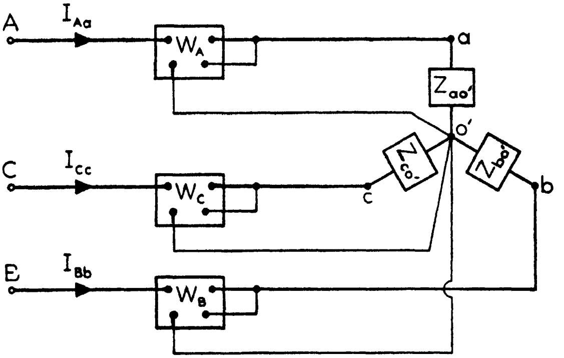
Fig. 14.15A. Measurement of single-phase power.

▶ $$Q = VI \sin(\theta_v - \theta_i), \qquad (14\text{-}73)$$

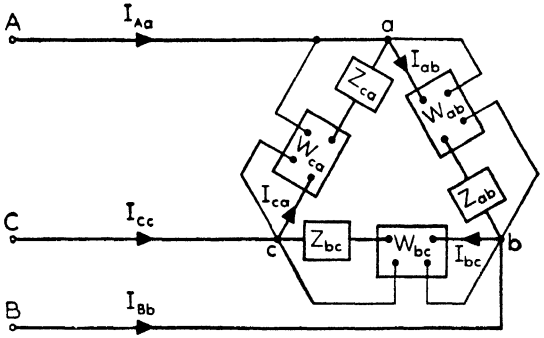
where: Q = *varmeter indication* = *reactive power*, in **vars**,

As a study of the figure indicates, each of the two coils of a wattmeter (and a varmeter) has two terminals, one of which is marked (±). The standard connection for such an instrument is to connect the current coil into one of the wires of the circuit in such a way that the "positive" direction of current flow is into the (±) terminal of the current coil, through the current coil, and then into the load. The (±) terminal of the potential coil then usually is connected to that side of the supply line which contains the current coil, and the other terminal of the potential coil is then connected to the other side of the supply line. If the angle $(\theta_v - \theta_i)$ is less than 90° in absolute value, the indication of the wattmeter is up scale, the power flow is to the load, and the power is said to be *positive*. If $(\theta_v - \theta_i)$ is between 90° and 180° in absolute value, then the wattmeter deflection tends to be down scale. To obtain a reading, one must reverse either the potential connections or the current connections to the instrument. Although now up scale, the resulting indication of the instrument must be classified as *negative.*

Article 14.14 pointed out that the total power delivered to a polyphase load can be found as the sum of the powers taken by the



(a) Wye-connected load.



(b) Delta-connected load.

Fig. 14.15B. Measurement of power in three-wire three-phase circuits by the N-wattmeter method.

individual phase loads. This procedure suggests placing *n* wattmeters in an *n*-phase circuit in such a way that the power of each phase is read on a wattmeter. If the standard connection scheme just outlined is employed for each wattmeter in conjunction with the standard three-phase indication of positive potential and current directions, then all the wattmeter indications will be positive for passive loads. Figure 14.15B shows such connections for a three-phase star-connected load and for a three-phase delta-connected load.

Simple as is the idea of the n-wattmeter method of measuring n-phase power, it is sometimes difficult to carry out in practice. Sometimes the common junction point of a star is inaccessible; sometimes the phase connections of a mesh cannot be broken open for the insertion of wattmeter current coils; it often even happens that only the n line terminals are available and it is not known what type of load connections there are beyond them. For these and other reasons, it is highly desirable to be able to measure total polyphase power in terms of line potentials and currents.

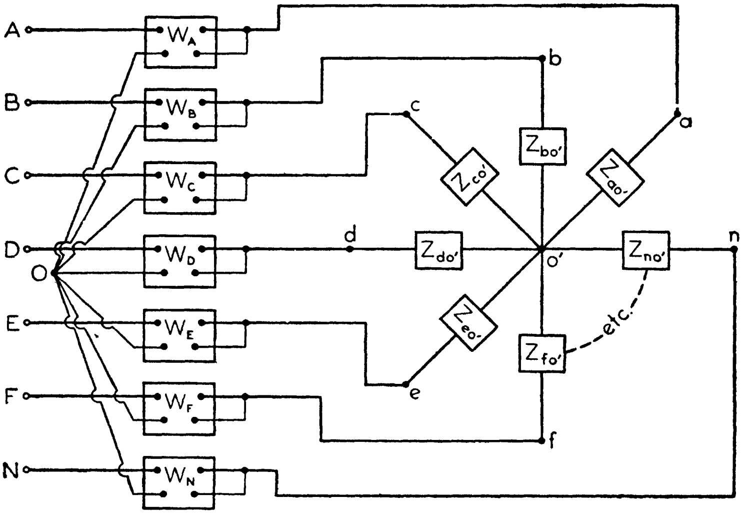


Fig. 14.15C. Derivation of (N-1)-wattmeter method of measuring power in an N-wire circuit.

In 1893, André Eugène Blondel published a theorem showing that if power is delivered to a load over n wires, the total power can be metered on $(n - 1)$ wattmeters with their current coils connected in the $(n - 1)$ wires, and their potential coils connected from the $(n - 1)$ wires to the nth wire. The total power then is the algebraic sum of the $(n - 1)$ individual wattmeter indications. There is no restriction as to wave form or frequency (including zero) of the potentials and currents, and the system need not be balanced or even "polyphase." The theorem can be proved in any one of several ways. One of the simplest follows.

Referring to Fig. 14.15C, we can write the following expression for the total average power delivered to the star-connected load shown:

$$P_T = \frac{1}{T}\int_0^T (v_{ao'}i_{Aa} + v_{bo'}i_{Bb} + \cdots + v_{no'}i_{Nn})\,dt. \tag{14-74}$$

Meanwhile, the sum of the indications of the n wattmeters shown is

$$W_T = \frac{1}{T}\int_0^T (v_{aO}i_{Aa} + v_{bO}i_{Bb} + \cdots + v_{nO}i_{Nn})\,dt. \tag{14-75}$$

It is clear that

$$v_{aO} = v_{ao'} - v_{Oo'}, \tag{14-76a}$$

$$v_{bO} = v_{bo'} - v_{Oo'}, \tag{14-76b}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$v_{nO} = v_{no'} - v_{Oo'}. \tag{14-76n}$$

Equation (14-75) may therefore be written as

$$W_T = \frac{1}{T}\int_0^T (v_{ao'}i_{Aa} + v_{bo'}i_{Bb} + \cdots + v_{no'}i_{Nn})\,dt - \frac{1}{T}\int_0^T v_{Oo'}(i_{Aa} + i_{Bb} + \cdots + i_{Nn})\,dt. \tag{14-77}$$

Since, from Kirchhoff's current law,

$$i_{Aa} + i_{Bb} + \cdots + i_{Nn} = 0, \tag{14-78}$$

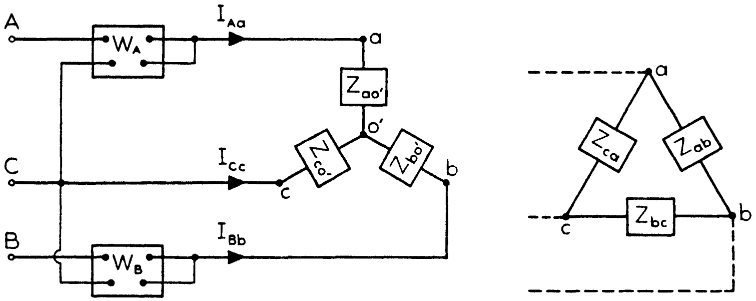
we conclude that

$$\blacktriangleright \quad W_T = P_T = \frac{1}{T}\int_0^T (v_{ao'}i_{Aa} + v_{bo'}i_{Bb} + \cdots + v_{no'}i_{Nn})\,dt. \tag{14-79}$$

Clearly, the sum of the n wattmeter indications is the total average power taken by the load. Inasmuch as no wave form, no relative magnitudes, and no relative phase angles are specified by equation (14-79), the relation is perfectly general. *It is also true no matter what the potential of point O relative to point o′.* If we make point O coincide with o', each wattmeter indicates the average power taken by a particular branch of the star. If the point O is placed on any one of the line wires, the power indication of the wattmeter in that line wire becomes equal to zero, rendering that wattmeter needless, and the total average power taken by the load is equal to the sum of the remaining $(n - 1)$ wattmeters. Inasmuch as any passive network of n terminals can be replaced by an equivalent star-connected network of n branches, this proof can be extended to cover any n-wire circuit, and we can state that $(n - 1)$ *wattmeters may be employed to read n-wire power.*

14.16. Measurement of Three-phase Power Quantities

Three-phase, three-wire power measurement may be accomplished by application of the three-wattmeter method, as indicated in Fig. 14.15B. The cases shown both involve the measurement of all individual phase powers. Each is an n-wire, n-wattmeter case. If a neutral wire is connected in part (a) of Fig. 14.15B, the situation is that of an n-wire, $(n - 1)$-wattmeter case. The neutral wire could just as well have contained one of the wattmeters, with one of the line wires serving as a common junction point, so far as the measurement of total average power is concerned. However, the individual wattmeters no longer would indicate the power taken by particular individual phases. The reading of one such wattmeter, in itself, then would have little or no significance.



(a) (b)

Fig. 14.16A. Two-wattmeter method of measuring three-wire three-phase power.

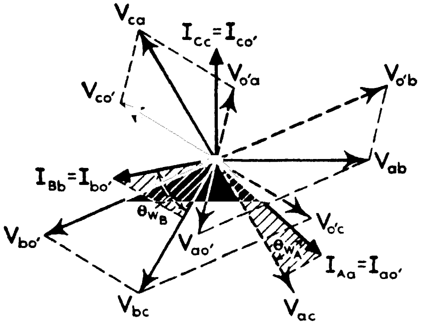
Figure 14.16A shows two three-wire three-phase circuits, one a wye- and one a delta-connected circuit, each of which is provided with two wattmeters properly connected to read the total average power by the two-wattmeter [$(n - 1)$-wattmeter] method. The choice of the line wire which will serve as the common junction of the potential elements is purely arbitrary.

A study of Fig. 14-16A reveals that the following general relations can be written for either the wye or the delta:

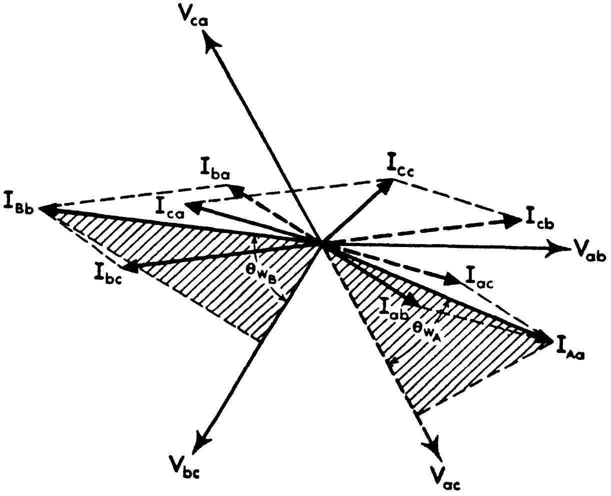
$$W_A = V_{ac} I_{Aa} \cos \theta \Big]_{V_{ac}}^{I_{Aa}} = V_{ac} I_{Aa} \cos \theta_{W_A}, \tag{14-80}$$

$$W_B = V_{bc} I_{Bb} \cos \theta \Big]_{V_{bc}}^{I_{Bb}} = V_{bc} I_{Bb} \cos \theta_{W_B}. \tag{14-81}$$

Vector diagrams applying to the two circuits are shown in Fig. 14.16B for two different sets of load impedances. It will be noted that in the case of the wye, phase potentials must be combined to



(a) Vector diagram for a particular wye-connected circuit [see Fig. 14.16A(a)].



(b) Vector diagram for a particular delta-connected circuit [see Fig. 14.16A(b)].

Fig. 14.16B. Vector diagrams for particular unbalanced circuits employing the two-wattmeter method of power measurement.

obtain line potentials, whereupon there is no particular difficulty in applying equations (14-80) and (14-81). Also, in the case of the delta, although phase currents must be combined to obtain line currents, there is no particular difficulty in applying these equations. In both of the examples, the particular values of load impedances in the particular cases shown are such that no wattmeter tends to read negatively because of a value of a wattmeter phase angle greater than 90° in absolute value. All such angles are less than 90° in absolute value in these examples, and therefore all wattmeter indications are positive. In other cases, one of the wattmeter indications might be negative. In each circuit, the algebraic sum of the two wattmeter indications is the total average power taken by the load impedances.

For the special case of a *balanced* three-wire three-phase load, either wye-connected or delta-connected, the following relations hold:

▶ $$P_{3\phi} = \sqrt{3}\, V_{\text{line}} I_{\text{line}} \cos \theta_{\text{phase}}, \tag{14-82}$$

and

▶ $$W_A = V_{\text{line}} I_{\text{line}} \cos (30° - \theta_{\text{phase}}), \tag{14-83a}$$

▶ $$W_B = V_{\text{line}} I_{\text{line}} \cos (30° + \theta_{\text{phase}}). \tag{14-83b}$$

Equation (14-82) is readily derived from the following relations which have already been derived for balanced circuits:

$$P_{3\phi} = 3P_{\text{phase}}, \tag{14-84}$$

$$V_{\text{line wye)}} = \sqrt{3}\, V_{\text{phase(wye)}}, \qquad \text{and} \qquad I_{\text{line(wye)}} = I_{\text{phase(wye)}}, \tag{14-85}$$

$$V_{\text{line(delta)}} = V_{\text{phase(delta)}}, \qquad \text{and} \qquad I_{\text{line(delta)}} = \sqrt{3}\, I_{\text{phase(delta)}}. \tag{14-86}$$

This derivation is left as a student exercise. Equations (14-83) may be derived for either a wye-connected or a delta-connected load by reference to a vector diagram of the type shown in Fig. 14.16C. The applicability of equations (14-83) to a delta-connected circuit should be apparent, for example, from a study of Fig. 14.16C. The identical equations result from a similar study of a balanced wye-connected load. It is instructive to note that equation (14-82) also can be obtained by adding equations (14-83).

If (14-83a) is divided by (14-83b), there results

$$\frac{W_A}{W_B} = \frac{V_{\text{line}} I_{\text{line}} \cos (30° - \theta)}{V_{\text{line}} I_{\text{line}} \cos (30° + \theta)} = \frac{\cos (30° - \theta)}{\cos (30° + \theta)}. \tag{14-87}$$

It should be clear that for each value of θ there is a different ratio of W_A to W_B. Consequently, as θ changes, the percentage of the total load read on each wattmeter changes. Recalling the fact that a wattmeter connected in the "standard" manner previously described deflects upscale as long as potential and current are within 90° of each other in phase, and that it deflects downscale if potential and current are out of phase by an angle falling between

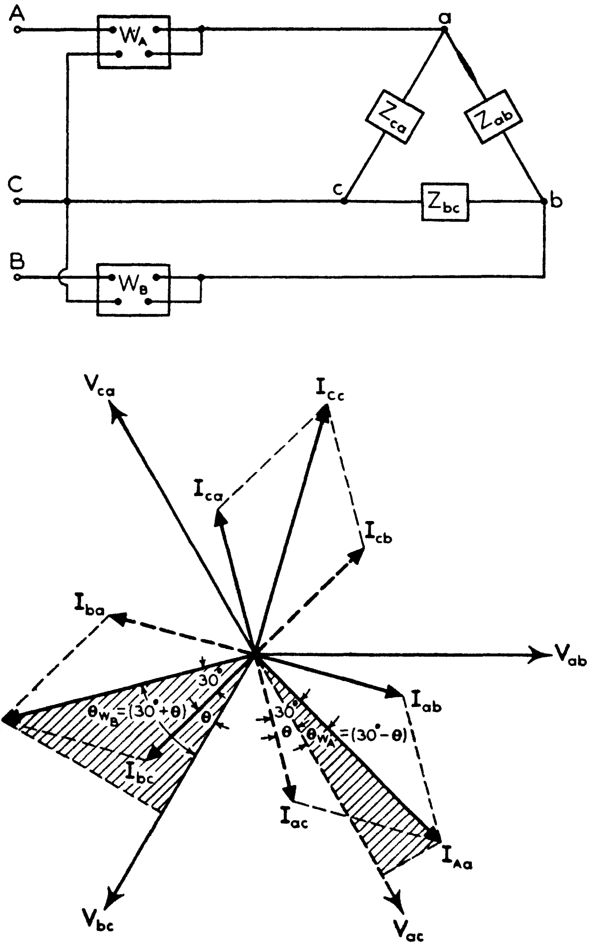


Fig. 14.16C. Two-wattmeter power measurement in a balanced delta-connected three-phase circuit.

90° and 180°, we can discern the validity of the following relations for the case under discussion:

(1) *When* $0° \leq \theta < 60°$, *both wattmeter indications are positive.*

(2) *When* $60° < \theta < 120°$, W_A *is positive and* W_B *is negative.*

(3) *When* $120° < \theta \leq 180°$, *both* W_A *and* W_B *are negative. If the angle* θ *is negative, the signs indicated above for* W_A *and* W_B *are interchanged. In all cases, the total active three-phase power is the algebraic sum of* W_A *and* W_B.

Often when the two-wattmeter method is used to measure three-phase power to an *unbalanced* load, it happens that at least one of the wattmeter indications is negative. Here, too, it is important to remember that total active three-phase power is the *algebraic* sum of W_A and W_B. We see, then, that it is important to be able to tell practically whether a given wattmeter indication is to be considered positive or negative.

There are several ways of determining practically whether a wattmeter reading should be considered positive or negative. One of these methods follows. Referring either to Fig. 14.16A or to Fig. 14.16C, open line wire *Aa*. All power must then be transferred to the load over line wires *Bb* and *Cc*. If wattmeter W_b is then connected so that it deflects upscale, we know that any upscale indications thereafter made by that wattmeter must be positive. Next reconnect line wire *Aa* and open line *Bb*; then connect wattmeter W_a so that it deflects upscale. Any upscale indications thereafter will be positive. Now close line *Bb*. If at any time either wattmeter tries to deflect negatively, power through the "channel" of that wattmeter must be considered negative. To obtain its value, either the potential or the current connections must be reversed.

EXAMPLE 14.16. The following unbalanced values of complex impedance, in ohms, apply to the delta-connected circuit of Fig. 14.16A: $\mathbf{Z}_{ab} = 8.66 + j5 = 10\underline{/30°}$; $\mathbf{Z}_{bc} = 3 + j4 = 5\underline{/53.1°}$; $\mathbf{Z}_{ca} = 5 - j5 = 7.07\underline{/-45°}$. A balanced symmetrical three-phase potential of 100 volts is applied.

(a) Assuming the phase sequence $\mathbf{V}_{ab}$-$\mathbf{V}_{bc}$-$\mathbf{V}_{ca}$ and utilizing $\mathbf{V}_{ab}$ as reference, find the indications of the two wattmeters. (b) Compare the sum $(W_A + W_B)$ with the sum of the I^2R values of the individual phases. (c) Compute the indications of similarly connected varmeters. (d) Compare the sum $(Q_A + Q_B)$ with the sum of the I^2X values of the individual phases. (e) Determine the three-phase vector power. (f) Draw the vector diagram for this case, clearly showing the quantities associated with the two meters.

The first step in the solution involves writing the expressions for the line (and phase) potentials:

$$\mathbf{V}_{ab} = 100 + j0 = 100\underline{/0^\circ} \text{ complex volts} = -\mathbf{V}_{ba},$$
$$\mathbf{V}_{bc} = -50 - j86.6 = 100\underline{/-120^\circ} \text{ complex volts} = -\mathbf{V}_{cb},$$
$$\mathbf{V}_{ca} = -50 + j86.6 = 100\underline{/120^\circ} \text{ complex volts} = -\mathbf{V}_{ac}.$$

The phase currents may next be found:

$$\mathbf{I}_{ab} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_{ab}} = \frac{100\underline{/0^\circ}}{10\underline{/30^\circ}} = 10\underline{/-30^\circ} = 8.66 - j5.0,$$

$$\mathbf{I}_{bc} = \frac{\mathbf{V}_{bc}}{\mathbf{Z}_{bc}} = \frac{100\underline{/-120^\circ}}{5\underline{/53.1^\circ}} = 20\underline{/-173.1^\circ} = -19.85 - j2.4,$$

$$\mathbf{I}_{ca} = \frac{\mathbf{V}_{ca}}{\mathbf{Z}_{ca}} = \frac{100\underline{/120^\circ}}{7.07\underline{/-45^\circ}} = 14.14\underline{/165^\circ} = -13.67 + j3.67 \text{ complex amperes.}$$

The line currents $\mathbf{I}_{Aa}$ and $\mathbf{I}_{Bb}$ can now be found:

$$\mathbf{I}_{Aa} = \mathbf{I}_{ab} - \mathbf{I}_{ca} = 22.33 - j8.67,$$
$$\mathbf{I}_{Bb} = \mathbf{I}_{bc} - \mathbf{I}_{ab} = -28.51 + j2.6 \text{ complex amperes.}$$

(a) The wattmeter readings then are, by the methods of Article 8.11,

$$\begin{aligned} W_A &= V_{ac_x}I_{Aa_x} + V_{ac_y}I_{Aa_y} = (50)(22.33) + (-86.6)(-8.67) \\ &= 1116 + 750 = 1866 \text{ watts,} \\ W_B &= V_{bc_x}I_{Bb_x} + V_{bc_y}I_{Bb_y} = (-50)(-28.51) + (-86.6)(2.6) \\ &= 1426 - 226 = 1200 \text{ watts.} \end{aligned}$$

(b) $W_A + W_B = 1866 + 1200 = 3066$ watts.

$$\begin{aligned} \Sigma I^2R &= (10)^2 \times 8.66 + (20)^2 \times 3 + (14.14)^2 \times 5 \\ &= 866 + 1200 + 1000 = 3066 \text{ watts.} \end{aligned}$$

The answers obtained by the two methods are seen to agree well within slide-rule precision.

(c) The varmeter readings are, by the methods of Article 8.11,

$$\begin{aligned} Q_A &= V_{ac_y}I_{Aa_x} - V_{ac_x}I_{Aa_y} = (-86.6)(22.33) - (50)(-8.67) \\ &= -1930 + 434 = -1496 \text{ vars,} \\ Q_B &= V_{bc_y}I_{Bb_x} - V_{bc_x}I_{Bb_y} = (-86.6)(-28.51) - (-50)(2.6) \\ &= 2465 + 130 = 2595 \text{ vars.} \end{aligned}$$

(d) $Q_A + Q_B = -1496 + 2595 = 1099$ vars.

$$\begin{aligned} \Sigma I^2X &= (10)^2(5) + (20)^2(4) + (14.14)^2(-5) \\ &= 500 + 1600 - 1000 = 1100 \text{ vars.} \end{aligned}$$

The answers obtained by the two methods are seen to agree within slide-rule precision.

(e) The three-phase vector power is

$$U'_{3\phi} = \sqrt{P^2_{3\phi} + Q^2_{3\phi}} = \sqrt{(3066)^2 + (1100)^2} = 3260 \text{ volt-amperes.}$$

(f) Figure 14.16B(b) is the vector diagram for this case.

14.17. Phase-sequence Effects

The consequences of reversing the phase sequence of emf's applied to a polyphase circuit are both interesting and important. The direction of rotation of three-phase induction and synchronous motors, for example, depends on the phase sequence of the applied potentials. The individual indications of the two wattmeters utilized to measure three-wire three-phase power by the two-wattmeter method change with a reversal of phase sequence, although the algebraic sum of their indications is unchanged. If the circuit is balanced, the two instruments interchange their indications when the phase sequence is reversed. These statements apply to two varmeters similarly connected.

In an unbalanced three-phase circuit, reversal of applied potential sequence will, in general, cause changes in both magnitudes and relative phase positions of line currents, although the total three-phase active and reactive power values and the total three-phase vector power remain unchanged in value. If sufficient unbalance exists, it is possible for the line currents and line potentials to have opposite phase sequences. For this reason, the phase sequence specified for a given circuit is understood to be the sequence of the applied line potentials.

In a balanced three-phase circuit, a reversal of phase sequence of applied potentials reverses the sequence of all potentials and currents but in no way alters their magnitudes. Except for symbolic notation, the vector diagram for a balanced circuit with one sequence of applied potentials looks exactly like the diagram applying to that circuit with the opposite sequence of applied potentials.

One method of checking the phase sequence of a given three-phase system is to observe the direction of rotation of a small three-phase motor connected to the system. If the motor rotation has previously been checked against a known phase sequence, the unknown sequence can be inferred.

A second method of checking phase sequence is pictured in Fig. 14.17A. An unbalanced wye-connected load consisting of two similar lamps and an inductor is connected to the three-phase line. The following rules then apply:

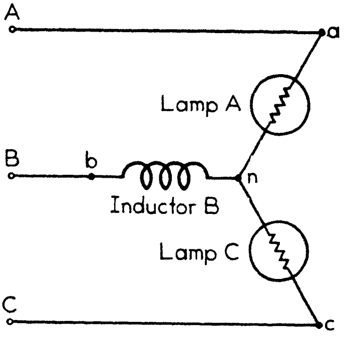


Fig. 14.17A. Two-lamp method of checking phase sequence.

If lamp A is brighter than lamp C, the phase sequence is $\mathbf{V}_{ab}$-$\mathbf{V}_{bc}$-$\mathbf{V}_{ca}$.

If lamp B is brighter than lamp A, the phase sequence is $\mathbf{V}_{ab}$-$\mathbf{V}_{ca}$-$\mathbf{V}_{bc}$.

In general terms, these rules may be combined in the statement that *phase rotation is: dim lamp, bright lamp, coil.* The difference in brightness is most readily discernible when the Q of the inductor is relatively high.

These statements can be verified by a circuit analysis which compares the magnitudes of the currents through (or of the potentials across) the two lamps. This analysis is left as a student exercise.

Example 14.17. Re-solve Example 14.16 for the case of reversed phase sequence of applied potentials.

The following expressions can be written for the applied potentials:

$$\mathbf{V}_{ab} = 100 + j0 = 100\underline{/0^\circ} \text{ complex volts} = -\mathbf{V}_{ba},$$
$$\mathbf{V}_{bc} = -50 + j86.6 = 100\underline{/120^\circ} \text{ complex volts} = -\mathbf{V}_{cb},$$
$$\mathbf{V}_{ca} = -50 - j86.6 = 100\underline{/-120^\circ} \text{ complex volts} = -\mathbf{V}_{ac}.$$

The phase currents are, in this case,

$$\mathbf{I}_{ab} = \frac{100\underline{/0^\circ}}{10\underline{/30^\circ}} = 10\underline{/-30^\circ} = 8.66 - j5.0 \text{ complex amperes,}$$
$$\mathbf{I}_{bc} = \frac{100\underline{/120^\circ}}{5\underline{/53.1^\circ}} = 20\underline{/66.9^\circ} = 7.86 + j18.4 \text{ complex amperes,}$$
$$\mathbf{I}_{ca} = \frac{100\underline{/-120^\circ}}{7.07\underline{/-45^\circ}} = 14.14\underline{/-75^\circ} = 3.66 - j13.68 \text{ complex amperes.}$$

The line currents $\mathbf{I}_{Aa}$ and $\mathbf{I}_{Bb}$ are, then,

$$\mathbf{I}_{Aa} = \mathbf{I}_{ab} - \mathbf{I}_{ca} = 5.0 + j8.68 \text{ complex amperes,}$$
$$\mathbf{I}_{Bb} = \mathbf{I}_{bc} - \mathbf{I}_{ab} = -0.8 + j23.4 \text{ complex amperes.}$$

(a) The wattmeter readings are, by the methods of Article 8.10,

$$W_A = (50)(5.0) + (86.6)(8.68) = 250 + 751 = 1001 \text{ watts},$$
$$W_B = (-50)(-0.8) + (86.6)(23.4) = 40 + 2025 = 2065 \text{ watts}.$$

Comparison of these results with those obtained for the original sequence (Example 14.16) shows that the individual indications change with a change of sequence but that the algebraic sum of the indications is the same as before:

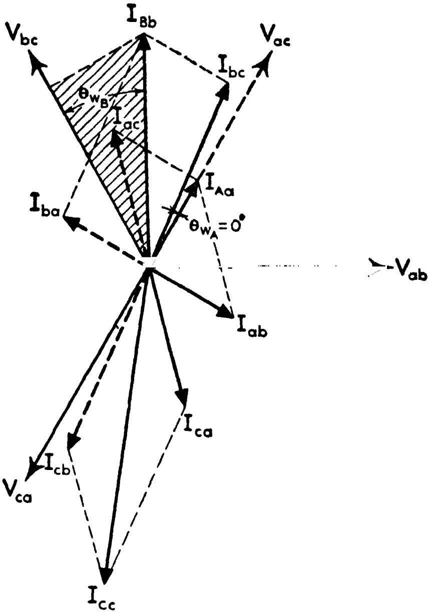


Fig. 14.17B. Vector diagram for Example 14.17.

(b) $W_A + W_B = 1001 + 2065 = 3066$ watts,

$$\Sigma I^2R = (10)^2 \times 8.66 + (20)^2 \times 3 + (14.14)^2 \times 5$$
$$= 3066 \text{ watts}.$$

The total power absorption by the load is unchanged by a change of sequence.

(c) The varmeter readings are, by the methods of Article 8.10,

$$Q_A = (86.6)(5.0) - (50)(8.68) = 433 - 434 = -1 \text{ var},$$
$$Q_B = (86.6)(-0.8) - (-50)(23.4) = -69.3 + 1170 = 1101 \text{ vars}.$$

The individual varmeter indications are seen to differ from the indications obtained with the original phase sequence, although the sum is the same as with the original sequence:

(d) $Q_A + Q_B = -1 + 1101 = 1100$ vars,
$\Sigma I^2X = (10)^2(5) + (20)^2(4) + (14.14)^2(-5) = 1100$ vars.

These answers are seen to agree with each other, as well as with those obtained for the original sequence.

(e) The three-phase vector power is the same as with the original sequence:

$$U'_{3\phi} = \sqrt{P^2_{3\phi} + Q^2_{3\phi}} = \sqrt{(3066)^2 + (1100)^2} = 3260 \text{ volt-amperes.}$$

(f) Figure 14.17B is the vector diagram for this case. For the original sequence, it should be compared with the diagram of Fig. 14.16B(b).

14.18. Copper Required to Transmit Power under Fixed Conditions

We shall now make an important comparison of the amount of copper required to transmit a given value of power a given distance with a given loss at a given highest potential between conductors, for circuits of different numbers of phases. It is clear that in all cases the total weight of copper will be directly proportional to the number of wires and to the cross-sectional area per wire. As a consequence, the total weight of copper will be directly proportional to the number of wires and inversely proportional to the wire resistance. Balanced operation is assumed in all cases. The subscripts indicate the number of phases, the potential and current symbols represent line values, the power-factor angle is that of the total power triangle, and the resistances are those of single line wires.

1. *Comparison of Three-Phase Three-Wire and Single-Phase Transmission.* Under the given conditions, the power transmitted by the three-phase circuit must equal that transmitted by the single-phase circuit:

$$P_3 = \sqrt{3}\, VI_3 \cos\theta = P_1 = VI_1 \cos\theta. \tag{14-88}$$

From this equation, we see that

$$\sqrt{3}\, I_3 = I_1. \tag{14-89}$$

The powers lost in transmission by the two systems must be equal:

$$I_3^2R_3 \times 3 = I_1^2R_1 \times 2. \tag{14-90}$$

From this equation, the ratio of the single-wire resistances of the two systems is found to be, by substitution of (14-89),

$$\frac{R_3}{R_1} = \frac{2I_1^2}{3I_3^2} = \frac{2 \times 3I_3^2}{3I_3^2} = \frac{2}{1}. \qquad (14\text{-}91)$$

Recalling that weight of copper is directly proportional to number of wires and inversely proportional to wire resistance, we may write

$$\frac{(\textit{Weight of three-phase copper})}{(\textit{Weight of single-phase copper})} = \frac{3}{2} \times \frac{R_1}{R_3} = \frac{3}{2} \times \frac{1}{2} = \frac{3}{4}. \qquad (14\text{-}92)$$

Under the given conditions of operation, a three-wire three-phase system requires only three-fourths as much copper as a single-phase system to transmit the same power. This is, of course, one of the major reasons for the predominance of three-phase power transmission over single-phase transmission in practice.

2. *Comparison of Three-Phase Three-Wire and Four-Phase Four-Wire Transmission.* Equating the powers transmitted by the two systems and letting V denote the highest wire-to-wire potential of the four-phase system, we may write

$$P_3 = \sqrt{3}\, VI_3 \cos\theta = P_4 = 4\left(\frac{V}{2} I_4 \cos\theta\right) = 2VI_4 \cos\theta. \qquad (14\text{-}93)$$

From this it is apparent that

$$\sqrt{3}\, I_3 = 2I_4. \qquad (14\text{-}94)$$

The powers lost in transmission by the two systems are equal:

$$I_3^2 R_3 \times 3 = I_4^2 R_4 \times 4. \qquad (14\text{-}95)$$

Combining (14-94) and (14-95) yields the following ratio of wire resistances:

$$\frac{R_3}{R_4} = \frac{4I_4^2}{3I_3^2} = \frac{4 \times \frac{3}{4} I_3^2}{3I_3^2} = \frac{1}{1}. \qquad (14\text{-}96)$$

The ratio of weights of copper for the two systems then is

$$\frac{(\textit{Weight of three-phase copper})}{(\textit{Weight of four-phase copper})} = \frac{3}{4} \times \frac{1}{1} = \frac{3}{4}. \qquad (14\text{-}97)$$

This is the same ratio found in the comparison of three-phase transmission with single-phase transmission. Similar analyses comparing three-phase transmission with transmission by n-wire n-phase systems in general show that, for the given conditions of operation, the three-phase system requires the least copper.

Similar analyses for the case of a fixed line-to-neutral (instead of line-to-line) potential reveal a fixed amount of copper required, regardless of the number of phases.

PROBLEMS

14-1. Find the potential between adjacent line wires of a symmetrical six-phase star connection, if the phase potential is 220 volts.

14-2. The potential between adjacent lines wires of a symmetrical twelve-phase star connection is 57 volts.
(a) Find the phase potential.
(b) Find the potential between alternate line wires.
(c) Draw a vector diagram to illustrate your method of solution.

14-3. A certain generator has six coils, each of whose generated emf's has an effective value of 57 volts. The emf's of adjacent coils are 60 electrical degrees apart.

(a) Draw a basic polar vector diagram for these emf's, and label the emf's E_{ab}, E_{cd}, E_{ef}, and so on, in sequence.
(b) Assuming a six-phase star connection,
  (1) Draw the wiring diagram.
  (2) Draw the polar vector diagram and the associated wave diagram.
  (3) Draw the string vector diagram.
  (4) Determine the potential between adjacent line wires.
  (5) Determine the potential between alternate line wires.
  (6) Write the complex expressions for all of the above emf's.
(c) Repeat (b) for a six-phase mesh connection.

14-4. Repeat Prob. 14-3 for all possible symmetrical phase connections which make use of all six coils.

14-5. Repeat Prob. 14-3 for all possible single-phase connections which make use of all six coils.

14-6. Two coils whose induced emf's are 90 electrical degrees apart and of equal effective values can be provided with "taps" (junction points between the ends of the windings) at any desired locations. If each induced emf equals 100 volts,
(a) Assuming a three-wire two-phase connection,
  (1) Draw the wiring diagram.
  (2) Draw the polar vector diagram and the associated wave diagram.
  (3) Draw the string vector diagram.
  (4) Determine the maximum effective potential between the adjacent wires.
  (5) Write the complex expressions for all the above emf's.
(b) Assuming a five-wire four-phase connection, repeat the steps of (a).

14-7. (a) By means of suitable wiring and vector diagrams, explain how the two coils of Prob. 14-6 could be used to obtain three-phase potentials from a two-phase supply line.

(b) Write the complex expressions for all vectors of (a).

14-8. The line-to-line potentials of a certain three-phase system are measured by a voltmeter, with the following results: $V_{ab} = 220$, $V_{bc} = 212$, $V_{ca} = 238$ volts. The phase sequence is known to be $\mathbf{V}_{ab}$-$\mathbf{V}_{bc}$-$\mathbf{V}_{ca}$.

(a) Using $\mathbf{V}_{ab}$ as reference, write the complex expressions for these potentials.

(b) Draw a polar vector diagram.

14-9. Solve Prob. 14-8 for the opposite phase sequence.

14-10. In the circuit of Fig. 14.09A, $\mathbf{Z}_{an} = 10 + j0$, $\mathbf{Z}_{bn} = 0 + j10$, $\mathbf{Z}_{cn} = 0 - j10$ ohms. Each line wire has an impedance $\mathbf{Z}_{\text{line}} = 1 + j2$ ohms. Also, the potentials applied to the input terminals of the line are: $\mathbf{V}_{AB} = 100\underline{/0^\circ}$, $\mathbf{V}_{BC} = 100\underline{/240^\circ}$, $\mathbf{V}_{CA} = 100\underline{/120^\circ}$ volts.

(a) Solve for all potential and current vectors, and express the answers in the polar form.

(b) Draw a complete vector diagram.

(c) Do the line currents have the same sequence as the applied emf's? Explain.

14-11. Solve Prob. 14-10 for the case of negligible line impedances and reversed sequence of applied emf's.

14-12. Add a neutral wire having the same impedance as the line wires to the circuit of Prob. 14-10 and solve.

14-13. Add a neutral wire of negligible impedance to the circuit of Prob. 14-11 and solve.

14-14. A balanced wye-connected load has in each phase an impedance, $\mathbf{Z}_Y = 3 - j4$ ohms. Balanced three-phase potentials of 100 volts appear at the terminals of this load. The load is connected to a generator by a line each wire of which has an impedance $\mathbf{Z}_{\text{line}} = 1 + j4$ ohms.

(a) Find the potential at the generator terminals.

(b) Draw a circuit diagram for one phase of this system, using a neutral wire of assumed negligible impedance to complete the current path. (Why is this assumption possible in this case?)

(c) Draw a vector diagram for the one phase of (b).

(d) Are the diagrams of (b) and (c) sufficient to give an understanding of this whole system? (Discuss.)

14-15. Repeat Prob. 14-14 for $\mathbf{Z}_Y = 5 + j3$ ohms.

14-16. In the circuit of Figure 14.11A, $\mathbf{Z}_{ab} = 10 + j0$ ohms, $\mathbf{Z}_{bc} = 0 + j10$ ohms, $\mathbf{Z}_{ca} = 0 - j10$ ohms. Each line wire has an impedance $\mathbf{Z}_{\text{line}} = 1 + j2$ ohms. Also, the potentials applied to the input terminals of the line are: $\mathbf{V}_{AB} = 100/0^\circ$ volts, $\mathbf{V}_{BC} = 100/240^\circ$ volts, $\mathbf{V}_{CA} = 100/120^\circ$ volts.

(a) Solve for all potential and current quantities and express the answers in the polar form.

(b) Draw a complete vector diagram.

(c) Do the line currents have the same phase sequence as the applied emf's? Explain.

14-17. Solve Prob. 14-16 for the case of negligible line impedances and reversed sequence of emf's.

14-18. A fuse in line wire *Aa* of Prob. 14-16 blows, thereby opening that line wire.

(a) Find the new complex expressions for all potentials and currents.

(b) Draw to scale a vector diagram for this new condition.

14-19. Repeat Prob. 14-18, except have the blown fuse in line wire *Cc*.

14-20. A balanced delta-connected load has in each phase an impedance $\mathbf{Z}_\Delta = 3 - j4$ ohms. Balanced three-phase potentials of 100 volts appear at the terminals of this load. The load is connected to a generator by a line each wire of which has an impedance $\mathbf{Z}_{\text{line}} = 1 + j4$ ohms.

(a) Find the potential at the generator terminals.

(b) Draw a complete circuit diagram.

14-21. Repeat Prob. 14-20 for $\mathbf{Z}_\Delta = 5 + j3$ ohms.

14-22. As a result of laboratory tests, the following information is available concerning the circuit of Fig. 14.11A:

(1) $V_{ab} = 190$, $V_{bc} = 126$, $V_{ca} = 140$ volts. The potential phase sequence is $\mathbf{V}_{ab}$-$\mathbf{V}_{bc}$-$\mathbf{V}_{ca}$.

(2) $I_{ab} = 10$, $I_{bc} = 7$, $I_{ca} = 10$, $I_{Aa} = 0.63$, $I_{Bb} = 15.9$, $I_{Cc} = 15.7$ A.

(3) Z_{ab} is capacitive, with resistance negligible.

(4) The load is passive.

Using $\mathbf{V}_{ab}$ as reference, determine the complex expressions for all potentials, currents, and impedances. (*Hint:* Try a preliminary graphical solution.)

14-23. Repeat Prob. 14-22 for the opposite sequence.

14-24. The following potentials appear at the terminals of the circuit of Fig. 14.12: $\mathbf{V}_{ab} = 100\underline{/0^\circ}$, $\mathbf{V}_{bc} = 100\underline{/240^\circ}$, $\mathbf{V}_{ca} = 100\underline{/120^\circ}$ volts.

The impedances have the following values: $\mathbf{Z}_{ab} = \mathbf{Z}_{bc} = 10 + j0$, $\mathbf{Z}_{ca} = 5 + j5$, $\mathbf{Z}_{an} = 0 - j5$, $\mathbf{Z}_{bn} = 5 + j0$, $\mathbf{Z}_{cn} = 3 + j2$ ohms.

(a) Find the values of all line currents.

(b) Draw a complete vector diagram.

14-25. Repeat Prob. 14-24 with the branch containing $\mathbf{Z}_{ab}$ removed.

14-26. (a) Determine the complex expressions of the vector powers for the individual impedances of Prob. 14-10.

(b) Determine the complex expression for the three-phase vector power of the load.

(c) Determine the complex expression for the three-phase vector power taken by the line wires.

(d) Determine the complex expression for the three-phase vector power taken by the whole system.

(e) Draw a power diagram showing how the power triangles of (a) combine to give the power triangle of (d).

(f) Draw another power diagram showing how the power triangles of (b) and (c) combine to give the power triangle of (d).

14-27. Repeat Prob. 14-26 for the circuit of Prob. 14-16.

14-28. A balanced three-phase motor load draws 500 kilowatts at 0.8 lagging power factor. In parallel with this is a balanced three-phase incandescent lamp load of 150 kilowatts.

(a) Find the over-all power factor of this combined load.

(b) Find the line currents if the combined load is supplied by a three-wire line at 230 volts and 60 cycles.

(c) Specify the ratings of a bank of delta-connected capacitors to be connected across the line to produce an over-all power factor of unity.

(d) Repeat (c) for a wye-connected bank.

14-29. (a) Find what value of delta-connected capacitance is required to raise to 0.9 lagging the power factor of a 2300-volt, 5000-kva, 0.6-lagging power-factor industrial load.

(b) What additional active power does this power factor improvement make available without increasing the total vector power?

14-30. The current coils of two wattmeters, W_A and W_B, are connected, respectively, in line wires *Aa* and *Bb* of the circuit of Prob. 14-16.

(a) Draw a circuit diagram showing the proper connections of the wattmeter potential coils for determining by the two-wattmeter method the total three-phase power taken by the load.

(b) Compute the indications of the two wattmeters and the total power.

(c) Check (b) by the I^2R method.

14-31. The current coils of two wattmeters, W_A and W_B, are connected, respectively, in line wires *Aa* and *Bb* of the circuit of Prob. 14-24.

(a) Draw a circuit diagram showing the proper connections of the wattmeter potential coils for determining by the two-wattmeter method the total three-phase power taken by the load.

(b) Compute the indications of the two wattmeters and the total power.

(c) Check (b) by the I^2R method.

14-32. The current coils of two wattmeters, W_A and W_B, are connected, respectively, in line wires *Aa* and *Bb* of the circuit of Prob. 14-20.

(a) Draw a circuit diagram showing the proper connections of the wattmeter potential coils for determining by the two-wattmeter method the total three-phase power taken by the load.

(b) Determine the indications of the two wattmeters and the total power.

(c) Check (b) by the I^2R method.

14-33. Repeat Prob. 14-32 for the circuit of Prob. 14-21.

14-34. Prepare a power factor chart for balanced three-phase circuits as follows:

(a) Utilizing equations (14-83), compute the ratios of *smaller indication* to *larger indication* for a series of values of power factor from 0 to 1.0, in 0.1 steps. Note that the algebraic sign of the indication must be carried along, with the result that half of the ratios will be negative.

(b) Graph the results of (a), using the ratio $(W_{\text{smaller}}/W_{\text{larger}})$ as abscissa, and *power factor* as ordinate. The resulting curve should resemble somewhat an integral sign.

(c) Study this chart in the light of the discussion of Article 14.16.

14-35. The indications of two wattmeters properly connected to determine total power in a balanced three-wire three-phase circuit are: $W_A = 5000$ watts, $W_B = 1500$ watts.

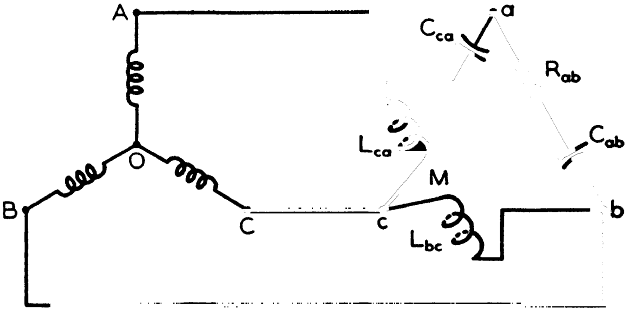
(a) Determine by computation the power factor of the load.

(b) Check the answer to (a) by the chart prepared in Prob. 14-34.

14-36. Referring to Fig. 14.17A, prove the statement made concerning the effect of phase sequence on the relative brightness of the two lamps.

14-37. Prove the concluding statement of Article 14.18 that a fixed amount of copper, regardless of the number of phases, is required to transmit a given value of power a given distance with a given loss at a fixed line-to-neutral potential.

14-38. In the accompanying circuit, $R_{ab} = 10$ ohms, $L_{bc} = L_{ca} = 20$ millihenrys, $k = 0.5$, $C_{ab} = C_{ca} = 200$ microfarads, $f = 60$ cycles per second, $\mathbf{E}_{OA} = 100\underline{/0^\circ}$, $\mathbf{E}_{OB} = 100\underline{/240^\circ}$, and $\mathbf{E}_{OC} = 100\underline{/120^\circ}$ volts. Assuming generator and supply-line impedances to be negligible, compute the line and phase currents.



Prob. 14-38.

14-39. Solve Prob. 14-38 with a resistance of 3 ohms in each line wire.

PART IV

NONSINUSOIDAL WAVES

CHAPTER 15

ANALYSIS OF NONSINUSOIDAL WAVES

15.01. Introduction

In all branches of electrical engineering nonsinusoidal waves[1] are encountered at least occasionally, and in some branches nonsinusoidal waves are more common than sinusoidal waves. To the communication engineer, for example, wave form is as much of a "commodity" as is electric energy to the power engineer. In many electronic circuits, nonsinusoidal waves of such distinctive forms as "sawtooth" waves and "square" waves are deliberately generated for specific purposes. This operation requires a knowledge of the properties of nonsinusoidal waves. In other circuits, the purest of sine waves are desired, and the elimination of any nonsinusoidal components becomes important. Frequently this operation, too, requires a knowledge of the properties of nonsinusoidal waves.

Undesired nonsinusoidal wave components can arise from a number of causes. The emf wave form of a commercial alternator is rarely exactly sinusoidal even at no load; and even if it were, the effect of armature reaction in distorting the flux distribution as the machine acquires load could result in a considerable distortion of the emf wave under loaded conditions. Because of the nonlinear magnetic characteristics of iron, a sinusoidal emf impressed on a circuit with an iron core may result in a nonsinusoidal current. As was pointed out in Article 4.28, nonlinear resistances can result in nonsinusoidal currents even though sinusoidal emf's are applied. Capacitors, also, can possess nonlinear characteristics.

Nonsinusoidal waves of specifically desired forms can be generated by a number of different means. Rotating machines of special design, electronic oscillators of special design, and circuit elements possessing particular nonlinear properties all find use in obtaining desired wave forms of a nonsinusoidal nature.

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[1] The term "wave" is used throughout this chapter in the restricted sense of *a recurrent function of time.* A review of Article 3.26 is suggested to clarify this point.

The reader may well wonder at this point how much of the theory so painstakingly developed in the previous chapters is applicable to nonsinusoidal waves, in view of the fact that it was all developed on the assumption of sinusoidal waves. The fact is that because it was developed for *sinusoidal* waves, all of this theory is, with proper interpretation, applicable to *nonsinusoidal* waves! The next article is the first step in explaining this seeming paradox.

15.02. Fourier Representation of Nonsinusoidal Waves

In 1822, Joseph Fourier showed in his *Analytical Theory of Heat* that any mathematical function which is periodic, single-valued, finite, and possessed of a finite number of discontinuities may be represented by the sum of a number of sinusoidal components of different frequencies. In mathematical form, this theorem can be stated in the form of a *Fourier series:*

$$\blacktriangleright \quad y = f(x) = B_0 + A_1 \sin x + B_1 \cos x + A_2 \sin 2x + B_2 \cos 2x + A_3 \sin 3x + B_3 \cos 3x + \cdots + A_n \sin nx + B_n \cos nx + \cdots \qquad (15\text{-}1)$$

or

$$\blacktriangleright \quad y = f(x) = B_0 + \sum_{n=1}^{\infty} A_n \sin nx + \sum_{n=1}^{\infty} B_n \cos nx. \qquad (15\text{-}2)$$

Virtually all of the nonsinusoidal waves encountered in electrical engineering come under this category. Except in special cases, an infinite number of components is theoretically required, although often only the first few are of significance.

It is apparent that, since each component of the wave is sinusoidal, each component can be handled individually by the methods previously outlined for handling sinusoidal waves. Even though the equation of a wave may be known in terms of some simpler function of x, it is usually advantageous to re-express it in terms of a Fourier series, because of the relative ease of manipulating the sinusoidal components.

The number of sinusoidal components that must be handled practically in a given problem can be cut in half by combining the sine and cosine term at each frequency into a single sine term containing an appropriate initial phase angle. Figure 15.02A illustrates this idea. By following this procedure, equation (15-2) can be rewritten as

$$\blacktriangleright \qquad y = C_0 + \sum_{n=1}^{\infty} C_n \sin (nx + \theta_n), \qquad (15\text{-}3)$$

where:

$$\mathbf{C}_n = C_n\underline{/\theta_n} = \sqrt{A_n^2 + B_n^2}\,\underline{\Big/\tan^{-1}\frac{B_n}{A_n}}$$
$$= A_n + jB_n.$$

Under this scheme, equation (15-1) becomes

$$\blacktriangleright \quad y = C_0 + C_1 \sin (x + \theta_1) + C_2 \sin (2x + \theta_2) + \cdots + C_n \sin (nx + \theta_n) + \cdots . \qquad (15\text{-}4)$$

By comparison with equation (15-1), or by letting $n = 0$ in equations (15-2) and (15-3), it becomes clear that $C_0 = B_0$.

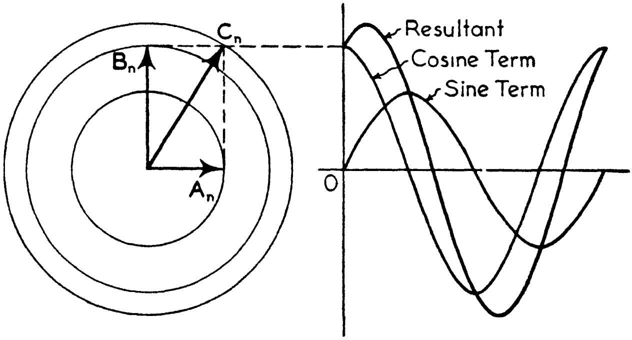


Fig. 15.02A. Sine and cosine components of a particular harmonic.

If the nonsinusoidal wave is a function of time, as in our representation of alternating currents and potentials, the wave can be represented as follows:

$$\blacktriangleright \quad y(t) = C_0 + C_1 \sin (\omega t + \theta_1) + C_2 \sin (2\omega t + \theta_2) + C_3 \sin (3\omega t + \theta_3) + \cdots + C_n \sin (n\omega t + \theta_n) + \cdots , \qquad (15\text{-}5)$$

where $y(t)$ may be a potential, a current, or some other periodically time-varying quantity in the circuit, and the capital letters represent the amplitudes of the components of different frequencies.

C_0 is, of course, the coefficient of a zero-frequency or "d-c" term. It has a constant value which is zero in most cases.

The sinusoidal components at different frequencies of equation (15-5) are termed the harmonics of the wave:

$$\begin{aligned}
C_1 \sin (\omega t + \theta_1) &= \text{the } \textit{fundamental}, \text{ or } \textit{first harmonic}, \\
C_2 \sin (2\omega t + \theta_2) &= \text{the } \textit{second harmonic}, \\
C_3 \sin (3\omega t + \theta_3) &= \text{the } \textit{third harmonic}, \\
&\cdots\cdots\cdots\cdots \\
C_n \sin (n\omega t + \theta_n) &= \text{the } \textit{nth harmonic}, \\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

Sometimes the third harmonic and all harmonics which are *odd multiples of the third*, such as the 9th, the 15th, and so on, are called *triplen* harmonics.

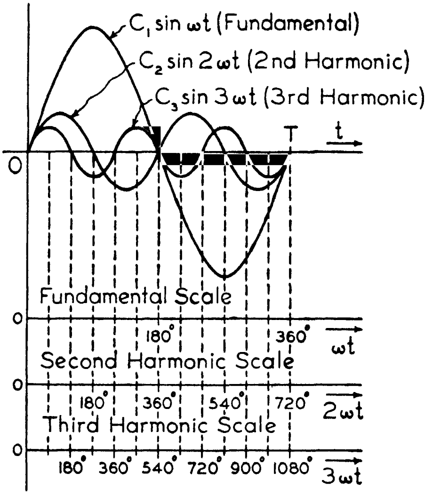


Fig. 15.02B. Fundamental, second, and third harmonics.

In applying an equation such as (15-5) it is important to realize that each harmonic must be plotted to its own angular scale and that the angular scale of the *n*th harmonic contains *n* times as many degrees or radians in a given time interval, as does the fundamental scale. This is another way of saying that *n* cycles of the *n*th harmonic occur in the period of one cycle of the fundamental. Figure 15.02B should make this clear. Similarly, in equation (15-5), the initial phase angle, θ_n, must be measured on the scale of the harmonic to which it belongs. It will, therefore, have a value *n* times as great as though it were measured on the fundamental scale. By the same reasoning, shifting the phase angle of a nonsinusoidal wave through a positive angle of α degrees shifts the fundamental through an angle α, the second harmonic through an angle 2α, the third harmonic through an angle 3α, and so on, and the *n*th harmonic through an angle $n\alpha$.

Example 15.02. Shift the following nonsinusoidal wave forward by 30 fundamental degrees:

$$i = 10 \sin (\omega t + 10^\circ) + 8 \sin (3\omega t - 30^\circ) + 5 \sin (5\omega t) - 2 \sin (7\omega t + 10^\circ) \text{ amperes.}$$

The new wave must have 30 *fundamental* degrees added to each of its terms:

$$\begin{aligned} i' &= 10 \sin (\omega t + 10^\circ + 30^\circ) + 8 \sin (3\omega t - 30^\circ + 3 \times 30^\circ) \\ &\qquad + 5 \sin (5\omega t + 5 \times 30^\circ) - 2 \sin (7\omega t + 10^\circ + 7 \times 30^\circ) \\ &= 10 \sin (\omega t + 40^\circ) + 8 \sin (3\omega t + 60^\circ) \\ &\qquad + 5 \sin (5\omega t + 150^\circ) - 2 \sin (7\omega t + 220^\circ) \text{ amperes.} \end{aligned}$$

or

$$i' = 10 \sin (\omega t + 40^\circ) + 8 \sin (3\omega t + 60^\circ) + 5 \sin (5\omega t + 150^\circ) + 2 \sin (7\omega t + 40^\circ) \text{ amperes.}$$

15.03. General Considerations Underlying Wave Analysis

The analysis of a given nonsinusoidal wave into its Fourier components consists of determining the coefficients B_0, A_1, B_1, A_2, B_2, . . . A_n, B_n . . . of equation (15-1). If desired, the coefficients C_0, C_1, C_2, . . . C_n, . . . and the angles θ_1, θ_2, θ_3, . . . θ_n, . . . of equation (15-5) can then be determined by application of (15-3). The operation needed to determine the A or B coefficients for a given harmonic consists of manipulating equation (15-1) in such a way as to eliminate all terms except the one containing the desired coefficient.

1. *Determination of Zero-frequency Term,* B_0. Multiplying equation (15-1) by dx and integrating between the limits 0 and 2π makes it possible to evaluate the zero-frequency term, B_0.

$$\int_0^{2\pi} y\,dx = \int_0^{2\pi} B_0\,dx + \int_0^{2\pi} A_1 \sin x\,dx + \int_0^{2\pi} B_1 \cos x\,dx + \cdots + \int_0^{2\pi} A_n \sin nx\,dx + \int_0^{2\pi} B_n \cos nx\,dx + \cdots. \tag{15-6}$$

From this equation,

$$\int_0^{2\pi} y\,dx = B_0 \int_0^{2\pi} dx + 0 = 2\pi B_0, \tag{15-7}$$

or

▶ $$B_0 = \frac{1}{2\pi} \int_0^{2\pi} y\,dx. \tag{15-8}$$

From this equation for the zero-frequency (d-c) term, it is seen that if the wave is symmetrical about the horizontal axis, $B_0 = 0$. This is often—though not always—the case.

2. *Determination of Sine-term Coefficient, A_n.* Multiplying equation (15-1) by ($\sin nx\,dx$) and integrating between the limits of 0 and 2π makes it possible to evaluate the coefficient, A_n:

$$\begin{aligned}\int_0^{2\pi} y \sin nx\,dx = \int_0^{2\pi} B_0 \sin nx\,dx + \int_0^{2\pi} A_1 \sin x \sin nx\,dx \\ + \int_0^{2\pi} B_1 \cos x \sin nx\,dx + \cdots + \int_0^{2\pi} A_n \sin^2 nx\,dx \\ + \int_0^{2\pi} B_n \cos nx \sin nx\,dx + \cdots . \qquad (15\text{-}9)\end{aligned}$$

It will be recognized that there are four different types of terms present. They are:

(a) $\int_0^{2\pi} \sin mx \sin nx\,dx = 0$, where m and n are different integers, including zero,

(b) $\int_0^{2\pi} \cos mx \sin nx\,dx = 0$, where m and n are different integers,

(c) $\int_0^{2\pi} \sin^2 nx\,dx = \frac{1}{2}\int_0^{2\pi} (1 - \cos 2nx)\,dx = \frac{2\pi}{2} = \pi$,

(d) $\int_0^{2\pi} \cos nx \sin nx\,dx = 0$.

Equation (15-9) then reduces to

$$\int_0^{2\pi} y \sin nx\,dx = A_n\pi, \qquad (15\text{-}10)$$

from which

▶ $$A_n = \frac{1}{\pi}\int_0^{2\pi} y \sin nx\,dx. \qquad (15\text{-}11)$$

3. *Determination of Cosine-term Coefficient, B_n.* Multiplying (15-1) by ($\cos nx\,dx$) and integrating between the limits of 0 and 2π makes it possible to evaluate the coefficient, B_n:

$$\begin{aligned}\int_0^{2\pi} y \cos nx\,dx = \int_0^{2\pi} B_0 \cos nx\,dx + \int_0^{2\pi} A_1 \sin x \cos nx\,dx \\ + \int_0^{2\pi} B_1 \cos x \cos nx\,dx + \cdots + \int_0^{2\pi} A_n \sin nx \cos nx\,dx \\ + \int_0^{2\pi} B_n \cos^2 nx\,dx + \cdots . \qquad (15\text{-}12)\end{aligned}$$

These terms all come under the category of one or another of the four types of terms previously outlined, except for terms of the type

$\int_0^{2\pi} \cos mx \cos nx\,dx = 0$, where m and n are different integers.

Equation (15-12) then reduces to

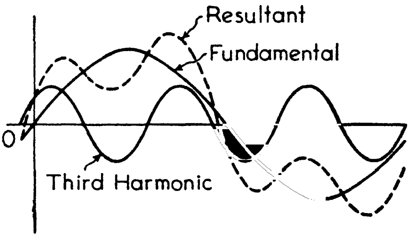
$$\int_0^{2\pi} y \cos nx \, dx = B_n \int_0^{2\pi} \cos^2 nx \, dx = B_n\pi. \qquad (15\text{-}13)$$

From this it is evident that, except when $n = 0$,

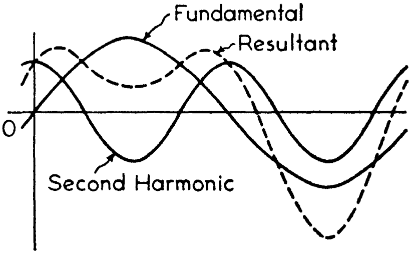
▶ $$B_n = \frac{1}{\pi} \int_0^{2\pi} y \cos nx \, dx. \qquad (15\text{-}14)$$

Equations (15-8), (15-11), and (15-14) are, then, the equations for determining the Fourier coefficients.

Usually, in wave analysis, an oscillographic record of the wave to be analyzed is first obtained. This is then analyzed, through the use of the equations derived above, into its component sinusoidal terms. The one major problem is to evaluate the integrals of these relations. If a simple expression can be written for y in terms of x, then the evaluation of these integrals is easy by analytical methods. If no such expression can be written, then one of various available graphical methods must be employed. The analytical method and two graphical methods are outlined in subsequent articles.



(a) Half-wave symmetry.



(b) Half-wave asymmetry.

Fig. 15.04A. Effect of even and odd harmonics on half-wave symmetry.

15.04. Simplifying Conditions

Before proceeding with a detailed study of methods of wave analysis, we may profitably consider a few special conditions which, if present in a given wave, materially reduce the labor of analysis.

Article 2.06 presented the following definition: "*A symmetrical alternating current is an alternating current of which all pairs of values that are separated by a half period have the same magnitude but the* *opposite sign.*" Figure 15.04A(a) shows a wave which possesses the *half-way symmetry* of the above definition. Figure 15.04A(b) shows a wave which is, by this definition, not symmetrical. It is possible to show that *a symmetrical wave cannot contain even harmonics.* The proof of this statement follows.

The ordinate at any time t of a general nonsinusoidal wave is, by equation (15-4),

$$\begin{aligned} y = C_0 &+ C_1 \sin(\omega t + \theta_1) + C_2 \sin(2\omega t + \theta_2) \\ &+ C_3 \sin(3\omega t + \theta_3) + C_4 \sin(4\omega t + \theta_4) \\ &+ C_5 \sin(5\omega t + \theta_5) + \cdots + C_n \sin(n\omega t + \theta_n) + \cdots . \end{aligned} \tag{15-15}$$

The ordinate at a time $(t' = t + T/2)$, one half cycle later, is the equivalent of (15-15) with each angle increased by π radians measured on the *fundamental* scale. Mathematically,

$$\begin{aligned} y' = C_0 &+ C_1 \sin(\omega t + \theta_1 + \pi) + C_2 \sin(2\omega t + \theta_2 + 2\pi) \\ &+ C_3 \sin(3\omega t + \theta_3 + 3\pi) + C_4 \sin(4\omega t + \theta_4 + 4\pi) \\ &+ C_5 \sin(5\omega t + \theta_5 + 5\pi) + \cdots \\ &\qquad + C_n \sin(n\omega t + \theta_n + n\pi) + \cdots . \end{aligned} \tag{15-16}$$

It is apparent that *when n is even,*

$$\sin(\alpha + n\pi) = \sin \alpha, \tag{15-17a}$$

and *when n is odd,*

$$\sin(\alpha + n\pi) = -\sin \alpha. \tag{15-17b}$$

It follows, then, that (15-16) can be rewritten as

$$\begin{aligned} y' = C_0 &- C_1 \sin(\omega t + \theta_1) + C_2 \sin(2\omega t + \theta_2) \\ &- C_3 \sin(3\omega t + \theta_3) + C_4 \sin(4\omega t + \theta_4) \\ &- C_5 \sin(5\omega t + \theta_5) + \cdots . \end{aligned} \tag{15-18}$$

A comparison of (15-15) and (15-18) makes it clear that the condition of half-wave symmetry, $y' = -y$, prevails only when both of the following characteristics are present:

(1) The zero-frequency term $C_0 = 0$,
(2) All other even harmonics are nonexistent.

Conversely, a wave possessing even harmonics cannot possess half-wave symmetry. Figure 15.04B should help to convince the reader of these points. *Advance knowledge that a wave which inspection shows to possess half-wave symmetry cannot contain even harmonics immediately reduces the problem of analyzing that wave to a search for only the odd harmonics.*

If the odd harmonics of a wave possessing half-wave symmetry start through zero at the same time as the fundamental, the wave possesses a "mirror symmetry" about a vertical line through the peak of the fundamental, and is said to possess *quarter-wave symmetry*. Figure 15.04B shows such a wave. The second harmonic sketched in for comparison could, it is quite clear, render the resultant wave totally asymmetrical if it were to be included as a component. This knowledge can be put to good use in a preliminary way in performing a wave analysis.

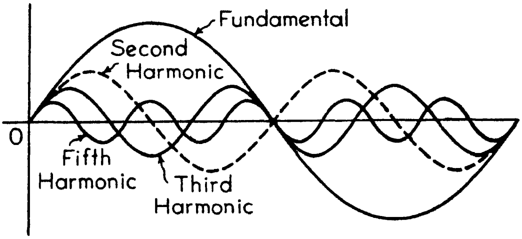


Fig. 15.04B. Effect of even and odd harmonics on quarter-wave symmetry.

15.05. Comparison of Wave Forms

While at this point, it is convenient for us to digress long enough to discuss the comparison of wave forms. Two waves, even though plotted to different scales, have the same wave form if all the following conditions are satisfied:

(1) The two waves have the same harmonics,

(2) The ratios of corresponding harmonics to respective fundamentals is the same, and

(3) The corresponding harmonics have the same initial phase angles relative to their fundamentals.

The method of testing two waves for identity of shape is to compare ratios of magnitudes for corresponding harmonics and then to shift one wave through whatever angle is necessary to make the waves coincide. If corresponding angles are the same and if the ratios of amplitudes are the same, then the two waves have the same form.

Example 15.05. Compare the forms of the following two waves:

$$v = 156 \sin(\omega t + 30^\circ) - 88 \sin(3\omega t - 120^\circ) + 15 \cos(5\omega t + 60^\circ) \text{ volts},$$
$$i = 15.6 \sin(\omega t - 60^\circ) + 8.8 \cos(3\omega t + 60^\circ) - 1.5 \sin(5\omega t + 60^\circ) \text{ amperes}.$$

Since the amplitudes of all harmonics of the current wave are 0.1 of the corresponding harmonics of the potential wave, the first two requirements are met. Now let us shift the current wave forward 90°:

$$\begin{aligned} i' &= 15.6 \sin(\omega t + 30^\circ) + 8.8 \cos(3\omega t + 330^\circ) - 1.5 \sin(5\omega t + 510^\circ) \\ &= 15.6 \sin(\omega t + 30^\circ) + 8.8 \sin(3\omega t + 420^\circ) - 1.5 \cos(5\omega t + 420^\circ) \\ &= 15.6 \sin(\omega t + 30^\circ) - 8.8 \sin(3\omega t + 600^\circ) + 1.5 \cos(5\omega t + 600^\circ) \\ &= 15.6 \sin(\omega t + 30^\circ) - 8.8 \sin(3\omega t - 120^\circ) + 1.5 \cos(5\omega t - 120^\circ). \end{aligned}$$

Corresponding angles are alike in the fundamental and in the third harmonic but not in the fifth harmonic, and so the waves are *not* of the same form.

15.06. Analytical Method

If, for any given wave, the equation of the dependent variable as a function of the independent variable can be written over a complete period, that wave can be analyzed through direct application of equations (15-8), (15-11), and (15-14). This procedure can be termed the *analytical method* for determining the Fourier components of a nonsinusoidal wave. It is not necessary that a single function of the independent variable cover the entire period. Several different functions can be used, one for each of several intervals into which the period can be divided. The complete integral from 0 to 2π then may be obtained as the sum of the integrals of the separate functions over their component intervals. The following example should serve to make this method clear.

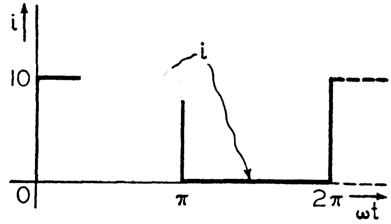


Fig. 15.06A. Rectangular wave of Example 15.06.

Example 15.06. Figure 15.06A represents the wave form of a particular emf. Employing numerical coefficients, perform the following operations: (a) Write, through the third harmonic terms, an equation of the same type as (15-1), describing this wave. (b) Express the answer to (a) in the form of (15-4). (c) Show by means of a sketch the manner in which the three sets of components combine to form the wave shown in the figure.

This wave can be represented by two simple equations. Between $\omega t = 0$ and $\omega t = \pi$, $i = 10$; between $\omega t = \pi$ and $\omega t = 2\pi$, $i = 0$. By equation (15-8),

$$B_0 = \frac{1}{2\pi}\int_0^{2\pi} i\,d(\omega t) = \frac{1}{2\pi}\left[\int_0^{\pi} 10\,d(\omega t) + \int_{\pi}^{2\pi} 0\,d(\omega t)\right]$$
$$= \frac{1}{2\pi}\left[10(\omega t)\right]_0^{\pi} + 0 = \frac{1}{2\pi} \times 10\pi = 5 \text{ amperes.}$$

Similarly, from equation (15-11),

$$A_n = \frac{1}{\pi}\int_0^{2\pi} i \sin n(\omega t)\,d(\omega t) = \frac{1}{\pi}\left[\int_0^{\pi} 10 \sin n(\omega t)\,d(\omega t) + 0\right]$$
$$= \frac{1}{\pi}\left[\frac{-10}{n}\cos n(\omega t)\right]_0^{\pi} = \frac{-10}{n\pi}[\cos n\pi - \cos 0].$$

For the fundamental, $n = 1$, and

$$A_1 = \frac{-10}{\pi}[-1 - 1] = \frac{+20}{\pi} = 6.37 \text{ amperes.}$$

For the second harmonic, $n = 2$, and

$$A_2 = \frac{-10}{2\pi}[1 - 1] = 0 \text{ amperes.}$$

For the third harmonic, $n = 3$, and

$$A_3 = \frac{-10}{3\pi}[-1 - 1] = \frac{30}{3\pi} = 2.12 \text{ amperes.}$$

Similarly, from equation (15-14),

$$B_n = \frac{1}{\pi}\int_0^{2\pi} i \cos n(\omega t)\,d(\omega t) = \frac{1}{\pi}\left[\int_0^{\pi} 10 \cos n(\omega t)\,d(\omega t) + 0\right]$$
$$= \frac{1}{\pi}\left[\frac{10}{n}\sin n(\omega t)\right]_0^{\pi} = \frac{10}{n\pi}[0 - 0] = 0 \text{ amperes.}$$

(a) We can now write the Fourier series by equation (15-1) is follows:

$$i = 5 + \frac{20}{\pi}\sin \omega t + \frac{20}{3\pi}\sin 3\omega t + \frac{20}{5\pi}\sin 5\omega t + \cdots$$
$$= 5 + 6.37 \sin \omega t + 2.12 \sin 3\omega t + 1.27 \sin 5\omega t + \cdots \text{ amperes.}$$

(b) Inasmuch as $B_n = 0$, $C_n = A_n$, and the foregoing equation is the solution to this part of the problem also.

(c) Figure 15.06B shows partial synthesis of the rectangular wave of the problem from the numerical answers obtained analytically.

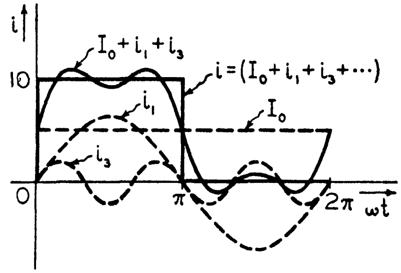


Fig. 15.06B. Fourier synthesis of rectangular wave of Example 15.06.

15.07. Graphical Method of Analysis

In Article 15.03 we developed the following equations for the coefficients of the sinusoidal components of a wave represented by a Fourier series:

$$B_0 = \frac{1}{2\pi}\int_0^{2\pi} y\,dx, \tag{15-19}$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi} y \sin nx\,dx, \tag{15-20}$$

and

$$B_n = \frac{1}{\pi}\int_0^{2\pi} y \cos nx\,dx. \tag{15-21}$$

Where the equation of y in terms of x is unknown, a graphical method of solution can be employed. This method, a logical outgrowth of the above three equations, amounts to a graphical evaluation of the above integrals.

Referring to Fig. 15.07A, which we shall assume to be the wave to be analyzed, we first divide the figure into m equal parts by the erection of m equally spaced ordinates. The interval between any two adjacent ordinates is

$$\Delta x = \frac{2\pi}{m}\text{ radians,} \tag{15-22}$$

and the distance from the origin to the ordinate at the center of the kth interval is

$$x = k\frac{2\pi}{m} - \frac{1}{2}\left(\frac{2\pi}{m}\right) = \left(k - \frac{1}{2}\right)\frac{2\pi}{m}\text{ radians.} \tag{15-23}$$

We realize that equation (15-19) expresses merely the area under the curve. This can be found simply by the use of a planimeter or by some such method as counting squares. The graphical evaluation of B_0 thus is obvious. In this case, the wave is half above and half below the axis, with the result that $B_0 = 0$.

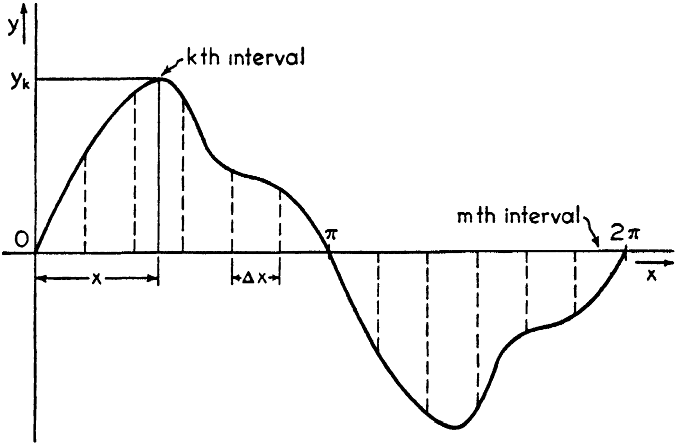


Fig. 15.07A. Construction of ordinates for graphical analysis.

The situation is not quite so simple for the A_n and B_n integrals, however. We must perform graphically the operations indicated by their equations.

Let us rewrite equation (15-20) as a summation:

$$A_n = \frac{1}{\pi} \sum_{x=0}^{x=2\pi} y \sin nx \cdot \Delta x. \tag{15-24}$$

Substituting (15-22) and (15-23) and inserting appropriate limits yields

$$A_n = \frac{1}{\pi} \sum_{k=1}^{k=m} y_k \sin n\left(k - \frac{1}{2}\right)\frac{2\pi}{m} \cdot \frac{2\pi}{m}, \tag{15-25}$$

or

▶ $$A_n = \frac{2}{m} \sum_{k=1}^{k=m} y_k \sin n\left(k - \frac{1}{2}\right)\frac{2\pi}{m}. \tag{15-26}$$

In similar manner,

$$B_n = \frac{1}{\pi} \sum_{x=0}^{x=2\pi} y \cos nx \cdot \Delta x, \tag{15-27}$$

leading to

$$\blacktriangleright \qquad B_n = \frac{2}{m} \sum_{k=1}^{k=m} y_k \cos n \left(k - \frac{1}{2}\right) \frac{2\pi}{m}. \qquad (15\text{-}28)$$

It is apparent that, in general, the larger the number of ordinates, *m*, the more nearly exact the values obtained for the coefficients.

Since equations (15-26) and (15-28) both call for the summation of series of terms, each of which must be individually computed, it becomes apparent that some form of tabular solution can be employed to advantage. A more or less standard form of analyzing table has been evolved for this purpose. As has been shown in a previous article, waves which have symmetrical positive and negative loops cannot contain even harmonics. Since most waves are of this type, many of these tables are set up only for odd harmonics. In the following pages, a tabular solution is employed for Example 15.07. The numerical values in the first four columns are standard for ordinates spaced every 5°. The numerical values in the last five columns apply only to the specific wave being analyzed and depend on measurements made on the wave. In this case, they apply to the wave of Fig. 15.07B. Since the wave consists of odd harmonics only, the two halves are alike except for polarity, and the summation need be taken over only the first 180°. The total summation over 360° can be obtained by multiplying that for the first 180° by 2. For this reason, only $(m/2)$ ordinates—covering only 180°—are taken. With such small intervals, employing ordinates at the end of each interval and using the angles corresponding to them, rather than employing the mid-ordinates and their angles, introduces an error which is negligible. Because of its greater simplicity, this scheme, too, is customarily employed in practice. The following example should help to clarify these ideas.

Example 15.07. Given the experimentally determined wave form shown in Fig. 15.07B, find, by graphical analysis, the Fourier equation for this wave. The wave is symmetrical, with only the positive loop shown.

(a) Ordinates are constructed every 5° as indicated in the figure. Since the wave is symmetrical, only the positive loop is needed, and only odd harmonics need be considered.

(b) Each ordinate is measured, and these values are entered in the *y* columns of the analyzing tables on the following pages.

(c) The products indicated by equations (15-26) and (15-28) are computed and entered in the tables, and the summations are found. From these, the values of the *A*'s and the *B*'s are found by the above-mentioned

TABLE 5
GRAPHICAL ANALYSIS OF NONSINUSOIDAL WAVE
Fundamental

1	2	3	4	5	6	7	8	9
Ordinate No.	Angle x	Sin x	Cos x	Meas. Ordinate y	Products y sin x		Products y cos x	
					+	−	+	−
1	5°	0.0872	0.9962	1.23	0.11	. .	1.23	.....
2	10	0.1736	0.9848	2.26	0.39	. .	2.22	.....
3	15	0.2588	0.9659	3.04	0.79	. .	2.93	.....
4	20	0.3420	0.9397	3.53	1.21	. .	3.32	.....
5	25	0.4226	0.9063	3.75	1.58	. .	3.40	.....
6	30	0.5000	0.8660	3.77	1.89	. .	3.26	.....
7	35	0.5736	0.8192	3.67	2.10	. .	3.01	.....
8	40	0.6428	0.7660	3.55	2.28	. .	2.72	.....
9	45	0.7071	0.7071	3.54	2.50	. .	2.50	.....
10	50	0.7660	0.6428	3.70	2.84	. .	2.38	.....
11	55	0.8192	0.5736	4.12	3.37	. .	2.36	.....
12	60	0.8660	0.5000	4.79	4.15	. .	2.40	.....
13	65	0.9063	0.4226	5.71	5.18	. .	2.41	.....
14	70	0.9397	0.3420	6.81	6.39	. .	2.33	.....
15	75	0.9659	0.2588	7.99	7.71	. .	2.06	.....
16	80	0.9848	0.1736	9.14	9.00	. .	1.59	.....
17	85	0.9962	0.0872	10.18	10.12	. .	0.89	.....
18	90	1.0000	0.0000	11.00	11.00	. .	0.00	0.00
19	95	0.9962	−0.0872	11.56	11.50	. .	.....	1.01
20	100	0.9848	−0.1736	11.84	11.63	. .	.....	2.06
21	105	0.9659	−0.2588	11.85	11.43	. .	.....	3.06
22	110	0.9397	−0.3420	11.64	10.91	. .	.....	3.98
23	115	0.9063	−0.4226	11.27	10.20	. .	.....	4.76
24	120	0.8660	−0.5000	10.79	9.34	. .	.....	5.40
25	125	0.8192	−0.5736	10.28	8.40	. .	.....	5.88
26	130	0.7660	−0.6428	9.74	7.46	. .	.....	6.26
27	135	0.7071	−0.7071	9.19	6.50	. .	.....	6.50
28	140	0.6428	−0.7660	8.62	5.54	. .	.....	6.60
29	145	0.5736	−0.8192	7.98	4.58	. .	.....	6.54
30	150	0.5000	−0.8660	7.23	3.62	. .	.....	6.25
31	155	0.4226	−0.9063	6.34	2.68	. .	.....	5.74
32	160	0.3420	−0.9397	5.28	1.81	. .	.....	4.96
33	165	0.2588	−0.9659	4.07	1.05	. .	.....	3.93
34	170	0.1736	−0.9848	2.74	0.47	. .	.....	2.69
35	175	0.0872	−0.9962	1.36	0 12	. .	.....	1.35
36	180	0.0000	−1.0000	0.00	0.00	. .	.....	0.00
					179.85	0	41.01	−76.97
					179.85		−35.96	

$$A_1 = \frac{179.85 \times 2}{36} = 9.990,$$

$$B_1 = \frac{-35.96 \times 2}{36} = -1.995,$$

$$C_1 = 9.99 - j1.995 = 10.19\underline{/-11.3^\circ}.$$

TABLE 5 (*Continued*)
Third Harmonic

1	2	3	4	5	6	7	8	9
Ordinate No.	Angle x	Sin $3x$	Cos $3x$	Meas. Ordinate y	Products y sin $3x$		Products y cos $3x$	
					+	−	+	−
1	5°	0.2588	0.9659	1.23	0.32	....	1.19	....
2	10	0.5000	0.8660	2.26	1.13	....	1.95	....
3	15	0.7071	0.7071	3.04	2.15	....	2.15	....
4	20	0.8660	0.5000	3.53	3.06	....	1.77	....
5	25	0.9659	0.2588	3.75	3.62	....	0.97	....
6	30	1.0000	0.0000	3.77	3.77	....	0.00	0.00
7	35	0.9659	−0.2588	3.67	3.54	....	.....	0.95
8	40	0.8660	−0.5000	3.55	3.08	....	.....	1.78
9	45	0.7071	−0.7071	3.54	2.50	....	.....	2.50
10	50	0.5000	−0.8660	3.70	1.85	....	.....	3.21
11	55	0.2588	−0.9659	4.12	1.07	....	.....	3.98
12	60	0.0000	−1.0000	4.79	0.00	0.00	.....	4.79
13	65	−0.2588	−0.9659	5.71	.....	1.48	.....	5.52
14	70	−0.5000	−0.8660	6.81	.....	3.41	.....	5.90
15	75	−0.7071	−0.7071	7.99	.....	5.65	.....	5.65
16	80	−0.8660	−0.5000	9.14	.....	7.91	.....	4.56
17	85	−0.9659	−0.2588	10.18	.....	9.84	.....	2.64
18	90	−1.0000	0.0000	11.00	.....	11.00	0.00	0.00
19	95	−0.9659	0.2588	11.56	.....	11.17	2.99	....
20	100	−0.8660	0.5000	11.84	.....	10.26	5.92	....
21	105	−0.7071	0.7071	11.85	.....	8.38	8.38	....
22	110	−0.5000	0.8660	11.64	.....	5.82	10.10	....
23	115	−0.2588	0.9659	11.27	.....	2.92	10.90	....
24	120	0.0000	1.0000	10.79	0.00	0.00	10.79	....
25	125	0.2588	0.9659	10.28	2.66	....	9.91	....
26	130	0.5000	0.8660	9.74	4.86	....	8.45	....
27	135	0.7071	0.7071	9.19	6.50	....	6.50	....
28	140	0.8660	0.5000	8.62	7.46	....	4.31	....
29	145	0.9659	0.2588	7.98	7.71	....	2.07	....
30	150	1.0000	0.0000	7.23	7.23	....	0.00	0.00
31	155	0.9659	−0.2588	6.34	6.13	....	.....	1.64
32	160	0.8660	−0.5000	5.28	4.58	....	.....	2.64
33	165	0.7071	−0.7071	4.07	2.87	....	.....	2.87
34	170	0.5000	−0.8660	2.74	1.37	....	.....	2.65
35	175	0.2588	−0.9659	1.36	0.35	....	.....	1.31
36	180	0.0000	−1.0000	0.00	0.00	....	.....	0.00
					77.81	−77.84	88.35	−52.59
					−0.03		+35.96	

$$A_3 = \frac{-0.03 \times 2}{36} = -0.00167,$$

$$B_3 = \frac{35.96 \times 2}{36} = 1.995,$$

$$\mathbf{C}_3 = 0.00167 + j1.995 = 1.995\underline{/90°}.$$

TABLE 5 (*Continued*)
Fifth Harmonic

1	2	3	4	5	6	7	8	9
Ordinate No.	Angle x	Sin $5x$	Cos $5x$	Meas. Ordinate y	Products y sin $5x$		Products y cos $5x$	
					+	−	+	−
1	5°	0.4226	0.9063	1.23	0.52	....	1.11	.....
2	10	0.7660	0.6428	2.26	1.73	....	1.45	.....
3	15	0.9659	0.2588	3.04	2.94	....	0.79	.....
4	20	0 9848	−0.1736	3.53	3.47	....	.....	0.61
5	25	0.8192	−0.5736	3.75	3.07	....	.....	2.15
6	30	0.5000	−0.8660	3.77	1.89	....	.....	3.26
7	35	0.0872	−0.9962	3.67	0.32	....	.....	3.66
8	40	−0.3420	−0.9397	3.55	.....	1.22	.....	3 34
9	45	−0.7071	−0.7071	3.54	.....	2.50	.....	2.50
10	50	−0.9397	−0.3420	3.70	.....	3.48	.....	1.27
11	55	−0.9962	0.0872	4.12	.....	4.10	0.36	.....
12	60	−0.8660	0.5000	4.79	.....	4.15	2.40	.....
13	65	−0.5736	0.8192	5.71	.....	3.27	4.66	.....
14	70	−0.1736	0.9848	6.81	.....	1.18	6.70	.....
15	75	0.2588	0.9659	7.99	2.07	....	7.71	.....
16	80	0.6428	0.7660	9.14	5.86	....	7.00	.....
17	85	0.9063	0.4226	10.18	9.21	....	4.31	.....
18	90	1.0000	0.0000	11.00	11.00	....	0.00	0.00
19	95	0.9063	−0.4226	11.56	10.46	....	.....	4.89
20	100	0.6428	−0.7660	11.84	7.60	....	.....	9.09
21	105	0.2588	−0.9659	11.85	3.06	....	.....	11.41
22	110	−0.1736	−0.9848	11.64	.....	2.02	.....	11.45
23	115	−0.5736	−0.8192	11.27	.....	6.46	.....	9.22
24	120	−0.8660	−0.5000	10.79	.....	9.35	.....	5.39
25	125	−0.9962	−0.0872	10.28	.....	10.22	.....	0.90
26	130	−0.9397	0.3420	9.74	.....	9.15	3.33	.....
27	135	−0.7071	0.7071	9.19	.....	6.50	6.50	.....
28	140	−0.3420	0.9397	8.62	.....	2.95	8.10	.....
29	145	0.0872	0.9962	7 98	0.69	....	7.96	.....
30	150	0.5000	0.8660	7.23	3.61	....	6.26	.....
31	155	0.8192	0.5736	6.34	5.18	....	3.64	.....
32	160	0.9848	0.1736	5.28	5.19	....	0.92	.....
33	165	0.9659	−0.2588	4.07	3.93	....	.....	1.06
34	170	0.7660	−0.6428	2.74	2.10	....	.....	1.76
35	175	0.4226	−0.9063	1.36	0.57	....	.....	1.23
36	180	0.0000	−1.0000	0.00	0.00	....	.....	0.00
					84.67	−66.55	73.20	−73.20
					18.12		0	

$$A_5 = \frac{18.12 \times 2}{36} = 1.005,$$

$$B_5 = \frac{0 \times 2}{36} = 0,$$

$$\mathbf{C}_5 = 1.005 + j0 = 1.005\underline{/0^\circ}.$$

TABLE 5 (*Continued*)
Seventh Harmonic

1	2	3	4	5	6	7	8	9
Ordinate No	Angle x	Sin $7x$	Cos $7x$	Meas Ordinate y	Products y sin $7x$		Products y cos $7x$	
					+	−	+	−
1	5°	0 5736	0 8192	1 23	0 71		1 01	
2	10	0 9397	0 3420	2 26	2 12		0 77	
3	15	0 9659	−0 2588	3 04	2 94			0 79
4	20	0 6428	−0 7660	3 53	2 27			2 70
5	25	0 0872	−0 9962	3 75	0 33			3 74
6	30	−0 5000	−0 8660	3 77		1 89		3 26
7	35	−0 9063	−0 4226	3 67		3 33		1 55
8	40	−0 9848	0 1736	3 55		3 50	0 62	
9	45	−0 7071	0 7071	3 54		2 50	2 50	
10	50	−0 1736	0 9848	3 70		0 64	3 64	
11	55	0 4226	0 9063	4 12	1 74		3 73	
12	60	0 8660	0 5000	4 79	4 15		2 40	
13	65	0 9962	−0 0872	5 71	5 68			0 50
14	70	0 7660	−0 6428	6 81	5 22			4 37
15	75	0 2588	−0 9659	7 99	2 06			7 71
16	80	−0 3420	−0 9397	9 14		3 12		8 56
17	85	−0 8192	−0 5736	10 18		8 34		5 83
18	90	−1 0000	0 0000	11 00		11 00	0 00	0 00
19	95	−0 8192	0 5736	11 56		9 45	6 62	
20	100	−0 3420	0 9397	11 84		4 05	11 10	
21	105	0 2588	0 9659	11 85	3 07		11 43	
22	110	0 7660	0 6428	11 64	8 92		7 49	
23	115	0 9962	0 0872	11 27	11 21		0 98	
24	120	0 8660	−0 5000	10 79	9 35			5 39
25	125	0 4226	−0 9063	10 28	4 35			9 13
26	130	−0 1736	−0 9848	9 74		1 69		9 59
27	135	−0 7071	−0 7071	9 19		6 50		6 50
28	140	−0 9848	−0 1736	8 62		8 49		1 50
29	145	−0 9063	0 4226	7 98		7 24	3 38	
30	150	−0 5000	0 8660	7 23		3 61	6 26	
31	155	0 0872	0 9962	6 34	0 55		6 32	
32	160	0 6428	0 7660	5 28	3 40		4 05	
33	165	0 9659	0 2588	4 07	3 93		1 05	
34	170	0 9397	−0 3420	2 74	2 57			0 94
35	175	0 5736	−0 8192	1 36	0 78			1 11
36	180	0 0000	−1 0000	0 00	0 00		0 00	0 00
					75 35	−75 35	73 35	−73 35
					0		0	

$A_7 = 0,$
$B_7 = 0,$
$C_7 = 0.$

equations. The equation of this particular wave may then be written in the form of equation (15-1), with the even harmonic terms omitted.

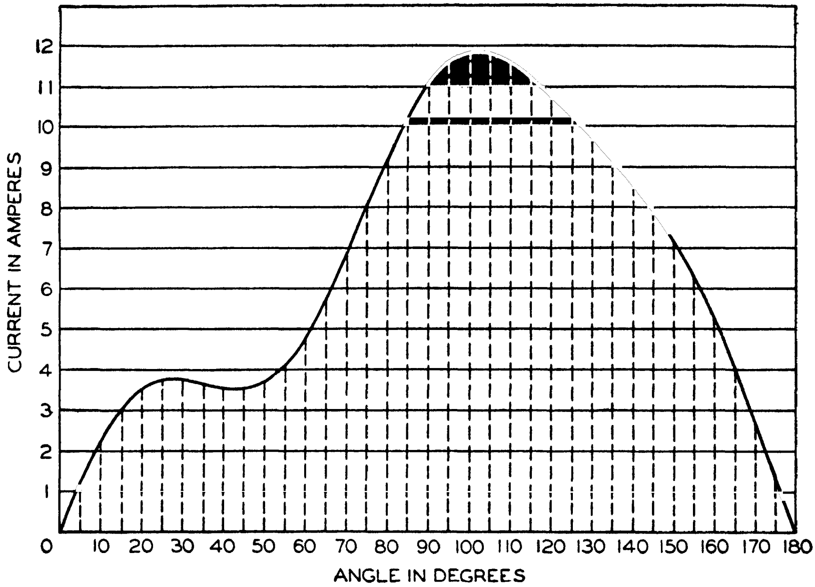


Fig. 15.07B. Positive half-cycle of a symmetrical non-sinusoidal wave.

Utilizing the values obtained from the foregoing tables, we may write the following equations for the wave of Fig. 15.07B:

$$y = 9.99 \sin \omega t - 1.995 \cos \omega t + 1.995 \cos 3\omega t + 1.005 \sin 5\omega t,$$

or $$y = 10.19 \sin (\omega t - 11.3°) + 1.995 \sin (3\omega t + 90°) + 1.005 \sin 5\omega t.$$

15.08. Fischer-Hinnen Method of Analysis

A method of wave analysis which reduces to a minimum the labor of determining the Fourier coefficients of nonsinusoidal wave components was first described in 1901. This method, known as the *Fischer-Hinnen method* (or *selected-ordinate method*) of analysis, is based on two easily demonstrated mathematical principles.

To demonstrate the first of these principles, let us erect m ordinates dividing into m equal intervals the 2π radians making up one complete cycle of the given wave. Now let us consider the nth harmonic. We shall measure those ordinates of this harmonic which occur at the beginning of each of these m intervals. These ordinates can be denoted as $y_{m1}, y_{m2}, y_{m3}, \ldots y_{mk}, \ldots y_{mm},$

where the first subscript, m, indicates the number of intervals into which the original wave was divided, and the second subscript indicates the position of the ordinate. Let us also denote a as the initial phase angle of the harmonic *measured on the fundamental scale,* and C_n as the amplitude of the harmonic. Figure 15.08A illustrates these definitions.

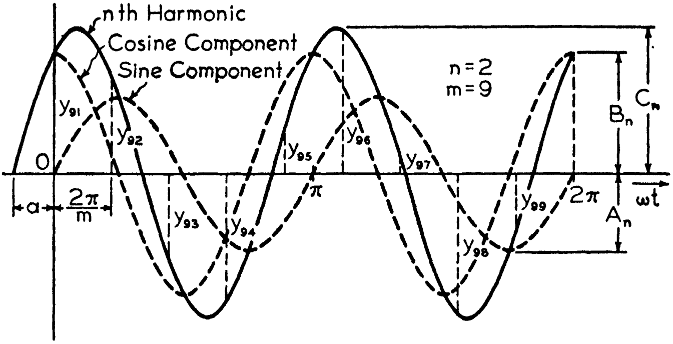


Fig. 15.08A. Fischer-Hinnen notation for the nth harmonic.

If the m ordinates of this nth harmonic are added algebraically, the sum can be expressed as follows:

$$\begin{aligned}\sum_{k=1}^{k=m} y_{mk} &= C_n \sin na + C_n \sin n\left(\frac{2\pi}{m} + a\right) + C_n \sin n\left(2\,\frac{2\pi}{m} + a\right)\\ &\quad + C_n \sin n\left(3\,\frac{2\pi}{m} + a\right) + \cdots\\ &\quad + C_n \sin n\left[(m-1)\,\frac{2\pi}{m} + a\right]\cdot\end{aligned} \tag{15-29}$$

Since $\sin(x + y) = (\sin x \cos y + \cos x \sin y)$, we can write

$$\begin{aligned}\sum_{k=1}^{k=m} y_{mk} &= C_n \sin na + C_n \sin n\,\frac{2\pi}{m} \cos na + C_n \cos n\,\frac{2\pi}{m} \sin na\\ &\quad + C_n \sin 2n\,\frac{2\pi}{m} \cos na + C_n \cos 2n\,\frac{2\pi}{m} \sin na + \cdots\\ &\quad + C_n \sin (m-1)n\,\frac{2\pi}{m} \cos na\\ &\quad + C_n \cos (m-1)n\,\frac{2\pi}{m} \sin na.\end{aligned} \tag{15-30}$$

Regrouping terms, we have

$$\sum_{k=1}^{k=m} y_{mk} = C_n \left\{ \sin na \left[1 + \cos n\frac{2\pi}{m} + \cos 2n\frac{2\pi}{m} + \cdots + \cos (m-1)n\frac{2\pi}{m} \right] \right\} + C_n \left\{ \cos na \left[\sin n\frac{2\pi}{m} + \sin 2n\frac{2\pi}{m} + \cdots + \sin (m-1)n\frac{2\pi}{m} \right] \right\}. \tag{15-31}$$

Since the sine of any whole number times 2π radians equals zero and since the cosine of any whole number times 2π radians equals unity, it is evident from (15-31) that *when (n/m) is a whole number:*

$$▶ \qquad \sum_{k=1}^{k=m} y_{mk} = m(C_n \sin na) = m(C_n \sin \theta_n), \tag{15-32}$$

where:

a = *initial phase angle of harmonic, on fundamental scale,*

$\theta_n = na$ = *initial phase angle of harmonic, on harmonic scale.*

From this equation we see that *for any harmonic such as the nth, the sum of m equally spaced ordinates is equal to m times the first ordinate when the order, n, of the harmonic is any whole number times the number of ordinates, m.* This is the first important mathematical principle.

The second important mathematical principle also can be derived simply, with the aid of the following trigonometrical formulas:

$$\cos x + \cos 2x + \cos 3x + \cdots + \cos (m-1)x = -\frac{1}{2} + \frac{\cos (m-1)x - \cos mx}{(1-\cos x)}, \tag{15-33}$$

and

$$\sin x + \sin 2x + \sin 3x + \cdots + \sin (m-1)x = \frac{\sin\left(\frac{m-1}{2}\right)x \cdot \sin\left(\frac{mx}{2}\right)}{\sin\left(\frac{x}{2}\right)}. \tag{15-34}$$

If $x = n(2\pi/m)$ and if (n/m) is not a whole number, equation (15-33) reduces to (-1) and equation (15-34) reduces to zero. Therefore, when (n/m) is *not* a whole number,

$$\blacktriangleright \quad \sum_{k=1}^{k=m} y_{mk} = C_n\{\sin na \cdot [1 - 1] + \cos na \cdot [0]\} = 0. \qquad (15\text{-}35)$$

In words, this second important mathematical principle states that *when the ratio of the crder, n, of the harmonic to the number of ordinates, m, is not a whole number, the sum of the ordinates is zero.*

The relations expressed by equations (15-32) and (15-35) can be used to determine the coefficients of the Fourier series for any periodic wave.

Referring to equation (15-1), it will be recalled that *each harmonic possesses two components which are in time quadrature: a sine component and a cosine component.* These were combined, in equation (15-4), into a single sine term for each frequency, having an initial phase angle, θ_n. Shifting the start of time measurement affects θ_n but has no effect on amplitude, C_n. Since the amplitudes, A_n and B_n, of the two components of C_n are (see Fig. 15.02A)

$$\blacktriangleright \quad A_n = C_n \cos \theta_n, \qquad (15\text{-}36)$$

and

$$\blacktriangleright \quad B_n = C_n \sin \theta_n, \qquad (15\text{-}37)$$

it follows that a shift in the start of time measurement does affect the values of A_n and B_n, although not the value of C_n. One point to remember is that regardless of where time measurement begins, there is always a sine component and a cosine component of each harmonic. The relative amplitudes may be affected by the start of time measurement, but not their resultant magnitude. A second point to remember is that *the first ordinate, y_{m1}, erected under the Fischer-Hinnen method, is the sum of the amplitudes of the cosine components of the various frequencies.* This point leads us to the next step in the development of this method.

Substituting (15-37) in (15-32), *when (n/m) is a whole number,* we have, for the nth harmonic,

$$\sum_{k=1}^{k=m} y_{mk} = (y_{m1} + y_{m2} + \cdots + y_{mm}) = mB_n. \qquad (15\text{-}38)$$

Substituting (15-37) in (15-35), *if (n/m) is not a whole number,* we have, for the nth harmonic,

$$\sum_{k=1}^{k=m} y_{mk} = (y_{m1} + y_{m2} + \cdots + y_{mm}) = 0 \times B_n = 0. \qquad (15\text{-}39)$$

In the preceding paragraph we learned that, for the resultant nonsinusoidal wave (assuming symmetry, and therefore no even harmonics),

$$y_{m1} = B_1 + B_3 + B_5 + \cdots + B_n. \qquad (15\text{-}40)$$

Now let us assume that we have divided one cycle of a given nonsinusoidal wave into three equal intervals by three ordinates. Then $m = 3$. From (15-39), it follows that the sum of the three distances along the ordinates to any harmonic which is not divisible by three is equal to zero. From (15-38), it follows that the sum of the three distances along the ordinates to any harmonic which is divisible by 3 is three times the amplitude of the cosine component of that harmonic. This could be restated to say that for any harmonic which is a multiple of the number of ordinates, 3, the amplitude of the cosine component of that harmonic is one-third the sum of the ordinates of that harmonic. It follows, then, *that the sum of the ordinates to the wave (which equals the sum of the sums of the ordinates to the harmonics) will be three times the sum of the amplitudes of the cosine terms of all harmonics which are multiples of three.* In mathematical form,

▶ $$(B_3 + B_9 + B_{15} + \cdots) = \tfrac{1}{3}(y_{31} + y_{32} + y_{33}), \qquad (15\text{-}41)$$

where the y's are the distances along the ordinates to the nonsinusoidal wave to be analyzed.

Similarly, if a nonsinusoidal wave is divided into five equal intervals by five equally spaced ordinates, y_{51}, y_{52}, y_{53}, y_{54}, and y_{55}, then,

▶ $$(B_5 + B_{15} + B_{25} + \cdots) = \tfrac{1}{5}(y_{51} + y_{52} + y_{53} + y_{54} + y_{55}). \qquad (15\text{-}42)$$

In like manner, division into seven intervals gives

▶ $$(B_7 + B_{21} + B_{35} + \cdots) = \tfrac{1}{7}(y_{71} + y_{72} + \cdots + y_{77}), \qquad (15\text{-}43)$$

and division into nine intervals gives

▶ $$(B_9 + B_{27} + B_{45} + \cdots) = \tfrac{1}{9}(y_{91} + y_{92} + \cdots + y_{99}). \qquad (15\text{-}44)$$

In practice, it is convenient to erect the first ordinate at the point where the curve to be analyzed crosses the axis of time. In this case, $y_{m1} = 0$, and equation (15-40) becomes

$$(B_1 + B_3 + B_5 + \cdots) = 0, \tag{15-45}$$

or

$$B_1 = (-B_3 - B_5 - \cdots). \tag{15-46}$$

In most current and potential waves met in practice, harmonics of higher order than the seventh are not important unless of sufficient magnitude that their presence can easily be detected by an inspection of the wave. Computations can then be restricted to the harmonics through the seventh except for cases of noticeable harmonics of higher order.

If harmonics of higher order than the seventh are negligible, from the foregoing equations we may write

▶ $$B_1 = -B_3 - B_5 - B_7, \tag{15-47}$$

▶ $$B_3 = \tfrac{1}{3}(y_{31} + y_{32} + y_{33}), \tag{15-48}$$

▶ $$B_5 = \tfrac{1}{5}(y_{51} + y_{52} + \cdots + y_{55}), \tag{15-49}$$

▶ $$B_7 = \tfrac{1}{7}(y_{71} + y_{72} + \cdots + y_{77}). \tag{15-50}$$

If the ninth harmonic is present as well as the third, fifth, and seventh, we may write

▶ $$B_9 = \tfrac{1}{9}(y_{91} + y_{92} + \cdots + y_{99}), \tag{15-51}$$

and then

$$B_3 + B_9 = \tfrac{1}{3}(y_{31} + y_{32} + y_{33}), \tag{15-52}$$

or

▶ $$B_3 = \tfrac{1}{3}(y_{31} + y_{32} + y_{33}) - B_9. \tag{15-53}$$

When the approximate equations numbered (15-47) through (15-50) are employed to analyze a wave, the base line may be appropriately divided to detect suspected higher harmonics. If they exist in appreciable magnitude, correction must be made for them by the method indicated for correcting for the ninth in equations (15-51) through (15-53).

The foregoing development provides means for finding the amplitudes of the cosine components of the various harmonics of a nonsinusoidal wave. We also must know the amplitudes of the sine components of those harmonics. These can be found by the same method applied to suitably chosen different ordinates.

For each harmonic, the maximum value of the sine term occurs one quarter of a cycle (measured on the scale of that harmonic) from the initial ordinate, y_{m1}. If the three ordinates of the wave previously mentioned as an example are shifted in a lagging direction through an angle of 90° measured on the third harmonic scale, then a series of equations similar to those developed for evaluating the B's can be developed for evaluating the A's. One major difference which will exist is in the fact that the algebraic signs of the series of coefficients will alternate in the equations corresponding to (15-41) through (15-44), as follows:

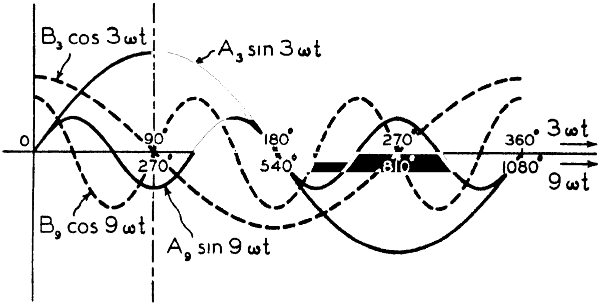


Fig. 15.08B. Effect of shifting vertical axis.

▶ $$(A_3 - A_9 + A_{15} - \cdots) = \tfrac{1}{3}(y'_{31} + y'_{32} + y'_{33}), \quad (15\text{-}54)$$

and

▶ $$(A_5 - A_{15} + A_{25} - \cdots) = \tfrac{1}{5}(y'_{51} + y'_{52} + \cdots + y'_{55}). \quad (15\text{-}55)$$

Also,

▶ $$(A_7 - A_{21} + A_{35} - \cdots) = \tfrac{1}{7}(y'_{71} + y'_{72} + \cdots + y'_{77}), \quad (15\text{-}56)$$

and

▶ $$(A_9 - A_{27} + A_{45} - \cdots) = \tfrac{1}{9}(y'_{91} + y'_{92} + \cdots + y'_{99}). \quad (15\text{-}57)$$

The reason for this alternation of sign arises from the fact that a shift of 90° of the third harmonic corresponds to a shift of 270° of the ninth harmonic, a shift of 450° of the fifteenth harmonic, and so on, thus throwing maximum values of waves of successively higher odd harmonic order alternately above and below the axis. Figure 15.08B shows this relationship for the third and ninth harmonics.

If the harmonics above the seventh are negligible,

▶ $$A_3 = \tfrac{1}{3}(y'_{31} + y'_{32} + y'_{33}), \tag{15-58}$$

▶ $$A_5 = \tfrac{1}{5}(y'_{51} + y'_{52} + \cdots + y'_{55}), \tag{15-59}$$

and

▶ $$A_7 = \tfrac{1}{7}(y'_{71} + y'_{72} + \cdots + y'_{77}). \tag{15-60}$$

If the ninth harmonic also is present, equation (15-58) must be corrected for its presence, as follows:

▶ $$A_9 = \tfrac{1}{9}(y'_{91} + y'_{92} + \cdots + y'_{99}) \tag{15-61}$$

and, by equation (15-54),

▶ $$A_3 = \tfrac{1}{3}(y'_{31} + y'_{32} + y'_{33}) + A_9. \tag{15-62}$$

The method of correcting for other higher-order harmonics, if they are present, is the same as for the ninth.

If the first ordinate, y_{m1}, of the group of ordinates for the cosine series is shifted through a negative angle of 90 fundamental degrees, its length becomes

$$Y'_{m1} = A_1 - A_3 + A_5 - A_7 + \cdots, \tag{15-63}$$

or

▶ $$A_1 = y'_{m1} - (-A_3 + A_5 - A_7 + A_9 - \cdots). \tag{15-64}$$

Equations (15-58) through (15-64) can be used in evaluating the coefficients of the sine components of the various harmonics of a nonsinusoidal wave.

When a wave is symmetrical, the positive and negative half cycles are identical except for algebraic sign; and it is necessary, in erecting ordinates for analysis, to work with only the first half cycle, since the length of any given ordinate for the second half cycle may be determined from a suitably placed ordinate in the first half cycle.

This possibility can be observed readily by reference to Fig. 15.08C, which depicts the ordinates needed to measure the cosine component of the third harmonic of a wave consisting of a fundamental and a third harmonic.

Let the ordinates marked 1, 2, and 3 at their upper ends be the ordinates for determining the cosine component of the third harmonic. Then we can write, since there are no higher multiples of the third harmonic present,

$$B_3 = \tfrac{1}{3}(y_{31} + y_{32} + y_{33}). \qquad (15\text{-}65)$$

Now, instead of dividing the whole wave into n ($=3$) intervals, *let the first half of the wave be divided into n intervals.* This corresponds to dividing the entire wave into $2n$ ($=6$) intervals. The encircled numbers at the bottoms of the ordinates in Fig. 15.08C indicate these $2n$ ordinates.

It is apparent from the figure that the second of the $2n$ ($=6$) ordinates is equal in magnitude but opposite in sign to the third of the n ($=3$) ordinates originally constructed. Therefore, equation (15-65) can be rewritten:

$$B_3 = \tfrac{1}{3}(y_{61} - y_{62} + y_{63}). \qquad (15\text{-}66)$$

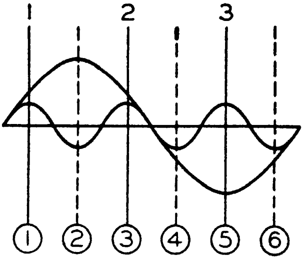


Fig. 15.08C. Ordinates for measurement of third harmonic.

If, now, the wave contains also multiples of the third harmonic, we may write

$$(B_3 + B_9 + B_{15} + \cdots) = \tfrac{1}{3}(y_{61} - y_{62} + y_{63}). \qquad (15\text{-}67)$$

In similar manner, if the fifth (or multiples of the fifth) harmonic is desired, we would divide the first half wave into five intervals (corresponding to dividing the entire wave into ten intervals). Then we could write

$$(B_5 + B_{15} + B_{25} + \cdots) = \tfrac{1}{5}(y_{101} - y_{102} + y_{103} - y_{104} + y_{105}). \qquad (15\text{-}68)$$

Likewise, after dividing the first half wave into seven parts, we could write

$$(B_7 + B_{21} + B_{35} + \cdots) = \tfrac{1}{7}(y_{141} - y_{142} + y_{143} - y_{144} + y_{145} - y_{146} + y_{147}). \qquad (15\text{-}69)$$

In general, we can state that if half of a symmetrical wave is divided into n equal intervals by n ordinates in the determination of the nth harmonic, the sum of the $(n - 1)/2$ ordinates with even numbers must be subtracted from the sum of the $(n + 1)/2$ ordinates with odd numbers in the determination of the equivalent sum of n ordinates equally spaced over a whole cycle.

It is possible to obtain from a single loop of a symmetrical wave not only the B's, or coefficients of the cosine components, but also the A's or coefficients of the sine components. Since we chose the number of intervals, m, equal to the order, n, of the harmonics, the n ordinates of the half wave under study all occur at the positive and negative peaks of the cosine component of the nth harmonic. Another set of n ordinates exactly centered in the intervals created by the original set of n ordinates can be erected and used in determining the values of the A's by equations (15-58) through (15-64).

The following convenient rule can be stated in summary of a method for determining both the A's and the B's for a symmetrical wave: *When utilizing the Fischer-Hinnen method to analyze a wave containing only odd harmonics, divide the positive loop into* $2n$ *equal intervals by* $2n$ *equally spaced ordinates, where* n *is the order of the highest harmonic being sought and the first ordinate is placed where the loop first crosses the axis of time. Those ordinates with odd numbers are used in determining the coefficients of the cosine components of the harmonics, whereas those with even numbers are used for determining the coefficients of the sine components of the harmonics. In utilizing these ordinates, those bearing odd numbers are used with alternate signs for determining* B_n*, and those with even numbers are used with alternate signs for determining* A_n*, the first ordinate being considered positive in each case. Thus,*

▶
$$B_n = \frac{1}{n}(y_1 - y_3 + y_5 - y_7 + \cdot \cdot \cdot + y_{(2n-1)}), \qquad (15\text{-}70)$$

and

▶
$$A_n = \frac{1}{n}(y_2 - y_4 + y_6 - y_8 + \cdot \cdot \cdot + y_{2n}). \qquad (15\text{-}71)$$

When employing the Fischer-Hinnen method for computing the B and A coefficients of a nonsinusoidal wave, it is necessary to keep in mind the fact that coefficients are obtained not singly but in groups. For this reason, it is always necessary to determine by inspection the highest order harmonic that is present in appreciable magnitude, in order that the coefficients given by approximate equations (15-47) through (15-50) and (15-58) through (15-60) may be corrected for the presence of higher-order harmonics.

The following numerical example should help to clarify these ideas.

Example 15.08. Analyze the wave of Figure 15.07B by the Fischer-Hinnen method.

Inasmuch as the wave is symmetrical, only the half cycle shown need be analyzed, and only odd harmonics need be investigated. Harmonics above the seventh will be ignored. In the investigation of the *n*th harmonic, $2n$ evenly spaced ordinates are erected in the half wave, with the first one at the point where the wave crosses the axis of time with positive slope. The results of measurement of these ordinates for each harmonic follow.

THIRD HARMONIC

$$(m = 2n = 6)$$

$$\begin{array}{ll@{\qquad\qquad}ll}
\multicolumn{2}{c}{\textit{Positive Signs}} & \multicolumn{2}{c}{\textit{Negative Signs}}\\
y_1 = 0.00 & y_2 = 3.77 & y_3 = 4.79 & y_4 = 11.00\\
y_5 = 10.79 & y_6 = 7.23 & & \\
\hline
10.79 & 11.00 & 4.79 & 11.00
\end{array}$$

By equation (15-70),

$$B_3 = \tfrac{1}{3}(y_1 - y_3 + y_5) = \tfrac{1}{3}(10.79 - 4.79) = \tfrac{1}{3}(6.00) = 2.00.$$

By equation (15-71),

$$A_3 = \tfrac{1}{3}(y_2 - y_4 + y_6) = \tfrac{1}{3}(11.00 - 11.00) = 0.$$

Then

$$\mathbf{C}_3 = 0.00 + j2.00 = 2.00\underline{/90^\circ}.$$

FIFTH HARMONIC

$$(m = 2n = 10)$$

$$\begin{array}{ll@{\qquad\qquad}ll}
\multicolumn{2}{c}{\textit{Positive Signs}} & \multicolumn{2}{c}{\textit{Negative Signs}}\\
y_1 = 0.00 & y_2 = 3.39 & y_3 = 3.63 & y_4 = 4.03\\
y_5 = 7.29 & y_6 = 11.00 & y_7 = 11.75 & y_8 = 10.16\\
y_9 = 8.09 & y_{10} = 4.80 & & \\
\hline
15.38 & 19.19 & 15.38 & 14.19
\end{array}$$

$$B_5 = \tfrac{1}{5}(y_1 - y_3 + y_5 - y_7 + y_9) = \tfrac{1}{5}(15.38 - 15.38) = 0.$$

$$A_5 = \tfrac{1}{5}(y_2 - y_4 + y_6 - y_8 + y_{10}) = \tfrac{1}{5}(19.19 - 14.19)$$

$$= \tfrac{1}{5}(5.00) = 1.00.$$

$$\mathbf{C}_5 = 1.00 + j0.00 = 1.00\underline{/0^\circ}.$$

SEVENTH HARMONIC

$$(m = 2n = 14)$$

$$\begin{array}{ll@{\qquad\qquad}ll}
\multicolumn{2}{c}{\textit{Positive Signs}} & \multicolumn{2}{c}{\textit{Negative Signs}}\\
y_1 = 0.00 & y_2 = 2.75 & y_3 = 3.78 & y_4 = 3.56\\
y_5 = 3.81 & y_6 = 5.58 & y_7 = 8.54 & y_8 = 10.99\\
y_9 = 11.90 & y_{10} = 11.10 & y_{11} = 9.89 & y_{12} = 8.43\\
y_{13} = 6.50 & y_{14} = 3.55 & & \\
\hline
22.21 & 22.98 & 22.21 & 22.98
\end{array}$$

$$B_7 = \tfrac{1}{7}(22.21 - 22.21) = 0.$$
$$A_7 = \tfrac{1}{7}(22.98 - 22.98) = 0.$$
$$\mathbf{C}_7 = 0 + j0.$$

FUNDAMENTAL

From equation (15-40),

$$B_1 = y_{m1} - (B_3 + B_5 + B_7) = 0 - (2 + 0 + 0) = -2,$$

and, from (15-64),

$$A_1 = y'_{m1} - (-A_3 + A_5 - A_7).$$

Since we are dealing with the fundamental, y'_{m1} is the ordinate at a position of 90 fundamental degrees lagging the position where the wave crosses the axis of time with positive slope. The ordinate at this point is 11.00. Then

$$A_1 = 11.00 - (-0 + 1.00 - 0) = 11.00 - 1.00 = 10.00,$$

and the following expression for $\mathbf{C}_1$ can be written:

$$\mathbf{C}_1 = 10 - j2 = 10.19\underline{/-11.3^\circ}.$$

The equation of the wave is, then, by (15-1),

$$y = 10.00 \sin \omega t - 2.00 \cos \omega t + 2.00 \cos 3\omega t + 1.00 \sin 5\omega t,$$

or, by (15-4),

$$y = 10.19 \sin (\omega t - 11.3^\circ) + 2.00 \sin (3\omega t + 90^\circ) + 1.00 \sin 5\omega t.$$

It will be observed that these equations agree with the answers obtained by the general graphical method. The saving in labor and time effected by the Fischer-Hinnen method is strikingly apparent.

15.09. Other Methods of Analysis

Other variations of the graphical method of analyzing a nonsinusoidal wave to determine its Fourier components have been developed in an effort to reduce the labor of making such analyses. These methods will not be discussed here, inasmuch as they offer, in general, very little if any advantage over the Fischer-Hinnen method.

An electronic laboratory instrument known as a *harmonic wave analyzer* is useful in laboratory analyses of wave forms. This device must, of course, be applied to the wave itself as it is being generated, and cannot be applied to an oscillogram. It also possesses the limitation of being able to find only the amplitudes—not the phase angles—of the various harmonics.

PROBLEMS

15-1. The emf generated in one phase of a three-phase generator is:

$$e_{ab} = 100 \sin (\omega t + 30°) + 20 \sin (3\omega t - 10°) + 5 \sin 5\omega t \text{ volts.}$$

The other two phases develop symmetrical emf's of the same wave form. Assuming the sequence e_{ab}-e_{bc}-e_{ca}, write the equations for e_{bc} and e_{ca}.

15-2. Shift the following wave through such an angle that the fundamental component has an initial phase angle of zero degrees:

$$e = 100 \sin (\omega t - 180°) + 50 \sin (5\omega t - 30°) + 10 \sin (7\omega t) \text{ volts.}$$

15-3. Shift the following wave through such an angle that the fifth harmonic has an initial phase angle of (-10) degrees:

$$i = 100 \sin (\omega t + 45°) + 75 \sin (3\omega t - 30°) + 50 \sin (5\omega t + 20°) \text{ amperes.}$$

15-4. Determine analytically whether the following waves possess half-wave symmetry, and then plot those waves:

(a) $e = 156 \sin (\omega t - 30°) + 28.3 \sin (3\omega t + 60°) - 10 \sin (5\omega t - 50°) + 6 \cos 7\omega t$ volts.

(b) $i = 7.8 \sin (\omega t + 60°) - 5 \sin (2\omega t + 30°) + 3 \sin (5\omega t - 50°) - 2 \sin (7\omega t + 180°)$ amperes.

15-5. Determine whether the waves of Prob. 15-4 possess "quarter-wave" symmetry.

15-6. (a) Write the equation of a nonsinusoidal wave which contains a fundamental and third, fifth, and seventh harmonics, and which possesses quarter-wave symmetry.

(b) Sketch the wave of (a).

15-7. Determine whether the following pairs of waves are of the same form:

$$e = 156 \sin (\omega t - 30°) + 28.3 \sin (3\omega t + 60°) - 10 \sin (5\omega t - 50°) \text{ volts,}$$
$$i = 7.8 \cos (\omega t - 120°) - 1.415 \sin (3\omega t + 120°) + 0.5 \cos (5\omega t + 50°) \text{ amperes.}$$

15-8. The following current flows in a linear circuit element:

$$i = 10 \sin \omega t + 3 \sin (3\omega t - 60°) - 0.7 \cos (5\omega t - 278°).$$

The third harmonic component of the resulting potential drop is:

$$v_3 = 21 \sin (3\omega t + 22°).$$

Write the equation of the entire wave of potential drop.

15-9. From zero to π radians, a certain wave of current has the value 6 amperes. From π to 2π radians it has the value (-12) amperes.

(a) Use the analytical method to determine through the fifth harmonic the Fourier series which represents this current as a function of angular displacement, α.

(b) Sketch the components of this wave, indicating the manner in which the components combine to approximate the original wave shape.

15-10. A certain *triangular wave* rises linearly from zero to its maximum value over the first quarter cycle, varies linearly to a numerically equally negative maximum value over the next *two* quarter cycles, and then varies linearly to zero over the remaining quarter cycle.

(a) Sketch this wave.

(b) Write, through the seventh harmonic, the Fourier series which represents this wave, expressing the various harmonic amplitudes in terms of the maximum value of the particular wave.

(c) Rewrite the answer to (b) in terms of the amplitude of the fundamental.

15-11. Repeat Prob. 15-10 for a *rectangular wave* which has maximum value for the first half cycle and a numerically equal negative maximum value for the second half cycle.

15-12. Write the following equation in terms of sine components only, one for each harmonic:

$v = 4.0 \sin \omega t - 3.0 \cos \omega t + 3 \sin 3\omega t + 10 \cos 3\omega t - 2 \sin 7\omega t + 5 \cos 7\omega t.$

15-13. Express the following wave in terms of its sine and cosine Fourier components:

$i = 5.0 \sin (\omega t - 36.9°) + 10.4 \sin (3\omega t + 73.3°) + 5.39 \sin (7\omega t + 111.8°).$

15-14. Determine the Fourier series which represents the unidirectional wave resulting from full-wave rectification of a sine wave. Figure 3.17A(b) represents such a wave.

15-15. Determine the Fourier series which represents the unidirectional wave resulting from half-wave rectification of a sine wave. The removal of alternate sine loops from the wave of Fig. 3.17A(b) yields this wave form.

15-16. The following table summarizes the data obtained by measurements on an oscillogram of a nonsinusoidal wave of potential:

Ordinate No.	Angle x	Meas. Ord. y	Ordinate No.	Angle x	Meas. Ord. y	Ordinate No.	Angle x	Meas. Ord. y
0	0°	0.0v	13	65°	82.7v	25	125°	65.0v
1	5	15.0	14	70	78.3	26	130	63.8
2	10	34.0	15	75	75.3	27	135	62.0
3	15	54.5	16	80	73.1	28	140	59.6
4	20	84.0	17	85	71.5	29	145	56.9
5	25	106.0	18	90	70.0	30	150	53.0
6	30	116.6	19	95	69.3	31	155	48.8
7	35	122.0	20	100	68.8	32	160	43.0
8	40	123.0	21	105	68.0	33	165	35.3
9	45	120.1	22	110	67.7	34	170	25.2
10	50	112.0	23	115	66.9	35	175	15.0
11	55	99.5	24	120	66.0	36	180	0.0
12	60	89.3	..	...	....	..	...	....

The wave possesses half-wave symmetry.

(a) Graph the wave.

(b) Determine by graphical analysis the Fourier equation for this wave through the seventh harmonic.

(c) Write the equation of the wave in terms of sine components only.

(d) Synthesize the components graphically and compare the resultant with the original wave.

15-17. (a) On coordinate paper draw an alternating wave whose positive and negative loops consist of semicircles.

(b) Determine by graphical analysis the Fourier equation for this wave through all significant harmonics. Express the amplitudes of the harmonics in terms of the radius of the semicircle, which must be, numerically, $\pi/2$.

(c) Synthesize the components graphically and compare the resultant with the original wave.

15-18. The following data were obtained from measurements made on the first quarter cycle of a current wave possessing both quarter-wave and half-wave symmetry:

Ordinate No.	Angle x	Meas. Ord. y	Ordinate No.	Angle x	Meas. Ord. y	Ordinate No.	Angle x	Meas. Ord. y
0	0°	0.0A	7	35°	3.8A	13	65°	12.2A
1	5	0.7	8	40	4.7	14	70	12.8
2	10	1.2	9	45	6.3	15	75	13.2
3	15	1.7	10	50	8.2	16	80	13.5
4	20	2.1	11	55	10.1	17	85	13.7
5	25	2.6	12	60	11.3	18	90	13.75
6	30	3.1						

(a) Graph the wave.

(b) Determine by graphical analysis the Fourier equation for this wave through the seventh harmonic.

(c) Synthesize the components graphically and compare the resultant with the original wave.

15-19. Analyze the wave of Prob. 15-16 by the Fischer-Hinnen method.

15-20. Analyze the wave of Prob. 15-17 by the Fischer-Hinnen method.

15-21. Analyze the wave of Prob. 15-18 by the Fischer-Hinnen method.

CHAPTER 16

NONSINUSOIDAL WAVES IN ELECTRIC CIRCUITS

16.01. Introduction

The first article of Chapter 15 pointed out many of the sources of nonsinusoidal waves in practical circuits and showed that such waves are not at all uncommon. The remainder of the chapter was devoted to methods of analyzing nonsinusoidal waves so that they might be accurately represented mathematically by Fourier series. It is the purpose of this chapter to demonstrate how, once such a Fourier series is determined for a given wave, the response of the circuit containing that wave may be determined.

16.02. Effective Value of a Nonsinusoidal Wave

It is logical to begin a study of circuit response to nonsinusoidal waves by investigating the response of a-c instruments to such waves. This means, of course, seeking a relation between the effective, or rms, value of the wave and the values of its harmonic components. The effective value of any wave, no matter how expressed, can be found through application of the "root-mean-square" formula, equation (8-7):

▶
$$I = \sqrt{\frac{1}{T}\int_0^T i^2\,dt}\cdot \tag{16-1}$$

The general Fourier representation of a nonsinusoidal alternating current is

▶
$$\begin{aligned} i = I_0 + I_{m_1}\sin(\omega t + \theta_{i_1}) + I_{m_2}\sin(2\omega t + \theta_{i_2}) \\ + I_{m_3}\sin(3\omega t + \theta_{i_3}) + \cdots + I_{m_n}\sin(n\omega t + \theta_{i_n}) + \cdots . \end{aligned} \tag{16-2}$$

The effective value of this current is, then, by (16-1),

$$I = \sqrt{\frac{1}{T}\int_0^T [I_0 + I_{m_1}\sin(\omega t + \theta_{i_1}) + I_{m_2}\sin(2\omega t + \theta_{i_2}) + \cdots]^2\,dt}. \tag{16-3}$$

This equation involves the integration of squared terms of the following type in the following manner:

$$\frac{1}{T}\int_0^T I_{m_n}^2 \sin^2(n\omega t + \theta_n)\,dt = \frac{I_{m_n}^2}{2} \tag{16-4}$$

It also involves the integration of products of terms of unlike frequency, as exemplified by the following relation in which n and k denote the two different frequencies:

$$\frac{1}{T}\int_0^T [I_{m_n}\sin(n\omega t + \theta_n) \times I_{m_k}\sin(k\omega t + \theta_k)]\,dt = 0. \tag{16-5}$$

Equation (16-3) becomes, upon application of these two results,

▶ $$I = \sqrt{I_0^2 + \frac{I_{m_1}^2 + I_{m_2}^2 + I_{m_3}^2 + \cdots + I_{m_n}^2 + \cdots}{2}}. \tag{16-6a}$$

In terms of effective values of the individual harmonics (recalling that the effective and maximum values of direct current are equal, whereas those of other frequencies are related by $\sqrt{2}$), (16-6a) becomes

▶ $$I = \sqrt{I_0^2 + I_1^2 + I_2^2 + I_3^2 + \cdots + I_n^2 + \cdots}. \tag{16-6b}$$

In similar manner,

▶ $$V = \sqrt{V_0^2 + \frac{V_{m_1}^2 + V_{m_2}^2 + V_{m_3}^2 + \cdots + V_{m_n}^2 + \cdots}{2}}, \tag{16-7a}$$

or

▶ $$V = \sqrt{V_0^2 + V_1^2 + V_2^2 + V_3^2 + \cdots + V_n^2 + \cdots}. \tag{16-7b}$$

Example 16.02. Find the effective value of the wave of Examples 15.08 and 15.09.

The Fourier expression for this wave was found to be

$$y = 10.19\sin(\omega t - 11.3°) + 2.00\sin(3\omega t + 90°) + 1.00\sin 5\omega t.$$

By (16-6a), the effective value is

$$Y = \sqrt{\frac{(10.19)^2 + (2.00)^2 + (1.00)^2}{2}} = \sqrt{\frac{109}{2}} = 7.38.$$

It is instructive to note that, since $C_n^2 = A_n^2 + B_n^2$, the same answer could just as easily have been obtained from the equation in the form of (15-1):

$$y = 10 \sin \omega t - 2 \cos \omega t + 2 \cos 3\omega t + 1 \sin 5\omega t,$$

from which

$$Y = \sqrt{\frac{(10)^2 + (-2)^2 + (2)^2 + (1)^2}{2}} = \sqrt{\frac{109}{2}} = 7.38.$$

16.03. Active Power Associated with Nonsinusoidal Waves

The general expression for average power in an a-c circuit is

▶
$$P = \frac{1}{T}\int_0^T vi\, dt. \tag{16-8}$$

The general forms of nonsinusoidal potential and current waves are

$$\begin{aligned} v = V_0 &+ V_{m_1} \sin (\omega t + \theta_{v_1}) + V_{m_2} \sin (2\omega t + \theta_{v_2}) + \cdots \\ &+ V_{m_n} \sin (n\omega t + \theta_{v_n}) + \cdots, \end{aligned} \tag{16-9}$$

and

$$\begin{aligned} i = I_0 &+ I_{m_1} \sin (\omega t + \theta_{i_1}) + I_{m_2} \sin (2\omega t + \theta_{i_2}) + \cdots \\ &+ I_{m_n} \sin (n\omega t + \theta_{i_n}) + \cdots. \end{aligned} \tag{16-10}$$

Then

$$\begin{aligned} P = \frac{1}{T}\int_0^T &\{[V_0 + V_{m_1} \sin (\omega t + \theta_{v_1}) + V_{m_2} \sin (2\omega t + \theta_{v_2}) + \cdots] \\ &\times [I_0 + I_{m_1} \sin (\omega t + \theta_{i_1}) + I_{m_2} \sin (2\omega t + \theta_{i_2}) + \cdots]\}\, dt. \end{aligned} \tag{16-11}$$

This equation involves product terms of like frequency of the following form:

$$\begin{aligned} \frac{1}{T}\int_0^T &[V_{m_n} \sin (n\omega t + \theta_{v_n}) \times I_{m_n} \sin (n\omega t + \theta_{i_n})]\, dt \\ &= \frac{V_{m_n} I_{m_n}}{2} \cos (\theta_{v_n} - \theta_{i_n}). \end{aligned} \tag{16-12}$$

It also involves product terms of unlike frequency of the following form:

$$\frac{1}{T}\int_0^T [V_{m_n} \sin (n\omega t + \theta_{v_n}) \times I_{m_k} \sin (k\omega t + \theta_{i_k})]\, dt = 0. \quad (16\text{-}13)$$

Equation (16-11) then becomes the following expression for average (and active) power with nonsinusoidal waves:

$$▶\quad P = \frac{V_{m_1}I_{m_1}}{2} \cos (\theta_{v_1} - \theta_{i_1}) + \frac{V_{m_2}I_{m_2}}{2} \cos (\theta_{v_2} - \theta_{i_2}) + \cdots + \frac{V_{m_n}I_{m_n}}{2} \cos (\theta_{v_n} - \theta_{i_n}) + \cdots . \quad (16\text{-}14)$$

In terms of effective values, this equation is

$$▶\quad P = V_1I_1 \cos (\theta_{v_1} - \theta_{i_1}) + V_2I_2 \cos (\theta_{v_2} - \theta_{i_2}) + \cdots + V_nI_n \cos (\theta_{v_n} - \theta_{i_n}) + \cdots . \quad (16\text{-}15)$$

If a given harmonic is present in one of the waves but absent from the other, it contributes nothing to the average power.

Example 16.03. If the potential drop

$$v = 141 \sin \omega t + 11 \sin (3\omega t + 30°) \text{ volts}$$

is produced by a current having the equation of the wave analyzed in the examples of Chapter 15 (the ordinate values in amperes), what is the average power being delivered to the circuit element across which the drop exists?

Since the current wave is

$$i = 10.19 \sin (\omega t - 11.3°) + 2 \sin (3\omega t + 90°) + 1 \sin 5\omega t \text{ amperes,}$$

we can write, by equation (16.14),

$$P = \frac{141 \times 10.19}{2} \cos (0° + 11.3°) + \frac{11 \times 2}{2} \cos (30° - 90°) + 0$$
$$= 704 + 5.5 + 0 = 709.5 \text{ watts.}$$

16.04. Reactive Power Associated with Nonsinusoidal Waves

"*The reactive power for the general case of an alternating current and an alternating potential difference where both have harmonics is the algebraic sum of the reactive powers corresponding to the sinusoidal harmonic components.*"[1] By equation (8-27), the reactive power,

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[1] *American Standard Definitions of Electrical Terms*, No. 05.21.050. New York: American Institute of Electrical Engineers, 1942.

Q, for the nonsinusoidal waves of potential and current expressed by equations (16-9) and (16-10) is

$$\blacktriangleright \quad Q = \frac{V_{m_1}I_{m_1}}{2}\sin(\theta_{v_1} - \theta_{i_1}) + \frac{V_{m_2}I_{m_2}}{2}\sin(\theta_{v_2} - \theta_{i_2}) + \cdots + \frac{V_{m_n}I_{m_n}}{2}\sin(\theta_{v_n} - \theta_{i_n}) + \cdots, \tag{16-16}$$

or

$$\blacktriangleright \quad Q = V_1I_1\sin(\theta_{v_1} - \theta_{i_1}) + V_2I_2\sin(\theta_{v_2} - \theta_{i_2}) + \cdots + V_nI_n\sin(\theta_{v_n} - \theta_{i_n}) + \cdots. \tag{16-17}$$

The similarity between this expression and the expression for average power, P, is evident. Here, too, if a given harmonic is present in one of the waves but is absent from the other, it contributes nothing to the reactive power associated with the waves.

EXAMPLE 16.04. Find the reactive power, Q, associated with the two waves of Example 16.03.

$$Q = \frac{141 \times 10.19}{2}\sin(11.3^\circ) + \frac{11 \times 2}{2}\sin(-60^\circ) = 123.8 \text{ vars (inductive)}.$$

16.05. Vector Power Associated with Nonsinusoidal Waves

"*The vector power when both the current and potential difference have harmonics is the vector sum of the vector powers of the harmonic components.*"[2] Mathematically,

$$\blacktriangleright \quad \begin{aligned} \mathbf{U}' &= \mathbf{U}_1' + \mathbf{U}_2' + \mathbf{U}_3' + \cdots + \mathbf{U}_n' + \cdots \\ &= \mathbf{V}_1\hat{\mathbf{I}}_1 + \mathbf{V}_2\hat{\mathbf{I}}_2 + \mathbf{V}_3\hat{\mathbf{I}}_3 + \cdots + \mathbf{V}_n\hat{\mathbf{I}}_n + \cdots \\ &= V_1I_1[\cos(\theta_{v_1} - \theta_{i_1}) + j\sin(\theta_{v_1} - \theta_{i_1})] \\ &\quad + V_2I_2[\cos(\theta_{v_2} - \theta_{i_2}) + j\sin(\theta_{v_2} - \theta_{i_2})] \\ &\quad + \cdots + V_nI_n[\cos(\theta_{v_n} - \theta_{i_n}) + j\sin(\theta_{v_n} - \theta_{i_n})] \\ &\quad + \cdots, \end{aligned} \tag{16-18}$$

and

$$\blacktriangleright \quad \mathbf{U}' = P + jQ = \sum_{n=0}^{n=\infty} P_n + j\sum_{n=0}^{n=\infty} Q_n. \tag{16-19}$$

Obviously, the magnitude of the vector power associated with two nonsinusoidal waves is

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[2] *Ibid.*, No. 05.21.055.

$$\blacktriangleright\quad U' = [(P_1 + P_2 + P_3 + \cdots + P_n + \cdots)^2 + (Q_1 + Q_2 + Q_3 + \cdots + Q_n + \cdots)]^{\frac{1}{2}}. \quad (16\text{-}20)$$

Example 16.05. Find the vector power, U', associated with the two waves of Example 16.03.

From the two preceding examples, $P = 709.5$ watts and $Q = 128.8$ vars. Then

$$\mathbf{U}' = P + jQ = 709.5 + j128.8 \text{ complex volt-amperes,}$$

and

$$U' = \sqrt{(709.5)^2 + (128.8)^2} = 721 \text{ volt-amperes.}$$

16.06. Apparent Power

We now have enough background to discuss the nature of *apparent power*, which may be defined as follows for a single-phase circuit: "*Apparent power at the two points of entry of a single-phase, two-wire circuit is equal to the product of the effective current in one conductor multiplied by the effective potential difference between the two points of entry.*"[3] Mathematically,

$$\blacktriangleright\quad U = VI = [(V_1^2 + V_2^2 + \cdots + V_n^2 + \cdots)(I_1^2 + I_2^2 + \cdots + I_n^2 + \cdots)]^{\frac{1}{2}}, \quad (16\text{-}21)$$

where:

U = *apparent power*, in **volt-amperes,**

V_n = *effective value of potential of the nth harmonic*, in **volts,**

I_n = *effective value of current of the nth harmonic*, in **amperes.**

Clearly, the apparent power is the same as the magnitude of the vector power, as defined by equation (16-20), *when both potential and current are sinusoidal.* The presence in either the potential or the current wave of harmonics not present in the other contributes to an increase of apparent power without contributing to an increase of vector power. Apparent power is, then, either equal to or greater than vector power.

Example 16.06. Find the apparent power, U, associated with the two waves of Example 16.03.

From Example 16.02, $I = 7.38$ amperes. The effective value of the potential is:

$$V = \sqrt{\frac{(141)^2 + (11)^2}{2}} = \frac{141.4}{\sqrt{2}} = 100 \text{ volts.}$$

---

[3] *Ibid.*, No. 05.21.060.

Then, from (16-21), the apparent power is:

$$U = VI = 100 \times 7.38 = 738 \text{ volt-amperes.}$$

16.07. Distortion Power

"*Distortion power is equal to the square root of the difference of the squares of the apparent power and the vector power.*"[4] Mathematically, the distortion power is

$$\blacktriangleright \qquad D = \sqrt{U^2 - (U')^2}, \qquad (16\text{-}22)$$

where: D = *distortion power*, in **distortion volt-amperes.**
If the waves of potential and current are sinusoidal, the distortion power is zero.

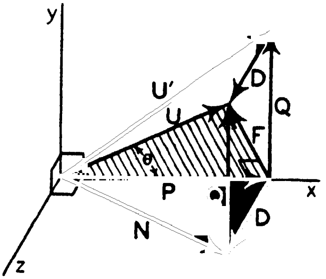
Example 16.07. Find the distortion power associated with the two waves of Example 16.03.

From the two previous examples, $U' = 721$ volt-amperes, and $U = 738$ volt-amperes. Then the distortion power is

$$D = \sqrt{(738)^2 - (721)^2} = 160.5 \text{ distortion volt-amperes.}$$

16.08. Geometric Power Diagram for a Single-phase Circuit

Figure 16.08 shows the geometric interpretation of the power quantities defined in the preceding articles. In this diagram, active power, reactive power, and distortion power are represented in the directions of three mutually perpendicular axes. Active power and reactive power may be either positive or negative, but distortion



P = active power
Q = reactive power
U' = vector power
U = apparent power
D = distortion power
F = fictitious power
N = non-reactive power

Fig. 16.08. Geometric power diagram for a single-phase circuit.

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[4] *Ibid.*, No. 05.21.065.

power is always considered as positive. In the case shown, all three are considered positive. Several other power quantities are defined by this figure and bear the names given in the key to the notation. Diagrams such as this are helpful in picturing the role which harmonics play in the power interchanges of a system.

16.09. Power Factor and Reactive Factor Associated with Nonsinusoidal Waves

Power factor and *reactive factor* associated with nonsinusoidal waves are defined in terms of the active, reactive, and apparent powers, as follows:

▶ $$\text{Power factor} = \frac{P}{U}$$

$$= \frac{\Sigma V_n I_n \cos(\theta_{v_n} - \theta_{i_n})}{\sqrt{V_1^2 + V_2^2 + \cdots + V_n^2 + \cdots}\sqrt{I_1^2 + I_2^2 + \cdots + I_n^2 + \cdots}}, \qquad (16\text{-}23)$$

▶ $$\text{Reactive factor} = \frac{Q}{U}$$

$$= \frac{\Sigma V_n I_n \sin(\theta_{v_n} - \theta_{i_n})}{\sqrt{V_1^2 + V_2^2 + \cdots + V_n^2 + \cdots}\sqrt{I_1^2 + I_2^2 + \cdots + I_n^2 + \cdots}}. \qquad (16\text{-}24)$$

16.10. Form Factor and Crest Factor

Form factor and *crest factor* have been defined previously in Articles 8.07 and 8.08:

$$\text{Form factor} = \frac{E}{E_{\text{ave}}} = \frac{\sqrt{\frac{1}{T}\int_0^T e^2\,dt}}{\frac{2}{T}\int_0^{T/2} e\,dt}, \qquad (16\text{-}25)$$

$$\text{Crest factor} = \frac{E_m}{E} = \frac{E_m}{\sqrt{\frac{1}{T}\int_0^T e^2\,dt}}. \qquad (16\text{-}26)$$

16.11. Equivalent Sine Waves

In practice, it is rarely that both current and potential are perfectly sinusoidal in form, although both are periodic. In many cases where the wave forms do not differ too widely from the sinusoidal, it is sufficiently accurate to replace the nonsinusoidal waves of potential or current by *equivalent sine waves.* These equivalent sine waves must be used with discretion, inasmuch as computations based upon them are in error by varying amounts which may run as high as several hundred per cent. It is interesting to note that some of the errors encountered in combining meter readings of alternating potentials and currents by vector methods arise from the fact that, in this procedure, sinusoidal waves are assumed where the waves may, in fact, be nonsinusoidal. An equivalent sine wave must:

(a) Have the same frequency as the fundamental of the wave which it replaces.

(b) Have the same effective value as the wave which it replaces.

(c) Have the same initial phase angle as the fundamental of the wave which it replaces, with the following qualification: The equivalent sine waves of potential and current associated with a circuit or part of a circuit must be out of phase by an *equivalent phase angle*, θ', such that

▶
$$\theta' = (\theta'_v - \theta'_i) = \cos^{-1} \frac{P}{VI}. \tag{16-27}$$

In the event that the equivalent phase angle θ' differs from the fundamental phase angle $(\theta_1 = \theta_{v_1} - \theta_{i_1})$, either the equivalent sine wave of current may be given the same initial phase angle as the fundamental current and the equivalent potential wave related to it by θ', or the equivalent sine wave of potential may be given the same initial phase angle as the fundamental potential and the current related to it by θ'. The algebraic sign of the equivalent phase angle is made the same as that of the angular phase difference $(\theta_1 = \theta_{v_1} - \theta_{i_1})$ of the fundamental.

The possibility that the equivalent phase angle may differ from the fundamental angular phase difference arises from the fact that the presence in one wave of harmonics absent from the other decreases the power factor by increasing (VI) without affecting P and without affecting the fundamental angular phase difference. When the fundamentals are in phase but the power factor is not unity, the sign of the equivalent phase angle is indeterminate.

Except for sinusoidal waves, equation (16-27) may have little significance.

EXAMPLE 16.11. Find the equivalent sine waves of potential and current for the following two waves:

$$v = 156 \sin (\omega t + 30°) + 50 \sin (3\omega t - 60°) + 2 \sin (5\omega t + 45°) \text{ volts},$$
$$i = 31.1 \sin (\omega t + 25°) + 5 \sin 5\omega t \text{ amperes}.$$
$$V = \sqrt{\frac{(156)^2 + (50)^2 + (2)^2}{2}} = 116 \text{ volts},$$
$$I = \sqrt{\frac{(31.1)^2 + (5)^2}{2}} = 22.3 \text{ amperes},$$
$$P = \frac{156 \times 31.1}{2} \cos 5° + \frac{2 \times 5}{2} \cos 45° = 2420 \text{ watts}.$$
$$\text{P.F.} = \frac{2420}{116 \times 22.3} = 0.935; \qquad \theta' = \cos^{-1} 0.935 = 20.78°.$$

By using the original fundamental of potential as reference,

$$v' = 116 \times \sqrt{2} \sin (\omega t + 30°) = 164 \sin (\omega t + 30°) \text{ volts},$$
$$i' = 22.3 \times \sqrt{2} \sin (\omega t + 30° - 20.78°) = 31.55 \sin (\omega t + 9.22°) \text{ amperes}.$$

By using the original fundamental of current as reference,

$$v'' = 164 \sin (\omega t + 25° + 20.78°) = 164 \sin 45.78° \text{ volts},$$
$$i'' = 31.55 \sin (\omega t + 25°) \text{ amperes}.$$

16.12. Deviation Factor

"*The deviation factor of a wave is the ratio of the maximum difference between corresponding ordinates of the wave and of the equivalent sine wave to the maximum ordinate of the equivalent sine wave when the waves are superposed in such a way as to make this maximum difference as small as possible.*"[5] Except in special cases, a deviation factor of 0.10 is permissible in the wave form of commercial electric machinery. Figure 16.12 illustrates this definition. Shifting one of these waves either way with respect to the other will increase one of the two "minimum maximum deviations," thereby producing a maximum deviation which is not as small as possible.

16.13. Addition and Subtraction of Nonsinusoidal Waves

The addition of two nonsinusoidal waves represented by Fourier series is accomplished by adding individually the components of the same frequency and then writing the Fourier equation for the resultant. Usually it is convenient to employ the complex expressions for the components in performing this operation. *Care must be taken to avoid adding terms of unlike frequency.* Subtraction consists of changing the sign of the term to be subtracted, and then adding. The following example illustrates this process.

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[5] *Ibid.*, No. 10.95.420.

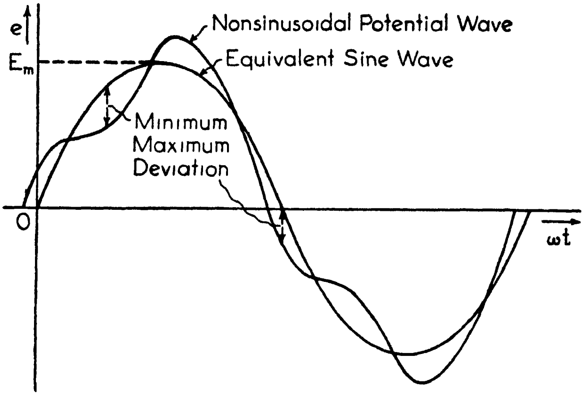


Fig. 16.12. Deviation from an equivalent sine wave.

Example 16.13. Given the following two nonsinusoidal waves:

$$i_A = 10 \sin \omega t + 5 \sin (3\omega t - 30^\circ) - 3 \sin (5\omega t + 60^\circ) \text{ amperes},$$
$$i_B = 20 \sin (\omega t - 30^\circ) + 10 \sin (5\omega t + 45^\circ) \text{ amperes}.$$

(a) Find the sum $i_S = i_A + i_B$. (b) Find the difference $i_D = i_A - i_B$.

Fundamental

$$\sqrt{2}\,\mathbf{I}_{A_1} = 10(\cos 0^\circ + j \sin 0^\circ) = 10 + j0,$$
$$\sqrt{2}\,\mathbf{I}_{B_1} = 20(\cos 30^\circ - j \sin 30^\circ) = 17.32 - j10,$$
$$\sqrt{2}\,\mathbf{I}_{S_1} = \sqrt{2}\,(\mathbf{I}_{A_1} + \mathbf{I}_{B_1}) = 27.32 - j10 = \underline{29.1\underline{/-21.48^\circ}},$$
$$\sqrt{2}\,\mathbf{I}_{D_1} = \sqrt{2}\,(\mathbf{I}_{A_1} - \mathbf{I}_{B_1}) = -7.32 + j10 = \underline{12.4\underline{/126.25^\circ}}.$$

Third Harmonic

$$\sqrt{2}\,\mathbf{I}_{A_3} = 5(\cos 30^\circ - j \sin 30^\circ) = 4.33 - j2.5,$$
$$\sqrt{2}\,\mathbf{I}_{B_3} = 0 + j0,$$
$$\sqrt{2}\,\mathbf{I}_{S_3} = \sqrt{2}\,\mathbf{I}_{D_3} = 4.33 - j2.5 = 5\underline{/-30^\circ}.$$

FIFTH HARMONIC

$$\sqrt{2}\,\mathbf{I}_{A_5} = -3(\cos 60^\circ + j\sin 60^\circ) = -1.5 - j2.6,$$
$$\sqrt{2}\,\mathbf{I}_{B_5} = 10(\cos 45^\circ + j\sin 45^\circ) = 7.07 + j7.07,$$
$$\sqrt{2}\,\mathbf{I}_{S_5} = \sqrt{2}\,(\mathbf{I}_{A_5} + \mathbf{I}_{B_5}) = 5.57 + j4.47 = 7.15\underline{/38.72^\circ},$$
$$\sqrt{2}\,\mathbf{I}_{D_5} = \sqrt{2}\,(\mathbf{I}_{A_5} - \mathbf{I}_{B_5}) = -8.57 - j9.67 = 12.91\underline{/-131.51^\circ}.$$

The equations of the resultants then are:

$$i_S = 29.1\sin(\omega t - 21.48^\circ) + 5\sin(3\omega t - 30^\circ) + 7.15\sin(5\omega t + 38.72^\circ) \text{ amperes},$$
$$i_D = 12.4\sin(\omega t + 126.25^\circ) + 5\sin(3\omega t - 30^\circ) + 12.91\sin(5\omega t - 131.51^\circ) \text{ amperes}.$$

16.14. Harmonics in Networks

The foregoing articles of this chapter have dealt with the interrelationships of nonsinusoidal potential, current, and power associated with a linear single-phase circuit. The mathematical relations which were established in these articles may be applied to any linear bilateral network through the medium of the principle of superposition. Under this principle, the effect of each harmonic can be evaluated independently and the results combined to yield the resultant nonsinusoidal currents and potential drops and the power exchanges throughout the circuit. All the general network theorems that have been developed in this book are applicable in evaluating individually the effects of the various harmonics, and the relations of the present chapter are applicable in determining the net effects of the harmonics acting together.

It is important to remember always that, although inductance and capacitance may be virtually independent of frequency, inductive reactance and capacitive reactance are functions of frequency and offer different complex values to the different harmonics. The result is that the nonsinusoidal potential drops of a circuit may differ widely in wave form from the nonsinusoidal currents with which they are associated.

The reader is reminded, further, of the fact that equivalent sine waves at best yield only approximately correct results when employed in circuit analyses, and in general they may lead to sizable errors. They should be used only with caution.

Example 16.14. The following emf is applied to the circuit of Fig. 16.14:

$$e = 141.4 \sin 377t + 28.3 \cos 1131t - 2 \sin (1885t - 30^\circ) \text{ volts.}$$

Write the equations for the following quantities as functions of time: (a) The potential drop from a to b. (b) The current i_C. (c) The current i_A. (d) The total active power.

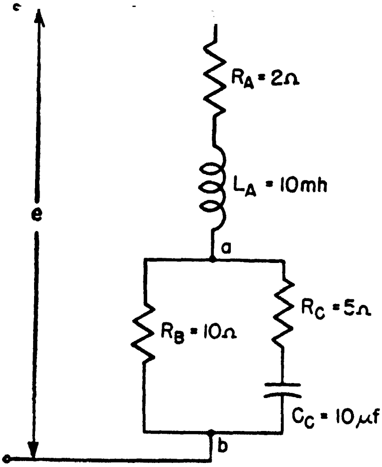


Fig. 16.14.

The impedance between a and b is

$$\mathbf{Z}_{ab} = \frac{\mathbf{Z}_B\mathbf{Z}_C}{\mathbf{Z}_B + \mathbf{Z}_C}.$$

The total impedance of the circuit is

$$\mathbf{Z}_T = \mathbf{Z}_A + \mathbf{Z}_{ab}.$$

The total current is

$$\mathbf{I}_T = \mathbf{I}_A = \frac{\mathbf{E}_T}{\mathbf{Z}_T}.$$

The drop from a to b is

$$\mathbf{V}_{ab} = \mathbf{I}_A\mathbf{Z}_{ab}.$$

The current through $\mathbf{Z}_C$ is

$$\mathbf{I}_C = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_C}.$$

The total fundamental power is

$$P_{T_1} = E_{T_1}I_{T_1} \cos (\theta_{v_1} - \theta_{i_1}).$$

These relations can be utilized for each harmonic in turn.

Fundamental

$$(\omega_1 = 377)$$

$$\mathbf{Z}_{A_1} = 2 + j377 \times 10 \times 10^{-3} = 2 + j3.77 = 4.26\underline{/62.04^\circ} \text{ complex ohms,}$$

$$\mathbf{Z}_{B_1} = 10 + j0 = 10\underline{/0^\circ} \text{ complex ohms,}$$

$$\mathbf{Z}_{C_1} = 5 + j\frac{-10^6}{377 \times 10} = 5 - j265.5 = 265.5\underline{/-88.9^\circ} \text{ complex ohms,}$$

$$\mathbf{Z}_{ab_1} = \frac{(10 + j0)(5 - j265.5)}{10 + j0 + 5 - j265.5} = 9.96 - j0.377$$

$$= 9.96\underline{/-2.2^\circ} \text{ complex ohms,}$$

$$\mathbf{Z}_{T_1} = (2 + j3.77) + (9.96 - j0.377) = 11.96 + j3.39$$

$$= 12.42\underline{/15.8^\circ} \text{ complex ohms,}$$

$$\sqrt{2}\,\mathbf{I}_{T_1} = \sqrt{2}\,\mathbf{I}_{A_1} = \frac{141.4\underline{/0^\circ}}{12.42\underline{/15.8^\circ}} = 11.39\underline{/-15.8^\circ} \text{ complex amperes,}$$

$$\sqrt{2}\,\mathbf{V}_{ab_1} = (11.39\underline{/-15.8^\circ})(9.96\underline{/-2.2^\circ}) = 113.2\underline{/-18^\circ} \text{ complex volts,}$$

$$\sqrt{2}\,\mathbf{I}_{C_1} = \frac{113.2\underline{/-18^\circ}}{265.5\underline{/-88.9^\circ}} = 0.426\underline{/70.9^\circ} \text{ complex amperes,}$$

$$P_{T_1} = \frac{141.4 \times 11.39}{2} \cos(0^\circ + 15.8^\circ) = 774 \text{ watts.}$$

THIRD HARMONIC

$$(\omega_3 = 1131)$$

$$\begin{aligned}
\mathbf{Z}_{A_3} &= 2 + j1131 \times 10 \times 10^{-3} = 2 + j11.31 = 11.50\underline{/10^\circ},\\
\mathbf{Z}_{B_3} &= 10 + j0 = 10\underline{/0^\circ},\\
\mathbf{Z}_{C_3} &= 5 + j\frac{-10^6}{1131 \times 10} = 5 - j88.4 = 88.5\underline{/-86.8^\circ},\\
\mathbf{Z}_{ab_3} &= \frac{(10 + j0)(5 - j88.4)}{10 + j0 + 5 - j88.4} = 9.80 - j1.10 = 9.85\underline{/-6.4^\circ},\\
\mathbf{Z}_{T_3} &= (2 + j11.31) + (9.80 - j1.10) = 11.80 + j10.21\\
&= 15.58\underline{/40.94^\circ} \text{ complex ohms,}
\end{aligned}$$

$$\sqrt{2}\,\mathbf{I}_{T_3} = \sqrt{2}\,\mathbf{I}_{A_3} = \frac{28.3\underline{/90^\circ}}{15.58\underline{/40.94^\circ}} = 1.82\underline{/49.06^\circ} \text{ complex amperes,}$$

$$\sqrt{2}\,\mathbf{V}_{ab_3} = (1.82\underline{/49.06^\circ})(9.85\underline{/-6.4^\circ}) = 17.9\underline{/42.66^\circ} \text{ complex volts,}$$

$$\sqrt{2}\,\mathbf{I}_{C_3} = \frac{17.9\underline{/42.66^\circ}}{88.5\underline{/-86.8^\circ}} = 0.2025\underline{/129.46^\circ} \text{ complex amperes,}$$

$$P_{T_3} = \frac{28.3 \times 1.82}{2} \cos(90^\circ - 49.06^\circ) = 19.44 \text{ watts.}$$

FIFTH HARMONIC

$$(\omega_5 = 1885)$$

$$\begin{aligned}
\mathbf{Z}_{A_5} &= 2 + j1885 \times 10 \times 10^{-3} = 2 + j18.85 = 18.91\underline{/84^\circ},\\
\mathbf{Z}_{B_5} &= 10 + j0 = 10\underline{/0^\circ},\\
\mathbf{Z}_{C_5} &= 5 + j\frac{-10^6}{1885 \times 10} = 5 - j53 = 53.1\underline{/-84.6^\circ},\\
\mathbf{Z}_{ab_5} &= \frac{(10 + j0)(5 - j53)}{10 + j0 + 5 - j53} = 9.50 - j1.75 = 9.66\underline{/-10.4^\circ},\\
\mathbf{Z}_{T_5} &= (2 + j18.85) + (9.50 - j1.75) = 11.50 + j17.10\\
&= 20.6\underline{/56.08^\circ} \text{ complex ohms,}
\end{aligned}$$

$$\sqrt{2}\,\mathbf{I}_{T_5} = \sqrt{2}\,\mathbf{I}_{A_5} = \frac{2\underline{/150^\circ}}{20.6\underline{/56.08^\circ}} = 0.0972\underline{/93.92^\circ} \text{ complex amperes,}$$

$$\sqrt{2}\,\mathbf{V}_{ab_5} = (0.0972/93.92^\circ)(9.66/-10.4^\circ) = 0.94/83.52^\circ \text{ complex volts,}$$

$$\sqrt{2}\,\mathbf{I}_{C_5} = \frac{0.94\underline{/83.52^\circ}}{53.1\underline{/-84.6^\circ}} = 0.0177\underline{/168.12^\circ} \text{ complex amperes,}$$

$$P_{T_5} = \frac{2 \times 0.0972}{2} \cos\,(150^\circ - 93.92^\circ) = 0.0542 \text{ watts.}$$

(a) The equation for the potential drop from a to b is

$$v_{ab} = 113.2 \sin\,(377t - 18^\circ) + 17.9 \sin\,(1131t + 42.66^\circ) + 0.94 \sin\,(1885t + 83.52^\circ) \text{ volts.}$$

(b) The current through branch C is

$$i_C = 0.426 \sin\,(377t + 70.9^\circ) + 0.2025 \sin\,(1131t + 129.46^\circ) + 0.0177 \sin\,(1885t + 168.12^\circ) \text{ amperes.}$$

(c) The total current is

$$i_A = 11.39 \sin\,(377t - 15.8^\circ) + 1.82 \sin\,(1131t + 49.06^\circ) + 0.0972 \sin\,(1885t + 93.92^\circ) \text{ amperes.}$$

(d) The total active power is

$$P_T = 774 + 19.44 + 0.0542 = 793.5 \text{ watts.}$$

16.15. Harmonics in Polyphase Circuits

The discussion of polyphase circuits in Chapter 14 was based on the assumption of sinusoidal emf's and currents. In practice, many factors, some of which have already been discussed, function to produce nonsinusoidal emf's and currents in polyphase circuits, just as in single-phase circuits. It is the purpose of this article to study the behavior of polyphase circuits in response to nonsinusoidal polyphase potentials and currents.

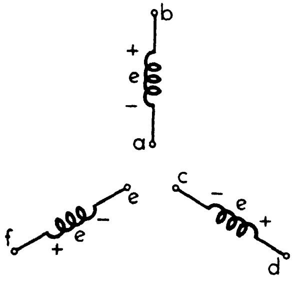


Fig. 16.15A. Windings of a three-phase generator.

Inasmuch as polyphase circuits are merely special forms of networks, the general solution to the problem of nonsinusoidal potentials and currents in a polyphase circuit is based on the principle of superposition. As we saw in the preceding article, the various harmonics are treated individually and the results combined. If the nonsinusoidal polyphase quantities are balanced and symmetrical, and if they exist in a balanced circuit, relations of special interest and importance can be developed. Because of the predominance of three-phase circuits, the study which follows will be restricted to *balanced* circuits of three phases.

Referring to Fig. 16.15A, which represents the armature windings of a three-phase synchronous generator, we may write the following expressions for the three balanced symmetrical emf's of similar wave form developed in the generator windings:

$$\begin{aligned} e_{ab} &= E_{m_1} \sin (\omega t + a_1) + E_{m_3} \sin (3\omega t + a_3) + E_{m_5} \sin (5\omega t + a_5) \\ &\qquad + E_{m_7} \sin (7\omega t + a_7) + \cdots , \end{aligned} \tag{16-28a}$$

$$\begin{aligned} e_{cd} &= E_{m_1} \sin (\omega t + a_1 - 120°) + E_{m_3} \sin (3\omega t + a_3 - 360°) \\ &\quad + E_{m_5} \sin (5\omega t + a_5 - 600°) + E_{m_7} \sin (7\omega t + a_7 - 840°) \\ &\qquad + \cdots , \end{aligned} \tag{16-29a}$$

$$\begin{aligned} e_{ef} &= E_{m_1} \sin (\omega t + a_1 - 240°) + E_{m_3} \sin (3\omega t + a_3 - 720°) \\ &\quad + E_{m_5} \sin (5\omega t + a_5 - 1200°) \\ &\quad + E_{m_7} \sin (7\omega t + a_7 - 1680°) + \cdots . \end{aligned} \tag{16-30a}$$

These can be rewritten as follows:

▶
$$\begin{aligned} e_{ab} &= E_{m_1} \sin (\omega t + a_1) + E_{m_3} \sin (3\omega t + a_3) \\ &\quad + E_{m_5} \sin (5\omega t + a_5) \\ &\quad + E_{m_7} \sin (7\omega t + a_7) + \cdots , \end{aligned} \tag{16-28b}$$

▶
$$\begin{aligned} e_{cd} &= E_{m_1} \sin (\omega t + a_1 - 120°) + E_{m_3} \sin (3\omega t + a_3) \\ &\quad + E_{m_5} \sin (5\omega t + a_5 - 240°) \\ &\quad + E_{m_7} \sin (7\omega t + a_7 - 120°) + \cdots , \end{aligned} \tag{16-29b}$$

▶
$$\begin{aligned} e_{ef} &= E_{m_1} \sin (\omega t + a_1 - 240°) + E_{m_3} \sin (3\omega t + a_3) \\ &\quad + E_{m_5} \sin (5\omega t + a_5 - 120°) \\ &\quad + E_{m_7} \sin (7\omega t + a_7 - 240°) + \cdots . \end{aligned} \tag{16-30b}$$

A study of these equations reveals the following facts regarding nonsinusoidal emf's in the windings of three-phase generators:

1. *The fundamental and all harmonics obtained by adding a multiple of 6 to the fundamental have the same sequence. (In this case, the ab-cd-ef sequence.)*
2. *The fifth harmonic and all harmonics obtained by adding a multiple of 6 to the fifth harmonic have the same sequence, which is the reverse of the fundamental sequence. (In this case, the ab-ef-cd sequence.)*
3. *The third harmonic and all multiples of the third harmonic (all "triplen" harmonics) are in phase.*

These facts are presented graphically in Fig. 16.15B.

If the windings of the generator under discussion are *delta-connected*, as indicated by Fig. 16.15C(a), we note (by reference to Fig. 16.15B) that the vector sum of the phase emf's at each nontriplen-harmonic frequency is zero, with the result that no current of nontriplen-harmonic frequency can circulate about the closed loop of the delta. The vector sum of the phase emf's at each triplen harmonic frequency is not zero, however, but is three times the emf of one phase at that frequency. Thus, a circulating current consisting entirely of triplen-harmonic components will flow through the closed loop of the delta. The component of this current due to each triplen harmonic is

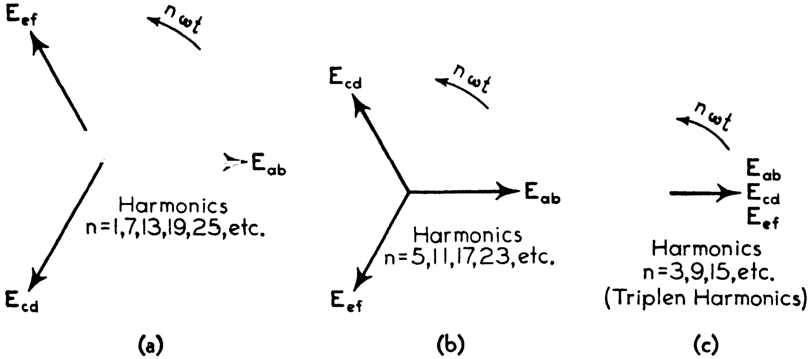


Fig. 16.15B. Sequences of the harmonic components of phase emf for a three-phase generator. (a) Fundamental sequence. (b) Fifth harmonic sequence. (c) Third harmonic sequence. System assumed completely balanced.

▶
$$I_{\Delta(\text{triplen})} = \frac{E_{\Delta(\text{triplen})}}{Z_{\Delta(\text{triplen})}} = \frac{3E_{\Delta(\text{triplen})_{\text{phase}}}}{3Z_{\Delta(\text{triplen})_{\text{phase}}}}, \tag{16-31}$$

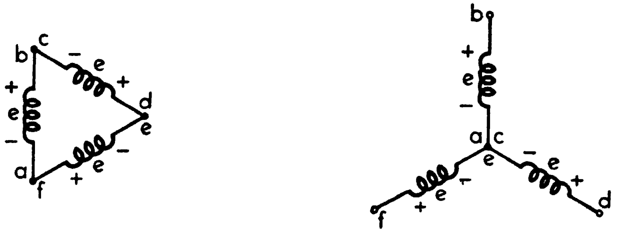
from which:

$$E_{\Delta(\text{triplen})_{\text{phase}}} = I_{\Delta(\text{triplen})} Z_{\Delta(\text{triplen})_{\text{phase}}}. \tag{16-32}$$

The net line-to-line potential component due to a given triplen-harmonic frequency is

▶
$$E_{\Delta(\text{triplen})_{\text{phase}}} - I_{\Delta(\text{triplen})} Z_{\Delta(\text{triplen})_{\text{phase}}} = 0. \tag{16-33}$$

The following general statement can, then, be made: *No triplen-harmonic components can appear in the line-to-line potentials of a* *balanced delta-connected generator, although there may circulate around the delta a current consisting entirely of triplen-harmonic components.* It must be observed, however, that if the continuity of the closed path is broken, no circulating current can flow, no IZ drops will occur, and the triplen-harmonic components will then appear in the line-to-line potentials. The foregoing statements apply equally well to the delta-connected secondary of a three-phase transformer.



(a) (b)

Fig. 16.15C. Two modes of connection of the three-phase generator windings of Fig. 16.15A. (a) Delta connection. (b) Wye connection.

If, now, the windings of the generator are *wye-connected,* as indicated by Fig. 16.15C(b), we note (by reference to Fig. 16.15B) that vector addition of phase emf's to yield line emf's results in the following facts:

For each nontriplen harmonic:

▶
$$E_{Y(\text{nontriplen})_{\text{line}}} = \sqrt{3}E_{Y(\text{nontriplen})_{\text{phase}}} \tag{16-34}$$

For each triplen harmonic:

▶
$$E_{Y(\text{triplen})_{\text{line}}} = 0. \tag{16-35}$$

Thus, we reach the interesting conclusion that *in three-phase generators, although there may be triplen harmonic emf's generated in the windings of the individual phases, there can be no triplen harmonic components in the line-to-line terminal potentials, with either a* Δ *connection or a* Y *connection of the phase windings.*

From the foregoing discussion it should be clear that no triplen harmonic currents will flow between generator and load of a three-wire three-phase system. A four-wire three-phase system is another case, however, as we shall show by reference to Fig. 16.15D.

In studying Fig. 16.15D, we note, first of all, that the line potentials must be devoid of any triplen-harmonic components. Because the neutral wire is present, however, each individual phase can now operate independently of the rest (ignoring potential drops due to neutral-wire impedance). There now exists in each phase a path through which triplen-harmonic current can flow in response to any triplen-harmonic components that may be present in the phase emf's of the generator. This triplen-harmonic current flows toward the load through the line wires and returns through the neutral wire. Since all three sets of triplen-harmonic currents are in time phase, we may write that, for any given triplen-harmonic frequency,

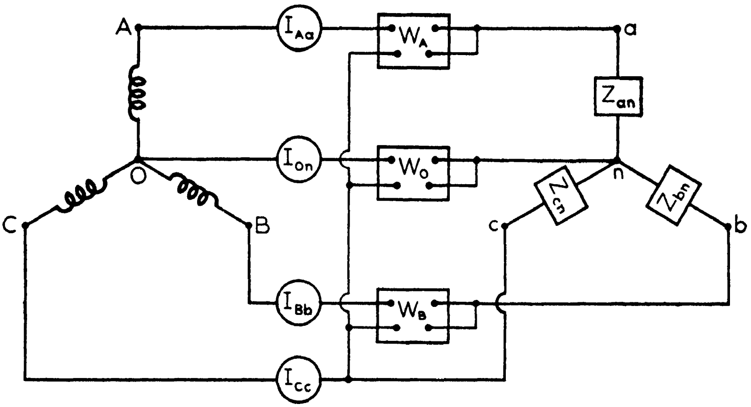


Fig. 16.15D. Circuit for study of triplen harmonics in a three-phase system.

▶ $$I_{On(\text{triplen})} = -(I_{Aa(\text{triplen})} + I_{Bb(\text{triplen})} + I_{Cc(\text{triplen})}). \tag{16-36}$$

If the loads are *balanced,*

▶ $$I_{\text{neutral}(\text{triplen})} = 3I_{\text{line}(\text{triplen})}. \tag{16-37}$$

A moment of reflection should make it clear that, although the individual phase *IZ* drops may contain triplen-harmonic components, in this circuit, too, the line potentials of the load cannot contain triplen-harmonic components.

By putting these facts together with the fact that a wattmeter responds only to those harmonics that are represented in *both* the potential and the current waves, it becomes apparent that the wattmeters as connected in Fig. 16.15D segregate indications of nontriplen harmonic power and triplen harmonic power, as follows:

▶ $W_A + W_B =$ total *non*triplen harmonic power, (16-38)

▶ $W_O =$ total triplen harmonic power, (16-39)

▶ $W_A + W_B + W_O =$ total three-phase power. (16-40)

The third statement is in accord with the $(n - 1)$ wattmeter method of measuring power in an n-wire circuit.

If the three wattmeters were so reconnected as to utilize the neutral wire as a common junction and the current coil of W_O were in wire C_c, each wattmeter would then indicate the total power (including triplen harmonic) taken by a given phase. The sum would again be the total power.

EXAMPLE 16.15. The load of Fig. 16.15D is a balanced resistance load. The ammeter in the neutral wire indicates 9 amperes and the wattmeter W_O indicates 450 watts. The generator supplies a line-to-line potential of 208 volts with the *a-b-c* sequence. Find: (a) The resistance per phase. (b) The indications of the other ammeters. (c) The phase potential at the load. (d) The indications of the other wattmeters.

(a) *Load resistance per phase:*

$$I_{\text{line(triplen)}} = \frac{I_{\text{neutral}}}{3} = \frac{9}{3} = 3 \text{ amperes} = I_{\text{phase(triplen)}},$$

$$P_{\text{phase(triplen)}} = \frac{P_{\text{total(triplen)}}}{3} = \frac{450}{3} = 150 \text{ watts} = R_{\text{phase}} I^2_{\text{phase(triplen)}}.$$

$$R_{\text{phase}} = \frac{P_{\text{phase(triplen)}}}{I^2_{\text{phase(triplen)}}} = \frac{150}{3^2} = 16.67 \text{ ohms}.$$

(b) *Ammeter indications.* By equation (16-6b),

$$I_{\text{line(nontriplen)}} = \sqrt{\Sigma I^2_{\text{line(nontriplen)}}} = \frac{V_{\text{phase(nontriplen)}}}{Z_{\text{phase}}} = \frac{208/\sqrt{3}}{16.67}$$
$$= 7.19 \text{ amperes},$$

$$\Sigma I^2_{\text{line(nontriplen)}} = (7\ 19)^2 = 51.7,$$

$$\Sigma I^2_{\text{line(triplen)}} = I^2_{\text{line(triplen)}} = (3)^2 = 9.$$

Also, by equation (16-6b),

$$I_{\text{line}} = \sqrt{\Sigma I^2_{\text{line(nontriplen)}} + \Sigma I^2_{\text{line(triplen)}}}$$
$$= \sqrt{51.7 + 9} = 7.8 \text{ amperes} = I_{Aa} = I_{Bb} = I_{Cc}.$$

(c) *Phase potential:*

$$V_{\text{phase}} = I_{\text{line}} R_{\text{phase}} = 7.8 \times 16.67 = 130 \text{ volts.}$$

It will be noted that this does *not* equal ($208/\sqrt{3} = 120$ volts).

(d) *Wattmeter indications:*

$$P_{\text{total (nontriplen)}} = 3R_{\text{phase}} I^2_{\text{line(nontriplen)}}$$
$$= 3 \times 16.67 \times (7.19)^2 = 2580 \text{ watts.}$$

We know that $(W_A + W_B) = P_{\text{total (nontriplen)}}$ and that, because it is a resistive load ($\theta = 0°$), $W_A = W_B$. It follows that

$$W_A = W_B = \frac{P_{\text{total (nontriplen)}}}{2} = \frac{2580}{2} = 1290 \text{ watts.}$$

PROBLEMS

16-1. Find the effective value of the following current:

$$i = 10 \sin \omega t + 5 \sin (2\omega t + 30°) + 3 \sin (3\omega t - 20°) \text{ amperes.}$$

16-2. Find the effective value of the following emf:

$$e = 100 \sin (\omega t - 45°) + 70 \sin 3\omega t + 20 \sin (5\omega t + 60°) \text{ volts.}$$

16-3. The equation for the plate current of a vacuum tube sometimes takes the following general form:

$$i = I_b + I_{m_1} \sin \omega t + I_{m_3} \sin 3\omega t \text{ amperes.}$$

If $I_b = 0.2$, $I_{m_1} = 0.06$, and $I_{m_3} = 0.004$ ampere,

(a) Find the maximum and minimum values of i.
(b) Find the average value of i.
(c) Find the effective value of i.
(d) Plot i.

16-4. The Fourier equation for a full-wave rectified sine wave like that of Fig. 3.17A(b) is:

$$e = \frac{2}{\pi} E_m - \frac{4}{3\pi} E_m \cos 2\omega t - \frac{4}{15\pi} \cos 4\omega t - \cdots$$
$$- \frac{4}{\pi} \frac{1}{(n^2 - 1)} E_m \cos n\omega t + \cdots .$$

(a) Graph the true wave, each loop of which has a peak value of E_m.

(b) Synthesize the first four components of the wave described by the given Fourier equation, and compare the result with the true wave.

(c) Determine the effective value of this wave from its Fourier series. Compare this answer with the effective value of an unrectified sine wave.

16-5. A potential $v = 100 \sin 377t + 50 \sin 1885t$ volts is applied to a series circuit consisting of 6 ohms of resistance and 8 ohms of 60-cycle reactance.

(a) Write the Fourier equation for current for the case in which the reactance is inductive.

(b) Write the Fourier equation for current for the case in which the reactance is capacitive.

(c) Plot the two waves of current to the same coordinates.

(d) Compare the results of (a) and (b), and draw any conclusions which you can concerning the effect of reactance in suppressing or emphasizing departure of waves from a nonsinusoidal form.

16-6. (a) Find the active power associated with the waves of Probs. 16-1 and 16-2.

(b) Find the reactive power associated with the same pair of waves.

16-7. A current $i = 10 \sin 377t + 5 \sin (1131t + 30°)$ amperes flows in a series circuit consisting of 5 ohms of resistance, 5 ohms of 60-cycle inductive reactance, and 8 ohms of 60-cycle capacitive reactance.

(a) Write the equation of the wave of total potential.

(b) Find the effective values of the two waves.

(c) Find the active power associated with the two waves.

(d) Find the reactive power associated with the two waves.

16-8. Given, the following waves of potential and current:

$$v = 100 \sin (\omega t + 60°) + 50 \sin (3\omega t - 30°) + 10 \sin 5\omega t \text{ volts,}$$
$$i = 15 \sin \omega t + 10 \sin (5\omega t - 45°) \text{ amperes.}$$

Find the following power quantities associated with these two waves: (*a*) Active power; (*b*) reactive power; (*c*) vector power; (*d*) apparent power; (*e*) distortion power; (*f*) power factor; (*g*) reactive factor; (*h*) draw the geometric power diagram for this situation; (*i*) plot the two waves.

16-9. Repeat Prob. 16-8 for the following two waves:

$$v = 250 \sin \omega t + 20 \sin 5\omega t \text{ volts,}$$
$$i = 10 \sin (\omega t - 30°) + 7 \sin (3\omega t + 60°) + 3 \sin (5\omega t + 120°) \text{ amperes.}$$

16-10. (a) Determine the equivalent sine waves of potential and current for Prob. 16-8.

(b) Plot the waves of (a) and compare with the plots of actual waves in Prob. 16-8.

16-11. Repeat the procedure of Prob. 16-10 for the two waves o Prob. 16-9.

16-12. Determine the deviation factor associated with the curren waves of Probs. 16-8 and 16-10.

16-13. Determine the deviation factor associated with the potentia waves of Probs. 16-9 and 16-11.

16-14. The following two currents approach a certain junction:

$$i_1 = 12 \sin (\omega t + 30°) + 8 \sin (3\omega t - 15°) + 5 \sin (5\omega t - 120°) \text{ amperes},$$
$$i_2 = 8 \sin (\omega t + 45°) + 5 \sin (3\omega t + 20°) + 2 \sin (5\omega t - 200°) \text{ amperes}.$$

(a) Find the equation of the current ($i_t = i_1 + i_2$) leaving the junction.

(b) Write the equations for the equivalent sine waves of currents i_1, i_2, and i_t, using the original fundamentals as references.

(c) Combine the equivalent sine waves for i_1 and i_2 to obtain a resultant equivalent sine wave. Compare this wave with the equivalent sine wave i_t obtained in (b),

(d) Compute the percentage error resulting from using equivalent sine waves to obtain the resultant equivalent sine wave in this problem.

16-15. Repeat Prob. 16-14 for the following two currents:
$i_1 = 10 \sin \omega t - 5 \sin (5\omega t + 120°)$ amperes,
$i_2 = 8 \sin (\omega t + 30°) + 6 \sin 3\omega t - 4 \sin (5\omega t + 20°)$ amperes.

16-16. Two impedances, $\mathbf{Z}_a$ and $\mathbf{Z}_b$, are connected in parallel, and a third impedance, $\mathbf{Z}_c$, is placed in series with the combination. The 60-cycle values of these impedances are: $\mathbf{Z}_a = 5 + j10$, $\mathbf{Z}_b = 2 - j12$, $\mathbf{Z}_c = 5 - j5$ ohms. The following emf is applied to this circuit:

$$e = 141.4 \sin 377t + 70.7 \sin (1131t + 30°) \text{ volts}.$$

(a) What are the indications of a-c ammeters placed in the various branches of the circuit to determine I_a, I_b, and I_c?

(b) Determine the complex expression for the total vector power.

(c) Determine the over-all power factor and reactive factor.

16-17. Solve Prob. 16-16 for the case in which the applied emf is:

$$e = 200 \sin 157t + 100 \sin 314t - 50 \sin (785t - 45°) \text{ volts}.$$

16-18. Carefully develop on paper all reasoning used to answer the following question: Is an impedance bridge that is balanced for a sinusoidal wave of a given frequency balanced for a nonsinusoidal wave whose fundamental is of that frequency?

16-19. The emf induced in phase OA of a three-phase wye-connected generator is:

$$e_{OA} = 110 \sin \omega t - 55 \sin (3\omega t + 60°) + 20 \sin (5\omega t - 30°) \text{ volts}.$$

If the phase sequence is e_{OA}-e_{OB}-e_{OC},

(a) Write the equations of the line potentials, e_{ab}, e_{bc}, e_{ca}.
(b) What is the effective value of phase potential?
(c) What is the effective value of line potential?

16-20. If the windings of the generator of Prob. 16-19 are reconnected in delta, what will be the equations of the line potentials, and what will be their effective values?

16-21. In the balanced delta-connected generator windings of Fig. 16.15C(a) under the condition of no load, the closed delta has been opened by removing the connection joining *b* and *c*. The following values are then read with an a-c voltmeter:

$$E_{ef} = 2400 \text{ volts,} \qquad E_{bc} = 2100 \text{ volts.}$$

What will be the reading if the voltmeter is placed between *f* and *c* under this condition?

16-22. In Example 16.15, a fuse to the left of the ammeter in line *Aa* (Fig. 16.15D) blows. Determine the indications of all instruments under this new condition.

16-23. Repeat Prob. 16-22, except place the blown fuse adjacent to load terminal *c* in line *Cc*.

PART V

TRANSIENTS

CHAPTER 17

TRANSIENTS

17.01. Introduction

In all the analyses presented thus far, the tacit assumption has been made that the circuits were operating under *steady-state conditions.* The criterion of such a condition is that all "meter-reading" (effective) values of potential and current are unvarying. The question naturally arises, What effects are introduced if some sudden change is made in the parameters or mode of operation of the circuit? Such changes might consist of the sudden insertion or removal of a circuit element, or the sudden insertion or removal of a source of emf.

As can be anticipated from a consideration of the nature of the basic phenomena involved, experiment shows two general facts of significance regarding the question that has been posed: (1) *The current in any inductance cannot change instantaneously,* and (2) *the charge (hence, the potential) of any capacitance cannot change instantaneously.* These statements lead us to the idea that, in general, a period of time is required for the transition of a practical circuit from one mode of operation to another. This transitional period is known as the *transient period.* Although transient periods are usually of short duration, it is during these periods that some of the most important and involved operating phenomena are encountered.

The general equation based on Kirchhoff's potential law for a circuit composed of constant series parameters can be written as follows:

$$e = L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt, \tag{17-1a}$$

or

▶
$$e = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C}. \tag{17-1b}$$

This is recognizable as an ordinary linear differential equation of the second order, in which e can be any function of time. Depending on the exact form of the equation and on the boundary conditions, many different forms of solution are possible. In general, these solutions can contain terms of two types: *steady-state terms*, which are either unidirectional terms of constant value or periodic terms of constant amplitude; and *transient terms*, which are either unidirectional terms of exponentially decreasing value, or periodic terms of exponentially decreasing amplitude. As will be seen shortly, *the transient terms can be regarded as physical quantities necessary to preserve the dynamic equilibrium expressed by Kirchhoff's potential law, during the period of transition from one mode of circuit operation to another.* The transient component serves as a "shock absorber" to dispose gradually of the initial discrepancy between the actual values of circuit variables and the values of those variables called for by the steady state.

Electric-circuit transients can be classified as *single-energy transients* and as *double-energy transients.* Single-energy transients occur in circuits in which energy storage occurs in only one form—either as inductive or as capacitive energy. These are R and L or R and C circuits. Double-energy transients appear in circuits in which energy storage can occur in both forms. Such circuits contain all three parameters, R, L, and C.

In general, transients accompany any change in the amount of energy stored in an electric circuit. The more prominent the resistance of the circuit, the more rapidly "excess" energy can be dissipated, and the less the severity of the transient.

Although many transients lead to troublesome problems in practice, in many other cases we find devices whose operation actually is based on recurring transient phenomena. Examples of troublesome transients are lightning and switching surges in transmission lines. Examples of useful applications of transients are certain types of electronic sweep circuits and vibrating-type generator-voltage regulators.

The following articles present transient analyses of various fundamental types of electric circuits.

17.02. Transients in Series R-L Circuits with Direct EMF

Consider the circuit of Fig. 17.02A, with the switch initially closed to the left. We shall assume that steady-state conditions prevail.

By this assumption, the current i is constant at a value $i_0 = E_0/R$. At a time $t = t_0$, the switch is suddenly thrown to the right, replacing E_0 by a different emf, E. Neglect the momentary disturbance introduced by interrupting the circuit while switching. A new steady-state condition, depending on the value of E, may be required. Mathematically, we can write the equation of dynamic equilibrium (Kirchhoff's potential law) at any instant as

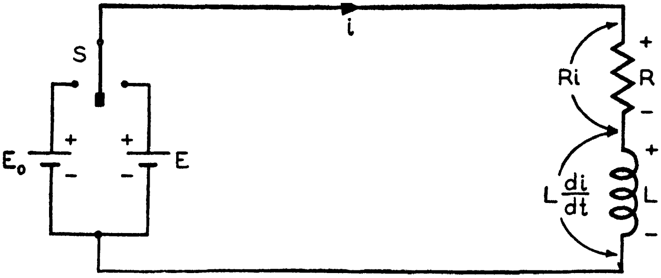


Fig. 17.02A. Circuit for study of transients in series R-L combination with direct emf.

$$E = L\frac{di}{dt} + Ri. \tag{17-2}$$

From this equation,

$$(E - Ri) = L\frac{di}{dt}, \tag{17-3}$$

and

$$\frac{di}{-(E - Ri)} = -\frac{1}{L}\,dt, \tag{17-4}$$

or

$$\frac{di}{i - \dfrac{E}{R}} = -\frac{R}{L}\,dt. \tag{17-5}$$

This equation can now be solved for the instantaneous current, i, by integrating and paying due regard to the initial conditions:[1]

$$\ln\left(i - \frac{E}{R}\right)\Big]_{i_0}^{i} = -\frac{R}{L}\,t\Big]_{t_0}^{t}, \tag{17-6}$$

or

$$\ln\left(i - \frac{E}{R}\right) - \ln\left(i_0 - \frac{E}{R}\right) = -\frac{R}{L}(t - t_0), \tag{17-7}$$

[1] ln = $\log_e$.

and
$$\ln \frac{i - \frac{E}{R}}{i_0 - \frac{E}{R}} = -\frac{R}{L}(t - t_0). \tag{17-8}$$

This equation can be written as

$$\frac{i - \frac{E}{R}}{i_0 - \frac{E}{R}} = \epsilon^{-\frac{R}{L}(t-t_0)}, \tag{17-9}$$

from which

$$i - \frac{E}{R} = \left(i_0 - \frac{E}{R}\right)\epsilon^{-\frac{R}{L}(t-t_0)}. \tag{17-10}$$

Solving this equation for i gives us *the general equation for current in an R-L series circuit activated by direct emf:*

▶ $$i = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right)\epsilon^{-\frac{R}{L}(t-t_0)}, \tag{17-11}$$

where:

i = *instantaneous current*, in **amperes,** *at time* t,
i_0 = *instantaneous current*, in **amperes,** *at time* t_0,
E = *constant applied emf*, in **volts,**
R = *total resistance of circuit*, in **ohms,**
L = *total inductance of circuit*, in **henrys.**

Equation (17-11) has the graphical interpretation indicated by Fig. 17.02B. As this curve shows, the equation consists of a *steady-state term* of value (E/R) and a *transient term* of value $(i_0 - E/R) \cdot \epsilon^{-\frac{R}{L}(t-t_0)}$. This transient term has an initial magnitude, when $t = t_0$, equal to the difference between E/R and i_0. As time increases, this difference decreases, with i approaching E/R as a limiting value. Since, theoretically, it would take an infinite time for the transient term to vanish completely, we might *denote the limiting value of* i, *corresponding to an infinite value of time, as* i_∞. Using this notation, and *denoting* $t_0 = 0$, in order to begin time measurement at the start of the transient, we can rewrite equation (17-11) as follows:

▶ $$i = i_\infty + (i_0 - i_\infty)\epsilon^{-\frac{R}{L}t}, \tag{17-12}$$

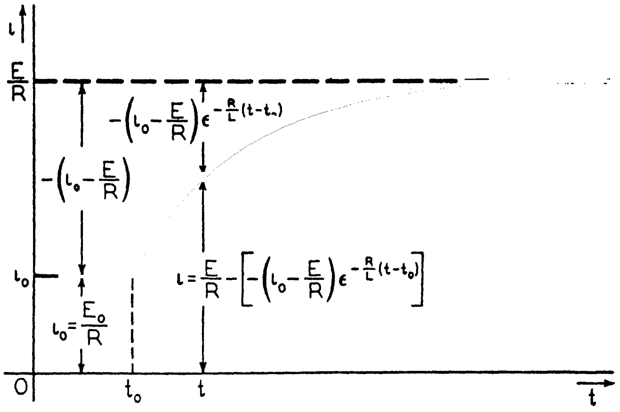

i
$\frac{E}{R}$
$-\left(i_0-\frac{E}{R}\right)\epsilon^{-\frac{R}{L}(t-t_0)}$
$-\left(i_0-\frac{E}{R}\right)$
i_0
$i_0=\frac{E_0}{R}$
$i=\frac{E}{R}-\left[-\left(i_0-\frac{E}{R}\right)\epsilon^{-\frac{R}{L}(t-t_0)}\right]$
O
t_0
t
t

Fig. 17.02B. A transient for a particular case of the circuit of Fig. 17.02A.

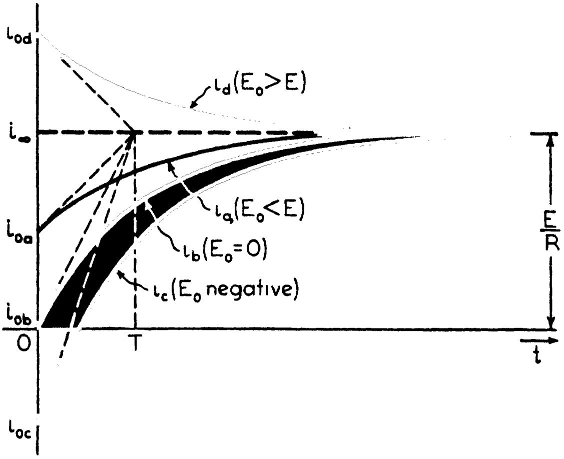


Fig. 17.02C. Curves represented by Eq. (17–12).

where:

i = *instantaneous current at time t,*
i_0 = *initial instantaneous value of current, when* $t = 0$,
i_∞ = *limiting value of current, as t approaches infinity.*

This procedure causes Fig. 17.02B to assume the appearance of curve i_a of Fig. 17.02C. The initial value of this curve is positive but less than the limiting value i_∞.

No restriction, either mathematical or physical, need be placed on the relative values of E_0 and E, so long as they are constant. Thus, E could be greater than E_0, equal to E_0, or of opposite polarity to E_0. Also, E_0 itself can have any value, either positive or negative, including zero. Curves illustrating all these cases are given in Fig. 17.02C. We shall discuss briefly two special cases which are frequently encountered in practice.

Case 1: $E_0 = 0$, $i_0 = 0$. In this case, the *initial value of the current is zero* and equation (17-12) becomes:

▶ $$i = i_\infty(1 - \epsilon^{-\frac{R}{L}t}) = \frac{E}{R}(1 - \epsilon^{-\frac{R}{L}t}). \qquad (17\text{-}13)$$

This is the standard equation for the growth from zero of current in a series R-L circuit. Curve i_b of Fig. 17.02C exemplifies this case.

Case 2: $E = 0$, $i_\infty = 0$. In this case, *the current is decaying from an initial value*, i_0, *to zero.* Equation (17-13) becomes:

▶ $$i = i_0\epsilon^{-\frac{R}{L}t} = \frac{E_0}{R}\epsilon^{-\frac{R}{L}t}. \qquad (17\text{-}14)$$

This is the standard equation for the decay to zero of current in a series R-L circuit. A transient term, alone, exists in this case. The curve of Fig. 17.02D exemplifies this equation.

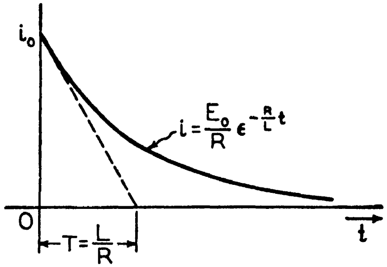


Fig. 17.02D. Decay of current to zero in a series R-L circuit.

Example 17.02. The following values apply to the circuit of Fig. 17.02A: $R = 5$ ohms, $L = 10$ millihenrys, $E = 100$ volts, $E_0 = -75$ volts. (The negative value of E_0 implies the battery to be connected with a reverse polarity to that shown in the figure.) Neglecting interruptions during switching, find the instantaneous value of the current, i, 6 milliseconds after the following sequence of events begins:

(1) The initially open switch is thrown to the left and held there for 2 milliseconds.

(2) Then the switch is thrown suddenly to the right and is held there for 2 milliseconds.

(3) Finally, the switch is thrown to the left again.

We shall use the subscripts 1, 2, and 3 to denote the position of the switch.

(1) *Initially open switch closed to left* ($\Delta t_1 = 2$ *ms*):

$$i_{0_1} = 0; \qquad i_{\infty_1} = \frac{E_0}{R} = \frac{-75}{5} = -15 \text{ amperes.}$$

Then, by (17-12),

$$i_{\text{final}_1} = -15 + (0 + 15)\epsilon^{-500\times 2\times 10^{-3}} = -9.47 \text{ amperes.}$$

(2) *Switch closed to right* ($\Delta t_2 = 2$ *ms*):

$$i_{0_2} = -9.47 \text{ amperes}; \qquad i_{\infty_2} = \frac{E}{R} = \frac{100}{5} = 20 \text{ amperes.}$$

$$i_{\text{final}_2} = 20 + (-9.47 - 20)\epsilon^{-500\times 2\times 10^{-3}} = 9.15 \text{ amperes.}$$

(3) *Switch again closed to left* ($\Delta t_3 = 2$ *ms*):

$$i_{0_3} = 9.15 \text{ amperes}; \qquad i_{\infty_3} = \frac{E_0}{R} = \frac{-75}{5} = -15 \text{ amperes.}$$

$$i_{\text{final}_3} = -15 + (9.15 + 15)\epsilon^{-500\times 2\times 10^{-3}} = -6.11 \text{ amperes.}$$

17.03. The Series R-L Time Constant, T = L/R

It now becomes convenient to define a general property of all circuits, the *time constant. The time constant, T, of a circuit is that value of time which will reduce the transient term to $\epsilon^{-1} = 1/\epsilon = 0.368 = 36.8$ per cent of its initial value.* Stated in another way, it is the time required for the transient to undergo $(1 - 0.368) = 0.632 = 63.2$ per cent of its total change. As will be shown subsequently, the time constant is the time required for the transient term to vanish if it were to decrease at a constant rate equal to that which existed at the start of the transient period. Graphically, it can be found by finding the intercept with the i_∞ line of a tangent to the curve at t_0. The time constants of all the curves of Figs. 17.02C and 17.02D have been indicated in this manner. *If the time constant is applied successively to an exponential growth or decay curve, each application represents the time required for the transient to go through* 63.2 *per cent of the remaining change.* Fig. 17.03 illustrates this statement. Using successive applications of the time constant thus provides a convenient means of quickly constructing graphically an exponential curve. Inasmuch as the value of the exponential merely *approaches* zero as a limit, it would theoretically take an infinite time for a transient term to disappear completely. Actually, for all practical purposes, the transient has vanished after the elapse of five or six time constants. At the end of five time constants, the transient would have a value of $(1/\epsilon)^5 = 0.00674$ of its initial value.

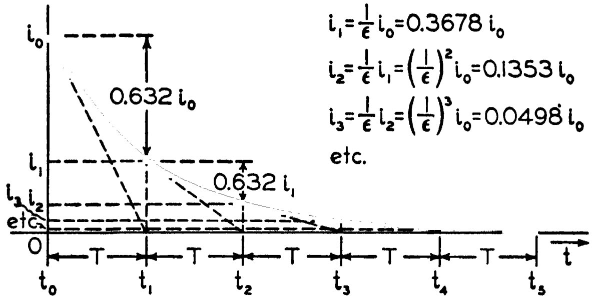

i_0
i_1
$i_3\ i_2$
etc.
0
0.632 i_0
0.632 i_1
$i_1 = \frac{1}{\epsilon} i_0 = 0.3678\, i_0$
$i_2 = \frac{1}{\epsilon} i_1 = \left(\frac{1}{\epsilon}\right)^2 i_0 = 0.1353\, i_0$
$i_3 = \frac{1}{\epsilon} i_2 = \left(\frac{1}{\epsilon}\right)^3 i_0 = 0.0498\, i_0$
etc.
T
T
T
T
T
t
t_0
t_1
t_2
t_3
t_4
t_5

Fig. 17.03. Successive applications of the time constant to the curve of a transient.

Applying the foregoing definition of the time constant to a series *R-L* circuit gives us, from equation (17-12),

$$\epsilon^{-\frac{R}{L}T} = \epsilon^{-1}. \tag{17-15}$$

It is apparent from this that

$$\frac{R}{L} T = 1. \tag{17-16}$$

The time constant for a series R-L circuit is seen to be

▶ $$T = \frac{L}{R}, \tag{17-17}$$

where:

T = *time constant*, in **seconds,**
L = *inductance*, in **henrys,** and
R = *resistance*, in **ohms.**

A somewhat similar time constant will be determined in the next article for the series *R-C* circuit.

The time constant of an *R-L* series circuit can be varied by varying either the inductance or the resistance of the circuit. With a given inductance, a small value of resistance results in a large time constant, and the resulting transient is of long duration. A large resistance, on the other hand, means a small time constant and a short transient. With a fixed applied potential, of course, an increased value of resistance would mean a "peak" current much lower in value. As an extreme case, an infinite resistance would produce a transient which resulted in zero current in zero time!

The equation of instantaneous potential can be found by substituting current obtained from equation (17-13) in equation (17-2). It will be noted that if an emf is suddenly applied to a series *R-L* circuit, inasmuch as current cannot change instantaneously, there is no initial *iR* drop, and all of the opposing potential is supplied by the inductance. This is the electrical analogue of mechanical inertia.

EXAMPLE 17.03. Determine the time constant for the circuit of Example 17.02 and employ this value in determining the solution.

$$T = \frac{L}{R} = \frac{10 \times 10^{-3}}{5} = 2 \times 10^{-3} \text{ second} = 2 \text{ milliseconds.}$$

This value, we realize, happens in this case to be the same as the time intervals of the three switch positions. We know, then, that during each of these intervals, the current value will make 63.2 per cent of the transition from i_0 to i_∞. The solution then becomes:

(1) *Initially open switch closed to left* ($\Delta t_1 = T$):

$$i_{f_1} = 0.632(i_{\infty_1} - i_{0_1}) = 0.632(-15 - 0) = -9.47 \text{ amperes.}$$

(2) *Switch closed to right* ($\Delta t_2 = T$):

$$i_{f_2} = i_{f_1} + 0.632(i_{\infty_2} - i_{f_1}) = -9.47 + 0.632(20 + 9.47) = 9.15 \text{ amperes.}$$

(3) *Switch again closed to left* ($\Delta t_3 = T$):

$$i_{f_3} = i_{f_2} + 0.632(i_{\infty_3} - i_{f_2}) = 9.15 + 0.632(-15 - 9.15) = -6.11 \text{ amperes.}$$

17.04. Transients in Series R-C Circuits with Direct EMF

Consider the circuit of Fig. 17.04A, with switch *S* initially closed to the left. We shall assume that steady-state conditions prevail. This means that the current *i* is zero and that the capacitor has become charged to a potential

$$v_0 = \frac{q_0}{C} = E_0. \qquad (17\text{-}18)$$

At a time $t = t_0$, switch S is suddenly thrown to the right, replacing E_0 by a different emf, E. Ignoring the momentary disturbance while the switch is open during switching, a new steady-state condition, depending on the value of E, may be required. Mathematically, we can write the equation of dynamic equilibrium (Kirchhoff's potential law) at any instant as follows:

$$e = \frac{q}{C} + R\frac{dq}{dt}. \tag{17-19}$$

From this equation it follows that

$$\left(E - \frac{q}{C}\right) = R\frac{dq}{dt}, \tag{17-20}$$

and

$$\frac{dq}{\left(E - \frac{q}{C}\right)} = \frac{1}{R}\,dt, \tag{17-21}$$

or

$$\frac{dq}{q - CE} = -\frac{1}{RC}\,dt. \tag{17-22}$$

Integrating, and paying due regard to the initial conditions, we have

$$\ln\,(q - CE)\Big]_{q_0}^{q} = -\frac{t}{RC}\Big]_{t_0}^{t}, \tag{17-23}$$

or

$$\ln\,(q - CE) - \ln\,(q_0 - CE) = -\frac{1}{RC}(t - t_0). \tag{17-24}$$

This can be written as

$$\ln\left(\frac{q - CE}{q_0 - CE}\right) = -\frac{1}{RC}(t - t_0), \tag{17-25}$$

or as

$$\frac{q - CE}{q_0 - CE} = \epsilon^{-\frac{1}{RC}(t - t_0)}, \tag{17-26}$$

from which

$$q - CE = (q_0 - CE)\epsilon^{-\frac{1}{RC}(t - t_0)}. \tag{17-27}$$

Solving equation (17-27) for q gives us the *general equation for charge in an R-C series circuit activated by a steady emf:*

▶

$$q = CE + (q_0 - CE)\epsilon^{-\frac{1}{RC}(t - t_0)}, \tag{17-28}$$

where:

q = *instantaneous charge on one conductor of capacitance,* in **coulombs,**

q_0 = *instantaneous charge,* in **coulombs,** *at time* t_0,

E = *constant applied emf,* in **volts,**

R = *total resistance of circuit,* in **ohms,**

C = *total capacitance of circuit,* in **farads.**

Figure 17.02B applies to this equation, if q is substituted for i.

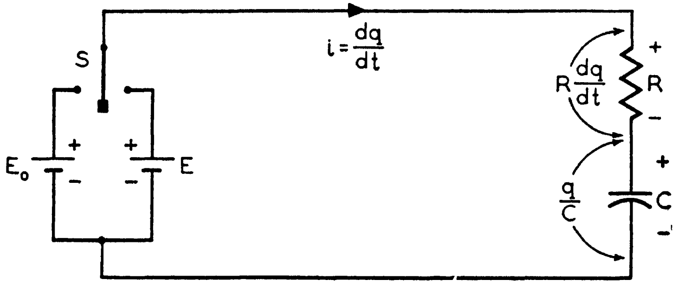


Fig. 17.04A. Circuit for study of transients in series R-C combination with direct emf.

Denoting $t_0 = 0$ and $q_\infty = CE$ (the value of charge after the transient has vanished), we may rewrite (17-28) as follows:

▶ $$q = q_\infty + (q_0 - q_\infty)\epsilon^{-\frac{t}{RC}}. \tag{17-29}$$

Figure 17.02C applies to this equation, if q is substituted for i and if $q_\infty = CE$. It should be realized that v_∞ (or q_∞) can have any value of either polarity relative to v_0 (or q_0). The general equation for current can be found by taking the derivative of (17-29):

▶ $$i = \frac{dq}{dt} = -\frac{1}{RC}(q_0 - q_\infty)\epsilon^{-\frac{t}{RC}} = -\frac{E_0 - E}{R}\epsilon^{-\frac{t}{RC}}. \tag{17-30}$$

We shall discuss briefly two special cases which are frequently encountered in practice.

Case 1: $E_0 = 0$; $q_0 = 0$. In this case, *the initial value of the charge is zero,* and equations (17-29) and (17-30) become, respectively,

▶ $$q = q_\infty(1 - \epsilon^{-\frac{t}{RC}}) = CE(1 - \epsilon^{-\frac{t}{RC}}), \tag{17-31}$$

and

▶
$$i = \frac{E}{R}\epsilon^{-\frac{t}{RC}}. \tag{17-32}$$

These are the standard equations for charging an initially uncharged capacitor.

Case 2: $E = 0$, $q_\infty = 0$. In this case, *the charge is decaying from an initial value*, $q_0 = CE_0$, *to zero*, and the current is decaying to zero from an initial value $i_0 = \frac{-E_0}{R}$. Equations (17-29) and (17-30) become, respectively,

▶
$$q = q_0\epsilon^{-\frac{t}{RC}} = CE_0\epsilon^{-\frac{t}{RC}}, \tag{17-33}$$

and

▶
$$i = -\frac{E_0}{R}\epsilon^{-\frac{t}{RC}}. \tag{17-34}$$

These are the standard equations for discharging a capacitor through resistance. It will be noted that the equation for charge, in this case, contains only a transient term. In both the charging and the discharging cases, the equation for current consists of only a tran-

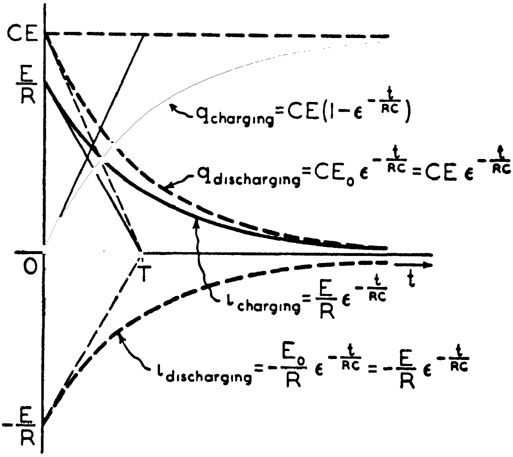


Fig. 17.04B. Charging and discharging of a capacitor through resistance.

sient term, that for discharging being of opposite sign to that for charging. This condition is logical, inasmuch as the discharge current must flow, physically, in an opposite direction to the charging current. Figure 17.04B illustrates this requirement.

17.05. The Series R-C Time Constant, T = RC

The time constant, T, in an R-C series circuit has all of the same general significance that it had in an R-L series circuit. It can be evaluated for the R-C case from the basic definition and from either equation (17-29) or equation (17-30):

$$\epsilon^{-\frac{T}{RC}} = \epsilon^{-1}, \tag{17-35}$$

$$\frac{T}{RC} = 1, \tag{17-36}$$

and *the time constant for a series R-C circuit* is

$$\blacktriangleright \qquad T = RC, \tag{17-37}$$

where:

T = *time constant,* in **seconds,**
R = *resistance,* in **ohms,** and
C = *capacitance,* in **farads.**

The time constant of an R-C series circuit can be varied by varying either the capacitance or the resistance of the circuit. With a given capacitance, a large resistance results in a large time constant. This statement means, physically, that a given potential will move charges at a relatively low rate because of the excessive opposition, resulting in a slow charge (or discharge) of the capacitor. On the other hand, if resistance is small, the time constant is small, and a given potential can move charges at a relatively high rate, resulting in a rapid charge (or discharge) of the capacitor. As an extreme case, zero resistance would result in an instantaneous charging of the capacitor to maximum value, but an infinite current would have to flow to accomplish the feat!

From equations (17-30) and (17-32), it will be noted that, inasmuch as charge on a capacitor plate cannot change instantaneously, if a potential is suddenly applied to an initially uncharged series R-C circuit, the initial capacitor potential is zero, and all of the opposing potential is supplied by the resistance in the form of iR

drop. This condition is analogous to that in which mechanical friction serves as the initial speed-limiting factor in the compression of a spring.

17.06. Transients in Series R-L-C Circuits with Direct EMF

The transients discussed thus far have been "single-energy" transients, because in each case there was but one type of energy-storing parameter present. We shall now consider transients in circuits containing both types of energy-storing parameters, L and C, as well as resistance. Let us refer to the circuit of Fig. 17.06A.

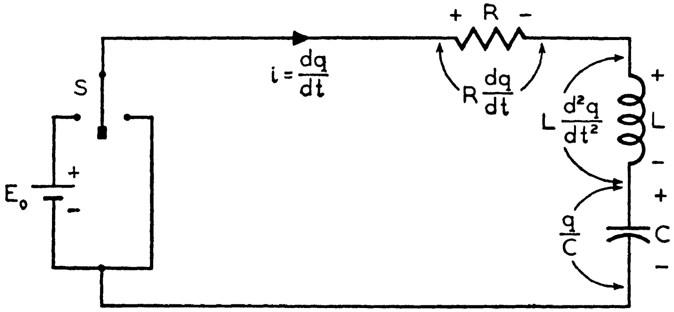


Fig. 17.06A. Circuit for study of transients in series R-L-C combination with direct emf.

To simplify the analysis, we shall make the assumption that when $t = 0$, $i = 0$. Any value which this quantity may actually have other than zero can be accounted for at the end of the derivation through use of the superposition principle.

The condition for dynamic equilibrium for this case is found, from equation (17-1a), to be

▶ $$E = L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt. \tag{17-38}$$

Differentiating, and noting that E is constant, we have

$$0 = L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C}. \tag{17-39}$$

The *auxiliary equation* is

$$Lm^2 + Rm + \frac{1}{C} = 0, \tag{17-40}$$

or

$$m^2 + \frac{R}{L} m + \frac{1}{LC} = 0. \tag{17-41}$$

The solution of this quadratic equation is

$$m = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = -a \pm b, \tag{17-42}$$

where

▶ $$a = \frac{R}{2L}, \quad \text{and} \quad b = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}. \tag{17-43}$$

The *complementary function* (the *transient term*) is

$$i_t = k_1 \epsilon^{(-a+b)t} + k_2 \epsilon^{(-a-b)t}. \tag{17-44}$$

The *particular integral* (the *steady-state term*) is zero, since the physical situation is one in which a capacitor is being charged by a direct emf, with a consequent decay of current to zero as full charge is attained. Mathematically,

$$i_{\text{s.s.}} = 0. \tag{17-45}$$

The complete solution is the sum of the particular integral and the complementary function:

$$i = i_t + i_{\text{s.s.}} = k_1 \epsilon^{(-a+b)t} + k_2 \epsilon^{(-a-b)t} + 0. \tag{17-46}$$

By imposing the known "boundary conditions," the constants k_1 and k_2 can be evaluated. These conditions are: when $t = 0$, $i = 0$; and when $t = 0$, $q = q_0$. Imposing the first condition on equation (17-46) gives

$$0 = k_1 + k_2, \quad \text{or} \quad k_2 = -k_1. \tag{17-47}$$

From equation (17-38),

$$\frac{1}{C}\int i\,dt = \frac{q}{C} = E - Ri - L\frac{di}{dt}, \tag{17-48}$$

or

$$q = CE - CRi - CL\frac{di}{dt}. \tag{17-49}$$

Substituting (17-46) in (17-49), we have

$$\begin{aligned} q = CE &- CR[k_1 \epsilon^{(-a+b)t} + k_2 \epsilon^{(-a-b)t}] \\ &- CL\,[k_1(-a+b)\epsilon^{(-a+b)t} + k_2(-a-b)\epsilon^{(-a-b)t}]. \end{aligned} \tag{17-50}$$

Imposing the second boundary condition on (17-50) and substituting equation (17-47) gives

$$k_1 = -k_2 = \frac{CE - q_0}{2CLb}. \tag{17-51}$$

Substitution from (17-43) converts this equation to

$$k_1 = -k_2 = \frac{CE - q_0}{2CL\sqrt{\dfrac{R^2}{4L^2} - \dfrac{1}{LC}}} = \frac{CE - q_0}{\sqrt{R^2C^2 - 4LC}}. \tag{17-52}$$

By substitution of this relation in equation (17-46), we obtain finally *the general expression for current in a series R-L-C circuit to which a direct emf has been suddenly applied:*

▶ $$i = \frac{CE - q_0}{\sqrt{R^2C^2 - 4LC}}\left[\epsilon^{(-a+b)t} - {}^{(-a-b)t}\right]. \tag{17-53}$$

Substitution of (17-48) and (17-52) in (17-50) gives the following *general expression for charge* in the same circuit:

▶ $$q = CE + (q_0 - CE)\left[\frac{RC + \sqrt{R^2C^2 - 4LC}}{2\sqrt{R^2C^2 - 4LC}}\epsilon^{(-a+b)t} - \frac{RC - \sqrt{R^2C^2 - 4LC}}{2\sqrt{R^2C^2 - 4LC}}\epsilon^{(-a-b)t}\right]. \tag{17-54}$$

This equation has the same general form as equation (17-29) but differs in that it contains two exponential factors instead of only one, because it describes a double-energy transient, rather than a single-energy transient. A similar comparison exists between (17-53) and (17-30).

A consideration of equation (17-43) reveals that the value of b will fall into one of three general cases, depending on the relative values of $(R^2/4L^2)$ and $(1/LC)$. The physical interpretations of these three cases are highly important.

Case 1: *Overdamped* (*Non-oscillatory*) *Case:* $(R^2/4L^2 > 1/LC)$. In this case, b is real and the exponents of equations (17-53) and (17-54) are real. As time approaches infinity, both exponential terms approach zero, and the charge on the capacitor approaches the value

CE. During the transient period, both the current and the charge are unidirectional, as indicated by Fig. 17.06B. The initial inrush of current is sufficiently limited by resistance and inductance that the capacitor does not overcharge, but comes gradually up to its final steady-state condition of charge.

Case 2: *Critically Damped* (*Limiting Non-oscillatory*) *Case:* ($R^2/4L^2 = 1/LC$). In this case, b is zero, and the exponents of equations (17-53) and (17-54) are equal but still real. As they stand, these equations are indeterminate under this condition.

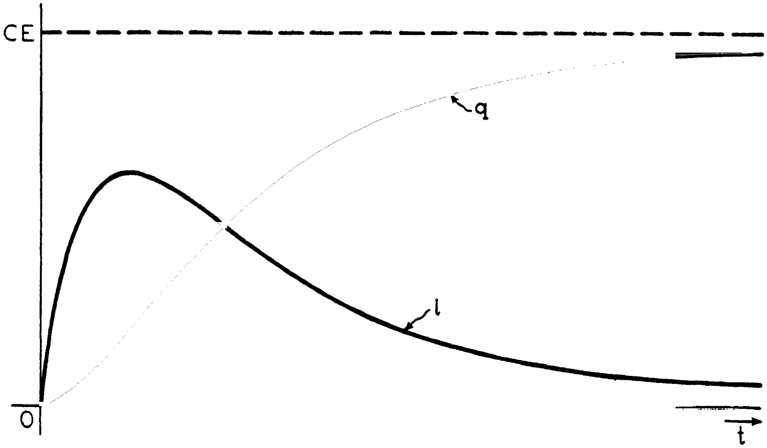


Fig. 17.06B. Transient resulting from sudden application of direct emf to an overdamped series R-L-C circuit.

Solution is possible by taking a limiting case of equations (17-62) and (17-63). The curves would be similar to those of Fig. 17.06B. Resistance has been reduced to the point where the charge builds up on the capacitor at the maximum rate possible without "overshooting the mark" and overcharging the capacitor, and without the consequent necessity of partially discharging before the steady-state condition can be attained. Energy stored in the inductance during the growth of current goes partially into the capacitor and is partially dissipated by resistance as the current decays and the transient dies.

Case 3: *Underdamped* (*Oscillatory*) *Case:* ($R^2/4L^2 < 1/LC$). In this case, b is imaginary. The evaluation of b can be accomplished by rewriting (17-43) as follows:

$$b = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \sqrt{(-1)\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}, \tag{17-55}$$

or

▶ $$b = j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = j\beta, \tag{17-56}$$

where $$\beta = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$

By substitution, equation (17-53) becomes

$$i = \frac{CE - q_0}{\sqrt{R^2C^2 - 4LC}}\left[\epsilon^{(-\alpha+j\beta)t} - \epsilon^{(-\alpha-j\beta)t}\right]. \tag{17-57}$$

This can be manipulated further, as follows:

$$i = \frac{CE - q_0}{\sqrt{R^2C^2 - 4LC}}\left[\epsilon^{-\alpha t} \cdot \epsilon^{j\beta t} - \epsilon^{-\alpha t} \cdot \epsilon^{-j\beta t}\right] \tag{17-58}$$

$$= \frac{CE - q_0}{\sqrt{R^2C^2 - 4LC}}\,\epsilon^{-\alpha t}\left[\epsilon^{j\beta t} - \epsilon^{-j\beta t}\right]. \tag{17-59}$$

By equation (5-20), this can be written as:

$$i = \frac{CE - q_0}{\sqrt{R^2C^2 - 4LC}}\,\epsilon^{-\alpha t}[(\cos\beta t + j\sin\beta t) - (\cos\beta t - j\sin\beta t)], \tag{17-60}$$

or $$i = \frac{CE - q_0}{\sqrt{R^2C^2 - 4LC}}\,\epsilon^{-\alpha t} \cdot (2j\sin\beta t). \tag{17-61}$$

The denominator of (17-61) is imaginary. Rewriting it in terms of j allows cancellation with the j of the numerator, and yields the following expression for current:

▶ $$i = \frac{2(CE - q_0)}{\sqrt{4LC - R^2C^2}}\,\epsilon^{-\alpha t} \cdot \sin\beta t. \tag{17-62}$$

By a similar process, the expression for charge becomes

▶ $$q = CE + \frac{2(q_0 - CE)\sqrt{LC}}{\sqrt{4LC - R^2C^2}}\,\epsilon^{-\alpha t} \cdot \sin(\beta t + \phi), \tag{17-63}$$

where $$\phi = \tan^{-1}\frac{\sqrt{4LC - R^2C^2}}{RC}.$$

A graphical representation of equations (17-62) and (17-63) for the case where $q_0 = 0$ is given in Fig. 17.06C.

Physically, the resistance is so low in this case that the current grows to a value such that when it decays, the energy of the collapsing inductive field overcharges the capacitor. The capacitor then discharges, returning energy to the inductance, the inductive field then collapses, returning energy to the capacitor, and so on; a series of oscillations occurring about the steady-state value. This process continues, with exponentially diminishing amplitude, until the last vestige of "extra" energy has been dissipated by the resistance.

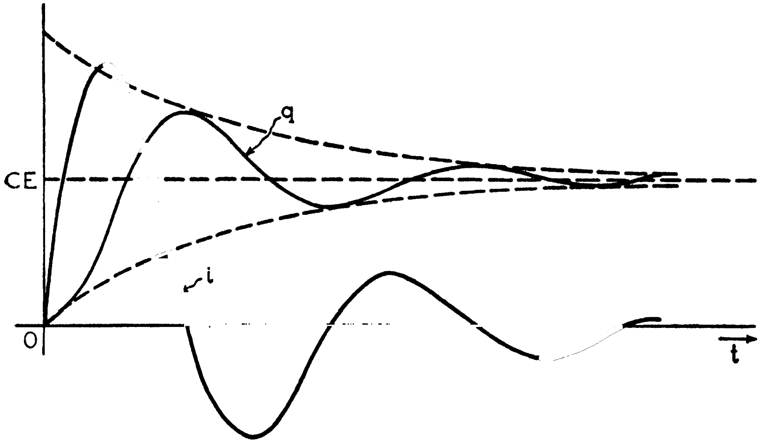


Fig. 17.06C. Transient resulting from sudden application of direct emf to an underdamped series R-L-C circuit.

The frequency of the oscillation, f_0, is obtained from either equation (17-62) or equation (17-63). Denoting T_0 as the period of the oscillation and noting that this is the time corresponding to a complete cycle of 2π radians, we may write

$$\beta T_0 = 2\pi \tag{17-64}$$

or

▶ $$T_0 = \frac{1}{f_0} = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}. \tag{17-65}$$

From this equation,

▶
$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}. \tag{17-66}$$

It is noted, by comparison with equation (9-67), that if resistance is zero, f_0 is the same as the a-c resonant frequency of the circuit.

If a steady state exists in the circuit of Fig. 17.06A with the switch thrown to the left, the capacitor is charged to a potential equal to the emf of the battery, E, and the current flow is zero. Let us now picture the events if the switch is thrown suddenly to the right. The capacitor can now be expected to discharge through the R-L-C series circuit. A moment of thought will reveal that this condition corresponds to the situations already discussed with the exceptions that now $E = 0$ and $q_0 \neq 0$. Equations (17-53) and (17-54) become, for this special case of *capacitor discharge through series resistance and inductance,*

▶
$$i = \frac{-q_0}{\sqrt{R^2C^2 - 4LC}}\left[\epsilon^{(-a+b)t} - \epsilon^{(-a-b)t}\right], \tag{17-67}$$

and

▶
$$q = q_0\left[\frac{RC + \sqrt{R^2C^2 - 4LC}}{2\sqrt{R^2C^2 - 4LC}}\,\epsilon^{(-a+b)t} - \frac{RC - \sqrt{R^2C^2 - 4LC}}{2\sqrt{R^2C^2 - 4LC}}\,\epsilon^{(-a-b)t}\right]. \tag{17-68}$$

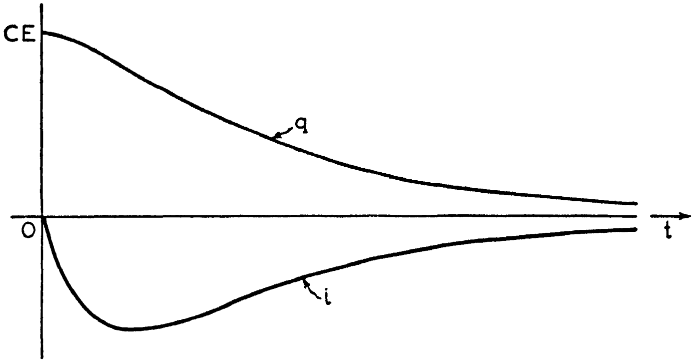
We can see that here, again, will be the three cases of overdamping, critical damping, and underdamping, depending on the relative values of $(R^2/4L^2)$ and $(1/LC)$. In these cases, the initial value of charge is q_0, and the final value of charge, q_∞, is zero. *For the oscillatory case,* equations (17-62) and (17-63) become, for discharge,

▶
$$i = \frac{-2q_0}{\sqrt{4LC - R^2C^2}}\,\epsilon^{-at}\sin\beta t, \tag{17-69}$$

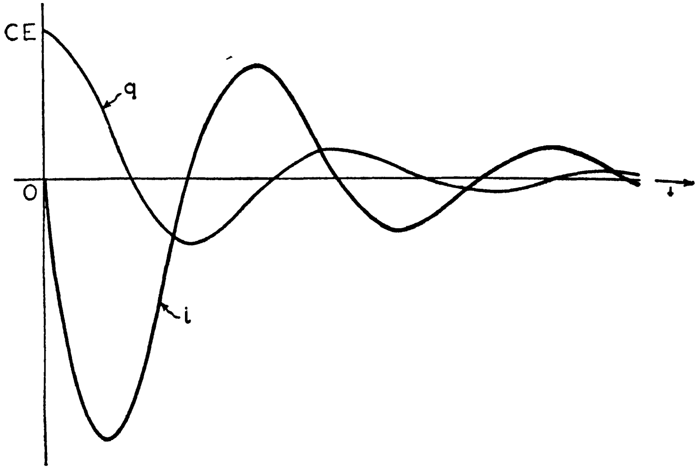
and

▶
$$q = \frac{2q_0\sqrt{LC}}{\sqrt{4LC - R^2C^2}}\,\epsilon^{-at}\sin(\beta t + \phi). \tag{17-70}$$

Curves for both non-oscillatory and oscillatory capacitor discharges are shown in Fig. 17.06D.



(a) Overdamped (Non-oscillatory) Case.



(b) Underdamped (Oscillatory) Case.

Fig. 17.06D. Discharging of capacitor through resistance and inductance in series.

17.07. Transients in Series R-L Circuits with Alternating EMF

If a sinusoidal alternating emf replaces the right-hand battery of Fig. 17.02A and if the switch has been open previous to the start of time measurement, then when the switch is suddenly closed to the right, the expression for dynamic equilibrium is

▶ $$e = E_m \sin(\omega t + \theta_e) = L\frac{di}{dt} + Ri, \tag{17-71}$$

or

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E_m}{L} \sin (\omega t + \theta_e). \qquad (17\text{-}72)$$

This equation is a linear differential equation of the first order, which has, in general, the form

$$\frac{dy}{dx} + f_1(x) \cdot y = f_2(x). \qquad (17\text{-}73)$$

The general solution of (17-73) is

$$y = \epsilon^{-\int f_1(x)dx} \int \epsilon^{\int f_1(x)dx} \cdot f_2(x)dx + k\epsilon^{-\int f_1(x)dx}. \qquad (17\text{-}74)$$

The solution of equation (17-72) then becomes

$$i = i_{s.s.} + i_t = \frac{E_m}{L} \epsilon^{-\frac{R}{L}t} \int \epsilon^{\frac{R}{L}t} \cdot \sin (\omega t + \theta_c)\, dt + k_1 \epsilon^{-\frac{R}{L}t}. \quad (17\text{-}75)$$

We know from our previous work that the steady-state solution is

$$i_{s.s.} = \frac{E_m}{Z} \sin (\omega t + \theta_i) = \frac{E_m}{Z} \sin (\omega t + \theta_e - \theta), \qquad (17\text{-}76)$$

where $Z = \sqrt{R^2 + \omega^2 L^2}$, and $\theta = \tan^{-1}\left(\frac{\omega L}{R}\right) = (\theta_e - \theta_i)$.

Equation (17-76) then becomes

$$i = \frac{E_m}{Z} \sin (\omega t + \theta_e - \theta) + k_1 \epsilon^{-\frac{R}{L}t}. \qquad (17\text{-}77)$$

Imposing the initial condition that when $t = 0$, then $i = 0$, equation (17-77) becomes

$$0 = \frac{E_m}{Z} \sin (\theta_e - \theta) + k_1. \qquad (17\text{-}78)$$

From this equation it develops that

$$k_1 = \frac{-E_m}{Z} \sin (\theta_e - \theta). \qquad (17\text{-}79)$$

By substitution in (17-77), we then can obtain the following *general equation for current in a series R-L circuit to which a sinusoidal emf has suddenly been applied:*

▶ $$i = i_{s.s.} + i_t = \frac{E_m}{Z} \sin (\omega t + \theta_e - \theta) - \frac{E_m}{Z} \sin (\theta_e - \theta) \epsilon^{-\frac{R}{L}t}, \qquad (17\text{-}80)$$

where:

$Z = \sqrt{R^2 + \omega^2 L^2}$ = *impedance of the circuit,* in **ohms,**

$\theta = \tan^{-1}\left(\frac{\omega L}{R}\right)$ = *steady-state power factor angle of the circuit,*

θ_e = *initial phase angle of the emf wave.*

From this equation, it is seen that the transient term depends on both the power factor of the circuit, as indicated by the presence of θ, and the phase angle, θ_e, of the emf wave at the moment the transient begins. If $\theta_e = \theta$, the transient term does not exist. This condition is logical, since it means that the application of emf occurs at a time when the steady-state current wave would be zero anyhow. No transient term is needed to adjust the actual value of zero current to a steady-state demand for some other value of current. If $\theta_e \neq \theta$, a transient term is necessary. Figure 17.07 shows such a case.

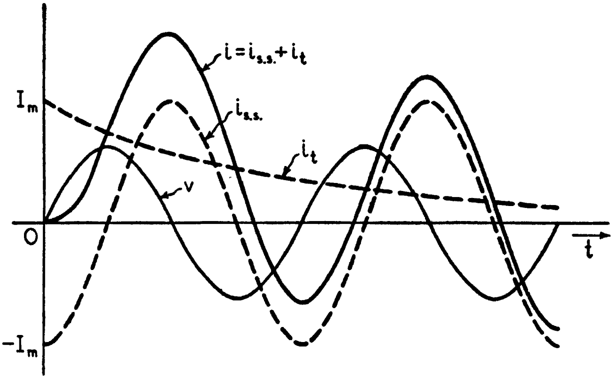


Fig. 17.07. Transient resulting from sudden application of alternating emf to a series R-L circuit.

In the case shown, the steady-state demand is, at the start of time measurement, for a value of current equal to the negative maximum of $i_{s.s.}$. In actuality, on the other hand, the initial value of current must be zero in a circuit containing inductance. The transient term appears in order to serve temporarily as an axis about which the a-c term can oscillate while the resistance is draining away the excess inductive energy associated with the transient.

17.08. Transients in Series R-C Circuits with Alternating EMF

If a capacitance replaces the inductance of the preceding article, we can write the following equation of dynamic equilibrium:

▶ $$e = E_m \sin(\omega t + \theta_e) = R\frac{dq}{dt} + \frac{q}{C}. \tag{17-81}$$

This can be expressed as

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{E_m}{R}\sin(\omega t + \theta_e). \tag{17-82}$$

The method of solving this equation is similar to the one employed in the preceding article. The solution is

$$q = q_{\text{s.s.}} + q_t = \frac{-E_m}{\omega Z}\cos(\omega t + \theta_e + \theta) + k_2 \epsilon^{-\frac{t}{RC}}, \tag{17-83}$$

where:

$$Z = \sqrt{R^2 + \frac{1}{\omega^2 C^2}},$$

$$\theta = \tan^{-1}\left(\frac{-1}{\omega CR}\right).$$

Imposing the initial condition that when $t = 0$, then $q = q_0$, we find k_2 to be

$$k_2 = q_0 + \frac{E_m}{\omega Z}\cos(\theta_e + \theta). \tag{17-84}$$

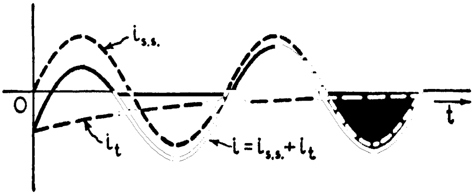
By substitution in (17-83), we obtain the following expression for *instantaneous charge in a series R-C circuit to which a sinusoidal emf has suddenly been applied:*

▶ $$q = \frac{-E_m}{\omega Z}\cos(\omega t + \theta_e + \theta) + \left[q_0 + \frac{E_m}{\omega Z}\cos(\theta_e + \theta)\right]\epsilon^{-\frac{t}{RC}}. \tag{17-85}$$

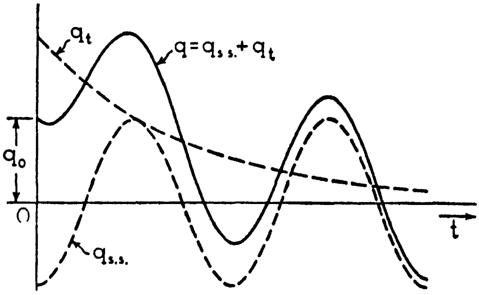
Differentiation of (17-85) yields the following expression for current:

▶ $$i = \frac{dq}{dt} = \frac{E_m}{Z}\sin(\omega t + \theta_e + \theta) - \left[\frac{q_0}{RC} + \frac{E_m}{\omega RCZ}\cos(\theta_e + \theta)\right]\epsilon^{-\frac{t}{RC}}. \tag{17-86}$$

Figure 17.08 shows these two relations in graphical form for a case where $q_0 \neq 0$. Recalling that the charge of a capacitance cannot change instantaneously, one can readily provide a physical interpretation of the phenomena pictured by these curves.



(a) Current.



(b) Charge.

Fig. 17.08. Transient resulting from sudden application of alternating emf to a series combination of resistance and initially charged capacitance.

17.09. Transients in Series R-L-C Circuits with Alternating EMF

If a sinusoidal alternating emf replaces the battery of Fig. 17.06A and if the switch has been initially open and the capacitance initially uncharged, when the switch is suddenly closed to the left the expression for dynamic equilibrium is

$$e = E_m \sin(\omega t + \theta_e) = L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt. \qquad (17\text{-}87)$$

Differentiating,

$$\omega E_m \cos(\omega t + \theta_e) = L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i, \qquad (17\text{-}88)$$

or

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = \frac{\omega E_m}{L}\cos(\omega t + \theta_e). \qquad (17\text{-}89)$$

This is a linear differential equation of the second order and first degree. The auxiliary equation is

$$m^2 + \frac{R}{L} m + \frac{1}{LC} = 0. \tag{17-90}$$

Then
$$m = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = -a \pm b, \tag{17-91}$$

where we have denoted

$$\left.\begin{aligned} a &= \frac{R}{2L} \\ b &= \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \end{aligned}\right\} \tag{17-92}$$

The transient term of the complete solution is

$$i_t = k_1 \epsilon^{(-a+b)t} + k_2 \epsilon^{(-a-b)t}. \tag{17-93}$$

The steady-state term of the complete solution is

$$i_{s.s.} = \frac{E_m}{Z} \sin (\omega t + \theta_e - \theta), \tag{17-94}$$

where:

$$Z = \sqrt{R^2 + \left(\omega L + \frac{-1}{\omega C}\right)^2}$$

$$\theta = \tan^{-1}\left(\frac{\omega L + \dfrac{-1}{\omega C}}{R}\right),$$

Then the complete expression for current is

$$i = i_{s.s.} + i_t = \frac{E_m}{Z} \sin (\omega t + \theta_e - \theta) + k_1 \epsilon^{(-a+b)t} + k_2 \epsilon^{(-a-b)t}. \tag{17-95}$$

Imposing the initial conditions $t = 0$, $i = 0$, and $q = 0$ on (17-87) results in

$$E_m \sin \theta_e = L \frac{di}{dt}. \tag{17-96}$$

Substituting (17-95) in (17-96) and placing $t = 0$ yields

$$E_m \sin \theta_e = L \left[\frac{\omega E_m}{Z} \cos (\theta_e - \theta) + k_1(-a + b) + k_2(-a - b)\right]. \tag{17-97}$$

Let us now denote

▶ $$\alpha_1 = (-a + b), \qquad \text{and} \qquad \alpha_2 = (-a - b). \tag{17-98}$$

Substitution of (17-98) in (17-97) and rearrangement of terms produces

$$k_1\alpha_1 + k_2\alpha_2 = \left[\frac{E_m}{L}\sin\theta_e - \frac{\omega E_m}{Z}\cos(\theta_e - \theta)\right]. \tag{17-99}$$

Imposing the initial conditions $t = 0$, $i = 0$, and $q = 0$ on (17-95) yields

$$0 = \frac{E_m}{Z}\sin(\theta_e - \theta) + k_1 + k_2, \tag{17-100}$$

or

$$k_1 + k_2 = \frac{-E_m}{Z}\sin(\theta_e - \theta). \tag{17-101}$$

Simultaneous solution of (17-99) and (17-101) results in the following values of k_1 and k_2:

$$k_1 = \left\{\frac{1}{2bL}\left[E_m\sin\theta_e - \frac{\omega L E_m}{Z}\cos(\theta_e - \theta) - \frac{E_m R}{2Z}\sin(\theta_e - \theta)\right] - \frac{E_m}{2Z}\sin(\theta_e - \theta)\right\}, \tag{17-102}$$

$$k_2 = \left\{\frac{-1}{2bL}\left[E_m\sin\theta_e - \frac{\omega L E_m}{Z}\cos(\theta_e - \theta) - \frac{E_m R}{2Z}\sin(\theta_e - \theta)\right] - \frac{E_m}{2Z}\sin(\theta_e - \theta)\right\}. \tag{17-103}$$

It will be observed that the term within the brackets of each of the above equations is a potential which is independent of time but dependent on the initial conditions. Let us denote this as E':

▶ $$E' = \left[E_m\sin\theta_e - \frac{\omega L E_m}{Z}\cos(\theta_e - \theta) - \frac{E_m R}{2Z}\sin(\theta_e - \theta)\right]. \tag{17-104}$$

The complete expression for the current wave then can be expressed as follows:

▶ $$i = i_{s.s.} + i_t = \frac{E_m}{Z}\sin(\omega t + \theta_e - \theta) + \left\{\frac{E'}{bL}\epsilon^{-at}\left[\frac{\epsilon^{bt} - \epsilon^{-bt}}{2}\right] - \frac{E_m}{Z}\sin(\theta_e - \theta)\epsilon^{-at}\left[\frac{\epsilon^{bt} + \epsilon^{-bt}}{2}\right]\right\}, \tag{17-105}$$

where:

$$E' = \left[E_m \sin \theta_e - \frac{\omega L E_m}{Z} \cos (\theta_e - \theta) - \frac{E_m R}{2Z} \sin (\theta_e - \theta)\right],$$

$$a = \frac{R}{2L},$$

$$b = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}},$$

$$Z = \sqrt{R^2 + \left(\omega L + \frac{-1}{\omega C}\right)^2}.$$

Three possible cases exist here, too, depending on the relative values of $(R^2/4L^2)$ and $(1/LC)$. A brief consideration of each of these cases follows.

Case 1: *Overdamped* (*Non-oscillatory*) *Case:* $(R^2/4L^2 > 1/LC)$. Since b is real, we can write

$$\left.\begin{aligned} \frac{\epsilon^{bt} - \epsilon^{-bt}}{2} &= \text{sirh}\, bt, \\ \frac{\epsilon^{bt} + \epsilon^{-bt}}{2} &= \cosh bt. \end{aligned}\right\}, \tag{17-106}$$

and the expression for current becomes

▶
$$i = i_{\text{s.s.}} + i_t = \frac{E_m}{Z} \sin (\omega t + \theta_e - \theta) + \left[\frac{E'}{bL} \cdot \epsilon^{-at} \sinh bt - \frac{E_m}{Z} \sin (\theta_e - \theta) \cdot \epsilon^{-at} \cosh bt\right]. \tag{17-107}$$

The fact that there are two transient terms is in accord with the fact that this is a double-energy transient.

Case 2: *Critically Damped* (*Limiting Oscillatory*) *Case:* $(R^2/4L^2 = 1/LC)$. This is a singular case which represents the transition from Case 1 to Case 3. The chance that it would occur in practice is extremely remote. For this reason, it does not warrant further discussion here.

Case 3: *Underdamped* (*Oscillatory*) *Case:* $(R^2/4L^2 < 1/LC)$. In this case b is imaginary, and we must again employ the notation of equation (17-56):

$$b = j\beta = j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}. \tag{17-108}$$

Recognizing that

$$\left.\begin{aligned}\frac{\epsilon^{j\beta t}-\epsilon^{-j\beta t}}{2j}&=\sin\beta t,\\ \frac{\epsilon^{j\beta t}+\epsilon^{-j\beta t}}{2}&=\cos\beta t.\end{aligned}\right\}\qquad(17\text{-}109)$$

we can obtain directly from equation (17-105) the following expression for current in an underdamped *R-L-C* circuit suddenly energized by a sinusoidal emf:

$$\blacktriangleright\quad i=i_{\text{s.s.}}+i_t=\frac{E_m}{Z}\sin(\omega t+\theta_e-\theta)$$
$$+\left[\frac{E'}{\beta L}\epsilon^{-\alpha t}\sin\beta t-\frac{E_m}{Z}\sin(\theta_e-\theta)\cdot\epsilon^{-\alpha t}\cos\beta t\right]\cdot\qquad(17\text{-}110)$$

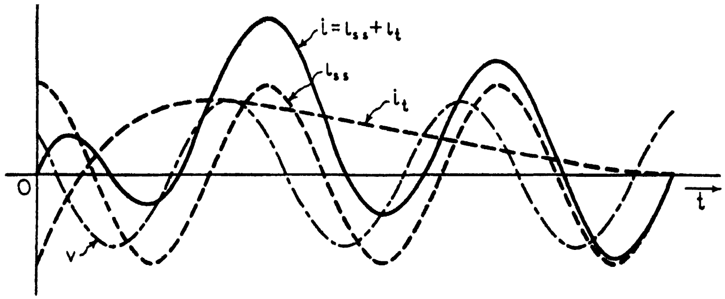
Since the two identically exponentially damped transient terms of this equation also vary sinusoidally with the same frequency, they can be combined to yield a single sinusoidal term containing a phase angle and an amplitude governed by the terms it replaces. Thus, we can rewrite equation (17-110) to obtain the following equation for an underdamped *R-L-C* series circuit suddenly energized by a sinusoidal emf:

$$\blacktriangleright\quad i=i_{\text{s.s.}}+i_t=\frac{E_m}{Z}\sin(\omega t+\theta_e-\theta)+I'\cdot\epsilon^{-\alpha t}\cdot\sin(\beta t-\gamma),\qquad(17\text{-}111)$$

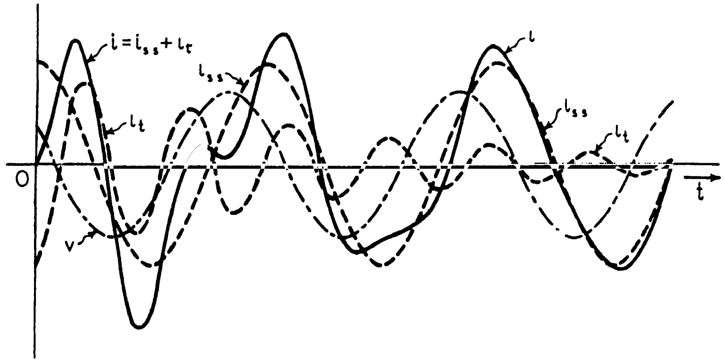
where:

$$I'=\sqrt{\left(\frac{E'}{\beta L}\right)^2+\left[\frac{E_m}{Z}\sin(\theta_e-\theta)\right]^2},$$
$$\gamma=\tan^{-1}\left[\frac{E_m\beta L\sin(\theta_e-\theta)}{E'Z}\right].$$

A study of this equation reveals that the frequency of the steady-state term is determined solely by the frequency of the applied emf, whereas the frequency of the transient term is determined entirely by the circuit parameters, *R*, *L*, and *C*. The frequency of the transient term may be either equal to, less than, or greater than the applied frequency. The damping factor, $\epsilon^{-\alpha t}$, soon damps out the transient term. Graphical interpretations of equations (17-107) and (17-111) for two particular cases appear in Fig. 17.09.



(a) Overdamped (Non-oscillatory) Case.



(b) Underdamped (Oscillatory) Case.

Fig. 17.09. Transient resulting from sudden application of alternating emf to a series combination of resistance, inductance, and initially uncharged capacitance.

17.10. A Word about Transients in Networks

The transient analysis of electric networks is based on the same general network theorems as the steady-state analysis: Kirchhoff's two laws, the loop method, the node method, superposition, Thévenin's theorem, and so on. However, instead of using the vector currents and potentials and the complex impedances which can be used in the steady-state analysis of networks, it is necessary to employ instantaneous currents and potentials and basic circuit parameters in the transient analysis. This is precisely what was done in the extremely simple cases of the present chapter. As these simple cases reveal, however, the mathematical treatment of transient behavior is considerably more involved than the mathematical treatment of steady-state behavior. Transient analysis calls for the solution of differential equations, whereas steady-state analysis calls for the solution of algebraic equations.

To simplify the general transient analysis of networks, several specialized mathematical methods (which are beyond the scope of this book) have been evolved. Even with these methods, the complete analysis of complicated networks may be extremely tedious. Several types of computing devices have been devised and are becoming ever more prominent in solving such problems. It is important to note, however, that a thorough understanding of the basic principles of circuit analysis which are presented in this book is a prerequisite to the use of the more specialized computational methods or the computing devices.

PROBLEMS

17-1. A 12-volt battery with an internal resistance of 1.5 ohms is connected in series with a noninductive resistor of 10 ohms and an inductor of 1.2 ohms of resistance and 100 millihenrys of inductance. The current in this circuit is at its steady-state value when suddenly one half of the cells of the battery are short-circuited. Graph the current from the instant of short circuit to an elapsed time of 1 second.

17-2. An inductor having a resistance of 5 ohms and an inductance of 1 henry is connected in series with a 5-ohm noninductive resistor. At the zero of time, a direct emf of 100 volts is suddenly impressed on this circuit, and 0.1 second later this potential is suddenly increased to 250 volts.

(a) Determine how long it will take for the current to reach 95 per cent of its final value.

(b) Graph the current over the time interval determined in (a).

17-3. An inductor having resistance R_1 and inductance L_1 is connected in parallel with a second inductor having a resistance R_2 and an inductance L_2. A noninductive resistor of resistance R_3 is connected in series with the parallel combination of inductors.

(a) Write the equations for the currents i_1, i_2, and i_3 as functions of time following the sudden application of a direct emf, E.

(b) Write the equations for i_1, i_2, and i_3 as functions of time following the sudden shorting out of E after the steady-state condition has been reached.

17-4. Review the solution of Prob. 8-2 as one example of the use of duals in the solution of transient circuit problems.

17-5. The d-c field circuit of a certain generator consists of the field winding itself, having resistance R_F and inductance L_F, in series with two series resistors having resistances R_1 and R_2, respectively, and a source of emf E. A voltage regulator of the vibrating type operates on the field circuit, alternately closing and opening contacts which short out series resistance R_1. The ratio of the time closed to the time open for this contactor determines the average value of the current through the field winding.

In a given case, $R_1 = 50$ ohms, $R_2 = 23$ ohms, $R_F = 27$ ohms, $L_F = 30$ henrys, and $E = 250$ volts. As set, the regulator contact is vibrating at a frequency of 6 cycles per second and has been in operation for a long time. The ratio of time open to time closed is: $t_c/t_0 = 0.2$.

(a) Between what limits is the field current varying?

(b) What is the average value of the field current?

17-6. A particular direct emf is suddenly applied to a certain inductor. If the resulting current undergoes an initial rate of change of 300 amperes per second and ultimately reaches a steady-state value of 10 amperes, what is the time constant of the circuit?

17-7. Determine the time constants of the following circuits: (a) The circuit of Prob. 17-1. (b) The circuit of Prob. 17-2. (c) The circuit of Prob. 17-5 with the contactor open. (d) The circuit of Prob. 17-5 with the contactor closed.

17-8. A 0.5-microfarad capacitor initially charged to a potential of 5000 volts is discharged through a 10-ohm resistor.

(a) What is the initial value of current?

(b) What is the initial rate of change of current?

(c) How long does it take for the capacitor potential to reach 4000 volts?

(d) What is the rate of change of current at the instant potential is 4000 volts?

(e) What is the time constant of this circuit?

17-9. The parameters of a certain series *R-C* circuit are: $R = 50$ ohms, $C = 100$ microfarads. The capacitor is initially uncharged.

(a) What time is required, following the sudden application of 50 volts, for the capacitor potential to reach 75 per cent of its steady-state value?

(b) If, at the moment $i = 0.75I_{s.s.}$, a switch is suddenly thrown, substituting for the 50-volt emf an emf of 100 volts and reverse polarity, find the value of capacitor potential after 10 additional milliseconds.

(c) At the end of (b), the switch is suddenly thrown to a third position which has the effect of substituting a resistance $R' = 1000$ ohms for the emf. What is the capacitor potential at the end of three additional milliseconds?

(d) Graph the curves of potential and current over the total interval embraced by the foregoing events.

17-10. A resistor of 20 ohms resistance and a capacitor of 200 microfarads capacitance are connected in series and a direct emf, $E = 200$ volts, is suddenly applied.

(a) What is the initial rate of change of current?

(b) What is the charge of the capacitor at $t = 0.01$ second?

(c) After the steady-state condition has been reached, a 50-ohm resistor is suddenly shunted across the capacitor. Write the equation for current in this resistor as a function of time.

17-11. In the circuit of Fig. 17.06A, $L = 10$ millihenrys, $C = 100$ microfarads, and $E = 250$ volts.

(a) Determine what value of resistance will produce critical damping. Call this R_{crit}.

(b) Compute the frequency of oscillation for $R = 0.5R_{\text{crit}}$.

(c) Assuming the capacitor to be initially uncharged, compute data for plotting three cycles of the charge and current curves for $R = 0.5R_{\text{crit}}$, following closure of the switch to the left.

(d) Assuming the capacitor to be initially uncharged, compute data for plotting charge and current curves for $R = 2R_{\text{crit}}$, over the same time interval as (c).

(e) Plot the curves of (c) and (d) on the same graph, and label them properly as belonging to the underdamped and the overdamped cases, respectively.

17-12. Repeat Prob. 17-11 for the case where the capacitor is initially fully charged to 250 volts and the switch is suddenly thrown to the right.

17-13. Solve Prob. 17-11 for the following values: $L = 20$ millihenrys, $C = 100$ microfarads, and $E = 250$ volts.

17-14. Solve Prob. 17-12 for the values of Prob. 17-13.

17-15. An emf, $e = 100 \sin (377t + 30°)$ volts, is suddenly applied to a circuit consisting of R and L in series. If $R = 10$ ohms and $L = 200$ millihenrys,

(a) Compute data for graphing i and its steady-state and transient components over 3 cycles of emf.

(b) Plot the data of (a).

17-16. Solve Prob. 17-15 for $e = 100 \sin 377t$ volts.

17-17. Solve Prob. 17-15 for $e = 100 \sin (377t - 30°)$ volts.

17-18. An emf, $e = 100 \sin (377t + 30°)$ volts, is suddenly applied to a circuit consisting of R and C in series. If $R = 100$ ohms and $C = 200$ microfarads,

(a) Compute data for graphing q and i and their transient and steady-state components over 3 cycles of emf.

(b) Plot the data of (a).

17-19. A typical lightning stroke can be simulated in the laboratory by means of a *surge generator*, in which charged capacitors replace the charged thundercloud of nature. The artificial lightning stroke consists of the discharge of these capacitors through R and L in series. With one such surge generator under a given condition of operation, the discharge path consists, essentially, of the following quantities in series: $R = 100$ ohms, $L = 294$ microhenrys, and $C = 0.00324$ microfarads. If the initial potential on the capacitor is 10,200 kilovolts,

(a) Determine the current equation.
(b) What is the maximum current?
(c) Determine the power equation.
(d) What is the maximum power, in kilowatts and in horsepower?
(e) Determine the energy released by the discharge.
(f) Graph the current of this discharge for 4 cycles.

17-20. Repeat Prob. 17-19 for $R = 300$ ohms. Graph the current over the same time interval.

17-21. An emf, $e = 100 \sin (377t + 30°)$ volts, is suddenly applied to a circuit consisting of R, L, and C, in series. If $R = 5$ ohms, $L = 200$ millihenrys, and $C = 200$ microfarads,

(a) Compute data for graphing i and its steady-state and transient components.

(b) What is the frequency of the transient term?
(c) Plot the data of (a).

17-22. Solve Prob. 17-21 for $R = 15$ ohms.

APPENDIX

TABLE A1

D-C Resistance of Standard Annealed Copper Wire—American Wire Gauge (B. and S.)

(Resistivity = 10.376 absolute circular-mil ohms per foot at 20°C)[1]

Gauge No.	Diameter in Mils d	Area in Circular Mils d^2	Absolute Ohms per 1000 Feet at 20°C	Pounds per 1000 Feet at 20°C
0000	460.00	211,600	0.04903	640.5
000	409.64	167,810	0.06183	507.9
00	364.80	133,080	0.07797	402.8
0	324.86	105,530	0.09832	319.5
1	289.30	83,694	0.1240	253.3
2	257.63	66,373	0.1564	200.9
3	229.42	52,634	0.1971	159.3
4	204.31	41,742	0.2486	126.4
5	181.94	33,102	0.3135	100.2
6	162.02	26,250	0.3953	79.46
7	144.28	20,816	0.4984	63.02
8	128.49	16,509	0.6285	49.98
9	114.43	13,094	0.7925	39.63
10	101.89	10,381	0.9994	31.43
11	90.742	8,234.0	1.261	24.92
12	80.808	6,529.9	1.589	19.77
13	71.961	5,178.4	2.004	15.68
14	64.084	4,106.8	2.526	12.43
15	57.068	3,256.7	3.186	9.858
16	50.820	2,582.9	4.018	7.818
17	45.257	2,048.2	5.067	6.200
18	40.303	1,624.3	6.388	4.917
19	35.890	1,288.1	8.055	3.899
20	31.961	1,021.5	10.16	3.092
21	28.462	810.10	12.81	2.452
22	25.347	642.40	16.15	1.945
23	22.571	509.45	20.37	1.542
24	20.100	404.01	25.68	1.223
25	17.900	320.40	32.39	0.9699
26	15.940	254.10	40.83	0.7692
27	14.195	201.50	51.50	0.6100
28	12.641	159.79	64.93	0.4837
29	11.257	126.72	81.87	0.3836
30	10.025	100.50	103.3	0.3042
31	8.928	79.70	130.2	0.2413
32	7.950	63.21	164.2	0.1913
33	7.080	50.13	207.0	0.1517
34	6.305	39.75	261.0	0.1203
35	5.615	31.52	329.2	0.09542
36	5.000	25.00	415.0	0.07568
37	4.453	19.83	523.4	0.06001
38	3.965	15.72	659.9	0.04759
39	3.531	12.47	832.2	0.03774
40	3.145	9.888	1050.	0.02993

---

[1] (1 Absolute ohm = 0.9995 international ohm.)

TABLE A2

D-C RESISTANCE OF BARE CONCENTRIC-LAY CABLES OF STANDARD ANNEALED COPPER

Area in Circular Mils	Absolute Ohms per 1000 Feet at 25°C	Pounds per 1000 Feet	Standard Concentric Stranding		
			No. of Strands	Strand Diameter in Mils	Outside Diameter in Mils
2,000,000	0.00539	6180	127	125.5	1631
1,500,000	0.00719	4630	91	128.4	1412
1,000,000	0.0108	3090	61	128.0	1152
900,000	0.0120	2780	61	121.5	1093
800,000	0.0135	2470	61	114.5	1031
700,000	0.0154	2160	61	107.1	964
600,000	0.0180	1850	61	99.2	893
500,000	0.0216	1540	37	116.2	814
400,000	0.0270	1240	37	104.0	728
350,000	0.0308	1080	37	97.3	681
300,000	0.0360	926	37	90.0	630
250,000	0.0431	772	37	82.2	575

TABLE A3

ALLOWABLE CURRENT-CARRYING CAPACITIES OF ENCLOSED INSULATED COPPER CONDUCTORS

(National Electrical Code, 1947)

Gauge No.	Area in Circular Mils	Current-carrying Capacity in Amperes for Two Types of Insulation	
		Rubber	Asbestos-Varnished-Cambric
14	4,106.8	15	25–30
12	6,529.9	20	30–35
10	10,381	30	40–45
8	16,509	40–45	50–60
6	26,250	55–65	70–80
4	41,742	70–85	90–105
3	52,634	80–100	105–120
2	66,373	95–115	120–135
1	83,694	110–130	140–160
0	105,530	125–150	155–190
00	133,080	145–175	185–215
000	167,810	165–200	210–245
0000	211,600	195–230	235–275
	250,000	215–255	270–315
	300,000	240–285	300–345
	350,000	260–310	325–390
	400,000	280–335	360–420
	500,000	320–380	405–470
	600,000	355–420	455–525
	700,000	385–460	490–560
	800,000	410–490	515–600
	900,000	435–520	555
	1,000,000	455–545	585–680
	1,500,000	520–625	700–785
	2,000,000	560–665	775–840

TABLE A4

RESISTIVITY, TEMPERATURE COEFFICIENT, AND INTERCEPT ON t-AXIS FOR CONDUCTORS

Material	Resistivity at 20°C in Absolute Microhm-Centimeters	Temperature Coefficient at 20°C	Intercept on t-Axis in Degrees C[1]
Aluminum (hard drawn)	2.829	+0.0039	−236.
Carbon	3500	−0.0005	
Constantan	49.	+0.000008	−125,000.
Copper (standard annealed)	1.725	+0.00393	−234.5
(hard drawn)	1.78	+0.00382	−242.
German silver	33.	+0.0004	−2,500.
Gold (pure)	2.44	+0.0034	−274.
Iala	49.	+0.000005	−200,000.
Iron	11–13.5	+0.0055	−162.
Lead (average)	22.	+0.0039	−236.
Manganin	42–74	+0.00003	−33,000.
Mercury	96.	+0.00089	−1,100.
Monel metal	42.	+0.002	−480.
Nichrome II	110.	+0.00016	−6,230.
Nickel	7.8	+0.006	−147.
Platinum	10.	+0.003	−310.
Silver (pure)	1.622	+0.00361	−247.
Steel (soft)	17.4	+0.0042	−218.
Tin	11.5	+0.0042	−220.
Tungsten	5.5	+0.0051	−176.
Zinc	6.3	+0.004	−230.

---

[1] This is the intercept of the extended linear portion in the vicinity of 20°C (see Fig. 4.17A). It is not the intercept of the actual curve.

TABLE A5

RELATIVE PERMITTIVITIES OF DIELECTRICS, $K = \epsilon/\epsilon_0$, FOR FREQUENCIES LESS THAN ONE MEGACYCLE

Materials	Temperature in Degrees C	K
Gases		
Air (1 atmosphere)	0	1.000585
(20 atmospheres)	0	1.0117
(40 atmospheres)	0	1.0218
(80 atmospheres)	0	1.0439
Carbon dioxide	0	1.000985
Hydrogen	0	1.000264
Water vapor	145	1.00705
Liquids		
Acetone	20	21.3
Alcohol (ethyl)	20	25.7
(methyl)	20	31.2
(propyl)	20	21.8
Benzene	20	2.28
Distilled water	20	81.1
Glycerin	15	56.2
Oil (castor)	20	4.67
(petroleum)	20	2.13
(transil oil)	20	2.5
Pyranol	20	5.0
Solids		
Asphalt	20	2.68
Bakelite	20	5.0
Glass (flint, density 4.5)	20	9.90
(lead, density 3.0–3.5)	20	5.4–8.0
Gutta-percha	20	2.78–4.9
Mica	20	4.5–7.5
Paper (kraft)	20	3.5
Paraffin	20	1.9–2.3
Polystyrene	20	2.6
Porcelain	20	4.5–6.9
Rubber	20	2.3–4.0
Shellac	20	2.7–3.7
Compounds (at power frequencies)		
Oil-impregnated paper (solid types)	20	3.7
(oil-filled types)	20	3.5
Varnished Cambric	20	5.5
Rubber	20	5.0

TABLE A6

Dielectric Strengths of Insulating Materials

(Average Values)

Material	*Kilovolts per Meter*
Air	3,000
Bakelite	21,000
Glass	35,000
Gutta-percha	14,000
Mica	50,000
Paraffin	29,000
Polystyrene	30,000
Porcelain	10,000
Pyranol	20 000
Rubber	70 000
Transil Oil	10,000

TABLE A7

Volume Resistivities for Solid Dielectrics

Material	*Volume Resistivity in Megohm-Centimeters*
Glass (plate)	2×10^{7}
Hard rubber (new)	1×10^{12}
Marble (Italian)	1×10^{5}
Mica (colorless)	2×10^{11}
Paraffin	1×10^{10}
Porcelain (unglazed)	3×10^{8}
Quartz (fused)	5×10^{12}
Shellac	1×10^{10}
Slate	1×10^{2}

TABLE A8

Miscellaneous Constants

Permeability of free space

$\mu_0 = 4\pi \times 10^{-7}$ **henry per meter** (rationalized mks units),
$\mu_0 = 1\,000$ unrationalized cgs electromagnetic units

Permittivity of free space:

$\epsilon_0 = 8\,854 \times 10^{-12} \simeq \frac{1}{36\pi} \times 10^{-9}$ **farad per meter** (rationalized mks units),

$\epsilon_0 = 1\,000$ unrationalized cgs electrostatic units.

Velocity of light (in rationalized mks units)

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}} = 2\,998 \times 10^{8} \simeq 3 \times 10^{8} \text{ meters per second.}$$

Logarithms:

$$\ln x = \log_e x = 2\,303 \log_{10} x = 2\,303 \log x.$$

TABLE A9
GREEK ALPHABET

Letter	Capital	Small
Alpha	Α	α
Beta	Β	β
Gamma	Γ	γ
Delta	Δ	δ
Epsilon	Ε	ϵ
Zeta	Ζ	ζ
Eta	Η	η
Theta	Θ	θ
Iota	Ι	ι
Kappa	Κ	κ
Lambda	Λ	λ
Mu	Μ	μ
Nu	Ν	ν
Xi	Ξ	ξ
Omicron	Ο	ο
Pi	Π	π
Rho	Ρ	ρ
Sigma	Σ	σ
Tau	Τ	τ
Upsilon	ϒ	υ
Phi	Φ	φ, ϕ
Chi	Χ	χ
Psi	Ψ	ψ
Omega	Ω	ω

TABLE A10

FACTORS RELATING RATIONALIZED MKS UNITS AND UNRATIONALIZED CGS UNITS

(A Quantities Unaffected by Rationalization)

Multiply Number of MKS Units Below	By	To Obtain Number of CGS Units Below
Ampere	10^{-1}	Abampere
Ampere per square meter	10^{-5}	Abampere per square centimeter
Coulomb	10^{-1}	Abcoulomb
Coulomb per cubic meter	10^{-7}	Abcoulomb per cubic centimeter
Farad	10^{-9}	Abfarad
Henry	10^{9}	Abhenry
Mho per meter	10^{-11}	Abmho per centimeter
Ohm	10^{9}	Abohm
Volt	10^{8}	Abvolt
Weber	10^{8}	Maxwell
Weber per square meter	10^{4}	Gauss
Joule	10^{7}	Erg
Newton	10^{5}	Dyne
Ampere	3×10^{9}	Statampere
Ampere per square meter	3×10^{5}	Statampere per square centimeter
Coulomb	3×10^{9}	Statcoulomb
Coulomb per cubic meter	3×10^{3}	Statcoulomb per cubic centimeter
Farad	9×10^{11}	Statfarad
Henry	$\frac{1}{9} \times 10^{-11}$	Stathenry
Mho per meter	9×10^{9}	Statmho per centimeter
Ohm ..	$\frac{1}{9} \times 10^{-11}$	Statohm
Volt	$\frac{1}{3} \times 10^{-2}$	Statvolt

(B Quantities Affected by Rationalization)

Multiply Number of Rationalized MKS Units Below	By	To Obtain Number of Unrationalized CGS Units Below
Ampere-turn	$4\pi \times 10^{-1}$	Gilbert
Ampere-turn per meter	$4\pi \times 10^{-3}$	Oersted

(In case the cgs units also are rational, the factors 4π are suppressed)

TABLE A11
Conversion Factors

Quantity	Symbol	Multiply Number of	By	To Obtain Number of
Length	l or s	Centimeters	3.281×10^{-2}	Feet
		Centimeters	0.3937	Inches
		Centimeters	0.01	Meters
		Centimeters	393.7	Mils
		Feet	30.48	Centimeters
		Feet	0.3048	Meters
		Feet	1.200×10^{4}	Mils
		Inches	2.540	Centimeters
		Inches	0.02540	Meters
		Inches	1000.	Mils
		Meters	100.0	Centimeters
		Meters	3.281	Feet
		Meters	39.37	Inches
		Meters	3.937×10^{4}	Mils
		Mils	2.540×10^{-3}	Centimeters
		Mils	0.001	Inches
Area	A or S	Circular mils	5.067×10^{-6}	Square centimeters
		Circular mils	7.854×10^{-7}	Square inches
		Square centimeters	1.973×10^{5}	Circular mils
		Square centimeters	0.1550	Square inches
		Square centimeters	10^{-4}	Square meters
		Square inches	1.273×10^{6}	Circular mils
		Square inches	6.452	Square centimeters
		Square inches	6.452×10^{-4}	Square meters
		Square meters	10^{4}	Square centimeters
		Square meters	1550.	Square inches
Force	F	Dynes	10^{-5}	Newtons
		Dynes	2.248×10^{-6}	Pounds
		Newtons	10^{5}	Dynes
		Newtons	0.2248	Pounds
		Pounds	4.448×10^{5}	Dynes
		Pounds	4.448	Newtons
Torque	T	Dyne-centimeters	10^{-7}	Newton-meters
		Dyne-centimeters	7.376×10^{-8}	Pound-feet
		Newton-meters	10^{7}	Dyne-centimeters
		Newton-meters	0.7376	Pound-feet
		Pound-feet	1.356×10^{7}	Dyne-centimeters
		Pound-feet	1.356	Newton-meters
Energy	W	British thermal units	1054.8	Joules (watt-seconds)
		British thermal units	2.930×10^{-4}	Kilowatt-hours
		Foot-pounds	1.356	Joules (watt-seconds)
		Foot-pounds	3.766×10^{-7}	Kilowatt-hours
		Horsepower-hours	0.7457	Kilowatt-hours
		Horsepower-hours	2.684×10^{6}	Joules
		Joules (watt-seconds)	9.480×10^{-4}	British thermal units
		Joules (watt-seconds)	0.7376	Foot-pounds
		Joules (watt-seconds)	3.725×10^{-7}	Horsepower-hours
		Joules (watt-seconds)	2.389×10^{-4}	Kilogram-calories
		Joules (watt-seconds)	2.778×10^{-7}	Kilowatt-hours
		Kilogram-calories	4186.	Joules
		Kilogram-calories	1.163×10^{-3}	Kilowatt-hours
		Kilowatt-hours	3413.	British thermal units
		Kilowatt-hours	2.655×10^{6}	Foot-pounds
		Kilowatt-hours	1.341	Horsepower-hours
		Kilowatt-hours	3.600×10^{6}	Joules
		Kilowatt-hours	860.0	Kilogram-calories

TABLE A11 (*Continued*)
CONVERSION FACTORS

Quantity	Symbol	Multiply Number of	By	To Obtain Number of
Power	P or p	Foot-pounds per second	1.356×10^{-3}	Kilowatts
		Foot-pounds per second	1.356	Watts
		Horsepower	0.7457	Kilowatts
		Horsepower	745.7	Watts
		Kilowatts	737.6	Foot-pounds per second
		Kilowatts	1.341	Horsepower
		Kilowatts	1000.	Watts
		Watts	0.7376	Foot-pounds per second
		Watts	1.341×10^{-3}	Horsepower
		Watts	10^{-3}	Kilowatts
Electric charge	Q or q	Abcoulombs	10.	Coulombs
		Coulombs	0.1	Abcoulombs
		Coulombs	2.998×10^{9}	Statcoulombs
		Statcoulombs	3.335×10^{-10}	Coulombs
Current	I or i	Abamperes	10.	Amperes
		Amperes	0.1	Abamperes
		Amperes	2.998×10^{9}	Statamperes
		Statamperes	3.335×10^{-10}	Amperes
Potential	E or e; V or v	Abvolts	10^{-8}	Volts
		Statvolts	299.8	Volts
		Volts	10^{8}	Abvolts
		Volts	3.335×10^{-3}	Statvolts
Resistance	R	Abohms	10^{-9}	Ohms
		Ohms	10^{9}	Abohms
		Ohms	1.112×10^{-12}	Statohms
		Statohms	8.988×10^{11}	Ohms
Resistivity	ρ	Microhm-centimeters	10^{-6}	Ohm-centimeters
		Microhm-centimeters	10^{-8}	Ohm-meters
		Microhm-centimeters	6.015	"Ohms per mil-foot"
		Ohm-centimeters	10^{6}	Microhm-centimeters
		Ohm-centimeters	0.01	Ohm-meters
		Ohm-centimeters	6.015×10^{6}	"Ohms per mil-foot"
		Ohm-meters	10^{8}	Microhm-centimeters
		Ohm-meters	100.	Ohm-centimeters
		Ohm-meters	6.015×10^{8}	"Ohms per mil-foot"
		"Ohms per mil-foot"	0.1662	Microhm-centimeters
		"Ohms per mil-foot"	1.662×10^{-7}	Ohm-centimeters
		"Ohms per mil-foot"	1.662×10^{-9}	Ohm-meters
Magnetic flux	Φ or φ	Kilolines (kilomaxwells)	1000.	Maxwells (lines)
		Kilolines (kilomaxwells)	10^{-5}	Webers
		Maxwells (lines)	10^{-8}	Webers
		Webers	10^{5}	Kilolines (kilomaxwells)
		Webers	10^{8}	Maxwells (lines)
Magnetic flux density	B	Gausses (lines per sq. cm.)	6.452	Lines per square inch
		Gausses (lines per sq. cm.)	10^{-4}	Webers per square meter
		Lines per square inch	0.1550	Gausses
		Lines per square inch	1.550×10^{-5}	Webers per square meter
		Webers per square meter	10^{4}	Gausses
		Webers per square meter	6.452×10^{4}	Lines per square inch
Magnetomotive force	$\mathcal{F}$	Ampere-turns	1.257	Gilberts
		Ampere-turns	12.57	Pragilberts
		Gilberts	0.7958	Ampere-turns
		Gilberts	10.	Pragilberts
		Pragilberts	7.958×10^{-2}	Ampere-turns
		Pragilberts	0.1	Gilberts

TABLE A11 (*Continued*)
CONVERSION FACTORS

Quantity	Symbol	Multiply Number of	By	To Obtain Number of
Magnetizing force (magnetic intensity)	H	Ampere-turns per cm.	2 540	Ampere-turns per inch
		Ampere-turns per cm.	1 257	Oersteds
		Ampere-turns per cm.	1257	Praoersteds
		Ampere-turns per inch	0 4950	Oersteds
		Ampere-turns per inch	495 0	Praoersteds
		Ampere-turns per meter	0 0254	Ampere-turns per inch
		Ampere-turns per meter	1 257 × 10^{-2}	Oersteds
		Ampere-turns per meter	12 57	Praoersteds
		Oersteds	2 021	Ampere-turns per inch
		Oersteds	1000	Praoersteds
		Praoersteds	2 021 × 10^{-3}	Ampere-turns per inch
		Praoersteds	10^{-3}	Oersteds
Inductance	L; M	Abhenrys	10^{-9}	Henrys
		Henrys	10^{9}	Abhenrys
		Henrys	1 112 × 10^{-12}	Stathenrys
		Stathenrys	8 988 × 10^{11}	Henrys
Electric intensity (potential gradient)	E	Volts per centimeter	2 540	Volts per inch
		Volts per centimeter	2.540 × 10^{-3}	Volts per mil
		Volts per meter	0 0254	Volts per inch
		Volts per meter	2 540 × 10^{-5}	Volts per mil
		Volts per inch	0 3937	Volts per centimeter
		Volts per inch	39 37	Volts per meter
		Volts per mil	393 7	Volts per centimeter
		Volts per mil	3 937 × 10^{4}	Volts per meter
Capacitance	C	Abfarads	10^{9}	Farads
		Farads	10^{-9}	Abfarads
		Farads	8.988 × 10^{11}	Statfarads
		Statfarads	1.112 × 10^{-12}	Farads

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